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The History and Calculation of Pi

By Herman H. Harris, Jr.

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The History and Calculation of Pi

by Herman H. Harris, Jr.*

I

A BRIEF HISTORY OF PI

The number *pi* occupies an unique place in the history of mathematics. While academicians, students, and laymen generally recognize the importance of pi and are aware of some of its many implications, there exists today a vagueness as to its historical significance, and an uncertainty concerning the numerous methods which, in the past, have been employed in its calculation. Whether defined as the ratio of the circumference of a circle to the diameter, or as the ratio of the area of a circle to the square on half the diameter, it has been the object of an intensive search by all nations from the earliest of times to the present day. It is significant that the number *pi* has wound itself through the structure of mathematics into the fabric of modern civilization.

The computation of pi is closely associated with the quadrature of the circle, one of three famous mathematical problems.[1] Its earliest recorded approximation is the number *three*, used by the Hebrews, Egyptians, and Babylonians, an approximation which was accepted for many centuries. One discovers references to this figure in the Old Testament, 1 *Kings* 7:23, and 2 *Chronicles* 4:22. In the latter source, one finds the following statement:

> Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits compass it round about.

However, the earliest traces of pi are to be noted in the Rhind Papyrus (1700 B.C.), now preserved in the British Museum, translated and explicated by Eisenlohr, in which is recorded ". . . the area of a circle is equal to that of a square whose side is the diameter diminished by one-ninth."[2] Such a qualification gives an approximation for pi as 3.1604, greater than 3.1416 by about 0.6%. Later, the so-called geometrical method of computing the value of pi was frequently employed, some mathematicians obtaining a closer approximation than others. This particular method consisted of inscribing in a circle, and circumscribing about the circle, a number of regular polygons. Next, by doubling in succession the number of sides and determing the perimeters or areas of the polygons, mathematicians were subsequently enabled to calculate an approximation. Furthermore, if this process were performed for a number of successive times, individuals discovered that they were able to obtain an even closer approximation. Among many who computed pi by this method were Archimedes,

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Francois Vieta, Adriaen Van Roomen, Ludolph Van Ceulen, Willebrord Snell, and Grienberger.

Today, it is generally conceded that the first truly scientific attempt to compute pi was undertaken by Archimedes (c. 240 B.C.), who discovered an approximation by means of the aforementioned method of circumscribing and inscribing regular polygons in a circle. In his efforts to obtain an approximation for pi, he recorded an upper and lower limit, and eventually concluded that pi was located between these limitations. He first used a regular hexagon circumscribed about a circle. Each time, thereafter, by determining the perimeter of the regular polygon (up to 96 sides), he resolved that pi was less than 3 1/7. Next, he determined a lower limit by inscribing regular polygons of six, twelve, twenty-four, forty-eight, and ninety-six sides into the circle, establishing for each successive polygon its subsequent perimeter, which he discovered to be always less than the circumference of the circle. From such an experiment, he concluded that the circumference exceeded three times its diameter by a part of which was less than 1/7, but more than 10/71 of the diameter.[3] Since 3 1/7 is greater than 3.1416 by about 0.04%, and is a smple number for ordinary computations, Archimedes' value for pi is still in common use, today. His approximation for pi is considerably closer, also, than that described in the biblical references.[4]

Claudius Ptolemy (c. 150 A.D.), a teacher in Athens and Alexandria, gave the world its next notable value for pi:

> His value for pi is given, in sexagesimal notation, as $3° \, 8' \, 30''$, which is equal to approximately 3.1416. This value was probably derived from the table of chords, which appears in his treatise. This table gives the lengths of the chords of a circle subtended by central angles of each degree and half degree. If the length of the chord of one degree central angle is multiplied by three-hundred-sixty, and the result divided by the length of the diameter of the circle, the value of pi is obtained.[5]

In the Eastern culture, the Chinese values of pi were 3 and the $\sqrt{10}$.[6] The most interesting extant record from the Chinese is, however, that of Tsu Ch'ung-chih in the fifth century, who found 31.415927 and 31.415926 for the limits of ten pi, from which he inferred, by a reasoning process unexplained in his work, that 22/7 and 355/113 were approximate values.[7]

An early Hindu mathematician, Aryabhata (c. 530), gave 62,832/20,000 as an approximate value, a figure that is equal to 3.1416:

> He showed that, if **a** is the side of a regular polyon of **n** sides inscribed in a circle of unit diameter, and if **b** is the side of a regular inscribed polygon of **2n** sides, then $\mathbf{b}^2 = \frac{1}{2} - \frac{1}{2}(1 - a^2)^{\frac{1}{2}}$. From the side of an inscribed hexagon, he found successively the sides of polygons of twelve, twenty-four, forty-eight, ninety-six, one hundred ninety-two, and three hundred eighty-four sides. The perimeter of the last is given as equal to $\sqrt{9.8684}$, from which his result was obtained by approximation.[8]

The most prominent Hindu mathematician of the seventh century was Brahmagupta (c. 650), who gave the $\sqrt{10}$ as a value for pi, equal to 3.1622, approximately 0.66% greater than the value, 3.1416.

> He obtained this value by inscribing in a circle of unit diameter regular polygons of twelve, twenty-four, forty-eight, and ninety-six sides, and calculating successively their perimeters, which he found to be $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, and $\sqrt{9.87}$, respectively; and to have [sic] assumed that as the number of sides is increased indefinitely the perimeter would approximate to $\sqrt{10}$.[9]

Bhaskara, also a Hindu mathematician (c. 1150), gave 3927/1205 equal to 3.14160. Furthermore, he cited 754/240, equivalent to 3.14166 for pi, but scholars question whether this last figure was cited only as an approximate value.[10]

In 1579, Francois Vieta, a French mathematician, determined pi correct to nine decimal places, by showing that ". . . pi was greater than $31415926535/10^{10}$, and less than $31415926537/10^{10}$."[11] He deduced this information from ". . . the perimeters of the incribed and circumscribed polygons of 6×2^{16} sides, obtained by repeated use of the formula, $2 \sin^2 \frac{1}{2} \theta = 1 - \cos \theta$"[12] In 1585, another Frenchman, Adriaen Anthonisz, produced the ratio 355/113, which is equal to 3.14159292, correct to six decimal places. It was apparently a lucky accident for Anthonisz, since all he had demonstrated was that pi was located between 377/120 and 333/106. He averaged the numerators and the denominators, thereafter, to obtain the approximate value for pi.[13]

Adriaen Van Roomen, a Dutch mathematician, in 1593 calculated the perimeter of the inscribed regular polygon of a 2^{30} sides from which he, in turn, determined the value of pi correct to fifteen decimal places.[14] Ludolph Van Ceulen, a German, computed pi to thirty-five decimal places by calculating the perimeter of a polygon having 2^{82} sides.[15] It was a Dutch physicist, Willebrord Snell, however, who, in 1621, devised a trigonometrical improvement over the so-called "classical method" of computing pi. From each pair of bounds on pi, established by the classical method of computation, he was able to obtain considerably closer bounds, even to duplicating Van Ceulen's earlier thirty-five decimal places. He accomplished this result by using polygons ". . . having only 2^{30} sides."[16] It is further significant to realize that the classical method, making use of such polygons, yielded only fifteen decimal places, and for polygons of ninety-six sides, two decimal places; whereas, ". . . Snell's improvement gives seven places."[17] Grienberger, in 1630, using Snell's refinement of method, carried the approximation to thirty-nine decimal places.[18]

While the geometric method of computation had been popularly relied upon for many centuries, it now became possible to calculate pi by other methods introduced into mathematics. One such means was the analytical method of computing by the convergent series, products, and continued fractions. Vieta (c. 1593), for example, discovered an interesting approximation for pi, using continued products for the result. His value may be obtained from the following formula:

$$2/\pi = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\ \sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\ \cdot\ \cdot\ \cdot$$

In 1650, John Wallis, an English mathematician, proved that

$$\pi/2 = 2/1 \cdot 2/3 \cdot 4/3 \cdot 4/5 \cdot 6/5 \cdot 6/7 \cdot \cdot \cdot \cdot$$ [30]

He quoted, in addition, a proposition which had been given a few years earlier by Viscount Brouncker, to the effect that

$$4/\pi = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cdot \cdot \cdot}}}$$ [21]

However, neither of these theorems was used to any large extent for future calculation. In 1668, James Gregory, a Scotch mathematician, derived a series which was used subsequently in connection with other relationships in calculating the value of pi. His series was as follows:

$$\text{arc tan } x = x - x^3/3 + x^5/5 - x^7/7 + \cdot \cdot \cdot \cdot$$ [22]

Later, in 1673, Gottfried Wilhelm Leibniz, a German mathematician, utilizing Gregory's series and permitting $x=1$, derived a subsequent series:

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdot \cdot \cdot \cdot$$ [23]

It is obvious that Leibniz' series converges rather slowly for an accurate value of pi. In 1699, Abraham Sharp, Englishman, also making use of Gregory's series and letting $x=\sqrt{1/3}$, derived the following series:

$$\pi/6 = \sqrt{\tfrac{1}{3}}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \cdot \cdot \cdot \cdot\right)$$ [24]

Sharp's series produces a result more usable than that for $\pi/4$, giving an approximation for pi to seventy-one decimal places. In 1706, John Machin, also an Englishman, by substituting Gregory's infinite series for *arc tan* 1/5 and *arc tan* 1/239, gave the expression,

$$\pi/4 = 4 \text{ arc tan } 1/5 - \text{arc tan } 1/239.$$ [25]

Such a convergent series is faster and more useful, and Machin succeeded in calculating pi correctly to one hundred decimal places. In 1873, another Englishman, William Shanks, computed pi to 707 decimal places, using Machin's formula; but in 1946, D. F. Ferguson found Shanks' computation to be in error in the 528th place.[26]

It was soon discovered that the value of pi might be determined experimentally by applications of the probability theory. For example, in 1760, Comte de Buffon devised his now-famous Needle Problem to determine pi by probability:

> On a plane a number of equidistant parallel straight lines, distance apart **a**, are ruled; and a stick of length **1**, which is less than **a**, is dropped on to the plane. The probability that it will fall so as to lie across one of the lines is $2\ 1/\pi a$. If the experiment is repeated many hundreds of times, the ratio of the number of favorable cases to the

whole number of experiments will be very nearly equal to this fraction; hence, the value of pi can be found.[27]

In the middle of the eighteenth century, mathematicians began anew to investigate the nature of the number *pi,* to determine if it were rational or algebraic or transcendental. The first such investigation of a fundamental importance was that undertaken by Johann Heinrich Lambert, a German, in 1761. Lambert obtained the two fractions,

$$\frac{e^x-1}{e^x-1} + \frac{1}{2/x} + \frac{1}{6/x} + \frac{1}{10/x} + \frac{1}{14/x + \cdots},$$

and

$$\tan x = \frac{1}{1/x} - \frac{1}{3/x} - \frac{1}{5/x} - \frac{1}{7/x - \cdots},$$

both are closely related with continued fractions obtained by Euler, but the convergence of which Euler had not established. As a result of an investigation of the properties of these continued fractions, Lambert established the following theorems: (1) if x is a rational number, different from zero, e^x cannot be a rational number. (2) If x is a rational number, different from zero, tan x cannot be a rational number. (3) If $x=\pi/4$, we have $\tan x=1$, and therefore $\pi/4$ cannot be a rational number, hence π cannot be a rational number.[28]

Following the discovery of the distinction between algebraic and transcendentals, man questioned to which of these two categories the number *pi* belonged. An algebraic irrational, first of all, is one which is a root of the equation of the form,

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = O,$$

where *n* is a rational number and $a_0, a_1, a_2, \cdots, a_n$ are rational. A transcendental number, on the other hand, is one for which such an equation is not satisfied. In 1882, Ferdinand Lindemann, a German mathematician, proved that the above equation could not hold, when x=e; n and a_0, $a_1, a_2, \cdots, a_n$ are algebraic numbers, not necessarily real. Euler had previously shown that $e^{i\pi} + 1 = O$. Now, if π is algebraic, then $i\pi$ is algebraic, and, thus, $e^n + 1 = 0$ is satisfied by $n = i\pi$, contradicting the theorem of Lindemann; hence it is proved that π is not algebraic, but transcendental.[29]

In addition, there are occasions today when it is desired to express pi to more than the well-remembered four decimal places. When this occurs, memory can be aided by the use of mnemonics, in which the number of letters in a word is the key to the appropriate digit involved. One example, taken from *School Science and Mathematics,* gives pi correct to twelve digits:

3 1 4 1 5 9

See, I have a rhyme assisting

2 6 5 3 5 9

My feeble brain its tasks resisting.[30]

Another mnemonics system, giving pi to thirty-one significant digits, appeared in the *Literary Digest,*[31] and was repeated in Moritz' *Memorabilia Mathematica*:

3 1 4 1 5 9
Now I, even I, would celebrate
2 6 5 3 5
In rhymes inapt, the great
8 9 7 9
Immortal Syracusan, rivaled nevermore,
6 2 3 8 4
Who in his wondrous lore,
6 2 6
Passed on before,
4 3 3 8 3 2 7 9
Left men his guidance how to circles mensurate.[32]

With the advent of the electronic calculating machines, man has achieved a calculation of an extraordinary magnitude. In 1949, an electronic calculator, known as ENIAC, gave pi to 2035 decimal places at the Army Ballistic Research Laboratories, Aberdeen, Maryland, in a matter of about seventy hours.[33] The most recent computation of pi was achieved by the Paris Data Processing Center in 1958, where, in only forty seconds, the IBM 704 computed pi to 707 decimal places, the number calculated by hand by William Shanks in 1873. The Paris 704, however, using the formula, $\pi/4=4$ arc tan $1/5-$arc tan $1/239$, extended this computation, within a period of one hour and forty minutes, to 10,000 decimal places, a result which makes use of less than one-half of the 20,500 decimal-place capacity for which the Paris 704 may be programmed.[34]

II

GEOMETRICAL METHODS FOR THE DETERMINATION OF PI

The history of the determination of the ratio of the circumference to the diameter of a circle falls into four periods, divided by fundamentally distinct differences with respect to method, immediate aims, and the advancement of mathematical knowledge. The first period embraces the time between the earliest record of the determinations of the ratio of the circumference to the diameter of a circle and the middle of the seventeenth century, a time span which shall be described as the geometrical period, hereafter. The main activity during these years consisted in the approximate determination of pi by calculation of the sides or areas of regular polygons inscribed and circumscribed to the circle. In the earlier reaches of the period, in spite of unavoidable difficulties, a number of surprisingly good approximations of pi was obtained. Later, geometric methods were devised by which approximations to the value of pi were achieved, requiring only a fraction of the labor involved previously in the earlier calculations. At the close of the period, geometric methods were developed to such a degree of exactness that no further advance could be hoped for along these lines. Further progress demanded the institution of more powerful methods.[35]

A Value of Pi in the Rhind Papyrus. One of the oldest known mathematical documents, the Rhind Papyrus (c. 1700 B.C.), contains an expression for pi, showing that the area of a circle is equal to that of a square whose side is the diameter diminished by one-ninth.[36] The document cites no reason for having taken one-ninth off the diameter, except for the fact that such a method appeared to lead to a satisfactory value of pi. Figure 1 shows a square presumably equal in area to that of a circle. By taking 1/9 off the diameter *AE* of the circle and by constructing a square upon the remainder *AB*, the area S, of the square *ABCD*, equals $64/81\ AE^2$; the area S, of the circle *O*, equals $1/4\ \pi\ AE^2$. The relation, $1/4\ \pi\ AE^2 = 64/81\ AE^2$, leads to the value for pi, $\pi = 256/81 = 3.1604$. This value is about 0.6% greater than the value 3.1416.

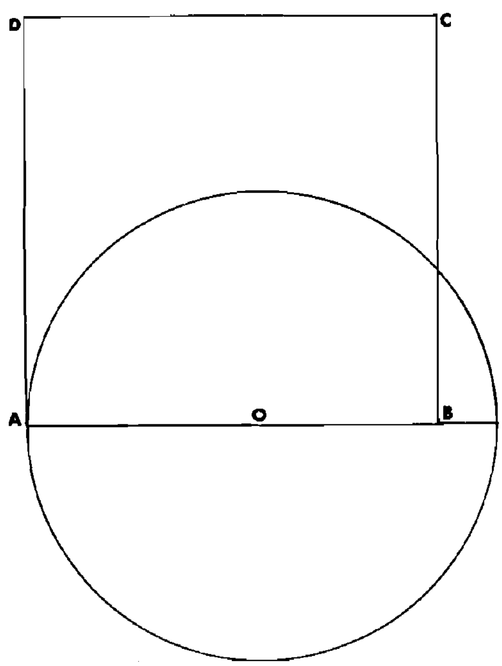

Figure 1
Square Approximately Equal in Area to a Circle

Calculation of Pi by Means of Perimeters of Circumscribed and Inscribed Regular Polygons. The circumference of a circle is approximately equal to the perimeter of a circumscribed or inscribed regular polygon of many sides. The value of pi is the ratio of the circumference to the diameter. In this relation, the magnitude of the error introduced in replacing the circumference by the perimeter of a circumscribed or inscribed regular polygon depends upon the number of sides of the polygon. In order to obtain the value of pi to several significant digits, the number of sides of the polygon must be progressively increased until the desired accuracy is reached. If c_n and i_n, respectively, denote sides of regular circumscribed and inscribed n-gons, the sides of corresponding regular circumscribed and incribed 2n-gons are given by the formulas,

$$\text{(I)} \quad c_{2n} = \frac{c_n \bullet i_n}{c_n + i_n},$$

and

$$\text{(II)} \quad i_{2n} = \tfrac{1}{2}\sqrt{2 \bullet c_{2n} \bullet i_n}\,.$$ [37]

The relation between the sides of regular circumscribed and inscribed polygons is illustrated in Figure 2. A_1B_1 and AB are corresponding sides of circumscribed and inscribed regular n-gons, and G_1H_1 and AH are corresponding sides of circumscribed and inscribed 2n-gons. If C_n and I_n denote, respectively, perimeters of regular circumscribed and inscribed polygons, the perimeters of circumscribed and inscribed regular n-gons, then, are given by the formulas, (III) $C_n = nc_n$, and (IV) $I_n = ni_n$. If C_{2n} denotes the perimeter of a regular circumscribed 2n-gon, the perimeter is given, then, by the formula,

$$\text{(V)} \quad \begin{aligned} C_{2n} &= 2n \bullet c_{2n} \\ &= 2n \frac{c_n \bullet i_n}{c_n + i_n} \\ &= 2 \frac{nc_n \bullet ni_n}{nc_n + ni_n}, \\ C_{2n} &= \frac{2 \bullet C_n \bullet I_n}{C_n + I_n}. \end{aligned}$$

If I_{2n} denotes the perimeter of a regular inscribed 2n-gon, the perimeter may be given, then, by the formula,

$$\text{(VI)} \quad \begin{aligned} I_{2n} &= 2n \bullet i_{2n} \\ &= 2n \bullet \tfrac{1}{2}\sqrt{2 \bullet c_{2n} \bullet i_n} \\ &= n\sqrt{2 \bullet \frac{C_{2n}}{2n} \bullet \frac{I_n}{n}}, \\ I_{2n} &= \sqrt{C_{2n} \bullet I_n}\,. \end{aligned}$$

In order to find the perimeters of regular circumscribed and inscribed polygons, it is convenient to start with such simple polygons as the equilateral triangle, the square, or the regular hexagon, applying the formulas for the perimeters of polygons with double the number of sides of the given polygons. If the original regular polygon has *n* sides, then *k* successive applications of the formulas, V and VI above, give perimeters of polygons whose number of sides equals $n \bullet 2^k$. In order to make successively better approximations of pi, one starts with the perimeters of circumscribed and inscribed regular hexagons; next, with the perimeters of regular circumscribed and inscribed dodecagons; then, in succession, with the perimeters of polygons with 24 sides, 48 sides, 96 sides, 192 sides, 384 sides, 768 sides, and 1,536 sides. In a unit circle, the perimeter of a regu-

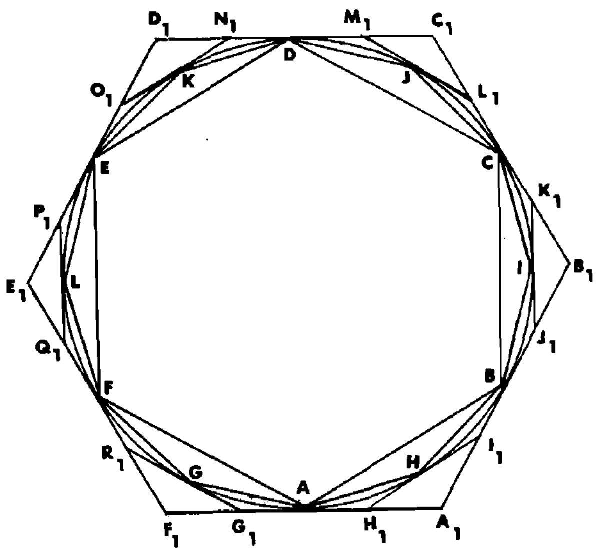

Figure 2

Circumscribed and Inscribed Regular Polygons

lar circumscribed hexagon C_6 equals $2\sqrt{3}$, and the perimeter of the regular inscribed hexagon I_6 equals 3. The perimeters of regular circumscribed and inscribed dodecagons are obtained by using formulas V and VI. Thus,

$$C_{12} = \frac{2 \bullet C_6 \bullet I_6}{C_6 + I_6}$$

$$= \frac{2(2\sqrt{3})\ (3)}{2\sqrt{3} + 3}$$

$$= 4\sqrt{3}\quad (2\sqrt{3} - 3)$$

$$= 3.2153903, \text{ approximately.}$$

Thus,

$$I_{12} = \sqrt{C_{12} \bullet I_6}$$

$$= \sqrt{12\sqrt{3}\ (2\sqrt{3} - 3)}$$

$$= \sqrt{9.6461709}$$

$$I_{12} = 3.10582885, \text{ approximately.}$$

The perimeters of regular circumscribed and inscribed polygons of 24 sides are obtained by the relations,

$$C_{24} = \frac{2 \bullet C_{12} \bullet I_{12}}{C_{12} + I_{12}}, \text{ and } I_{24} = \sqrt{C_{24} \bullet I_{12}}.$$

The above values and those obtained by the continuation of this process are presented in Table I, which contains the results of applying formulas

TABLE I

PERIMETERS OF CIRCUMSCRIBED AND INSCRIBED REGULAR POLYGONS

Number of sides	Perimeters of Circumscribed Polygons	Perimeters of Inscribed Polygons
6	3.4641016	3.0000000
12	3.2153903	3.1058285
24	3.1596599	3.1326286
48	3.1460862	3.1393502
96	3.1427146	3.1410319
192	3.1418730	3.1414524
384	3.1416627	3.1415576
768	3.1416101	3.1415838
1536	3.1415970	3.1415904

(V) and (VI) in eight successive steps, following the calculation of the perimeters of the hexagon. The eighth step yields the values of the perimeters of regular circumscribed and inscribed polygons of 1,536 sides. The perimeters are indicated to seven decimal places in the Table, and the perimeter of the circumscribed and inscribed regular polygon of 1,536 sides gives an approximation value of pi, correct to five decimal places. If greater accuracy were to be desired, the process should be continued further, and each calculation should be carried out to an appropriate number of decimal places. This process of circumscribing and inscribing regular polygons to determine a more accurate value of pi was carried on for a number of years. In 1579, Francois Vieta, a Frenchman, considered polygons of $6 \cdot 2^{16}$ sides–*i.e.*, 393,216 sides–finding pi correct to nine places. In 1593, Adriaen Van Roomen found pi correct to 15 decimal places by computing the perimeter of a regular circumscribed polygon of 2^{30} sides –*i.e.*, 1,073,741,824 sides. In 1610, Ludolph Van Ceulen found pi correct to 35 places by determining the perimeter of a polygon of 2^{62} sides, or 4,611,686,018,427,387,904 sides.[38] He devoted a considerable part of his life to this task, and his achievement was inscribed upon his tombstone, and it is frequently referred to in Germany as "the Ludolphian number."

Quadratrix of Dinostratus. About the year 425 B.C., Hippias invented a curve known as the Quadratrix, often associated with the name of Dinostratus (c. 350 B.C.), who studied the curve with care and who showed

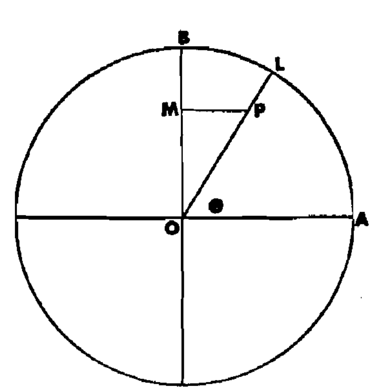

Figure 3

Construction For Pi

that the use of the curve gives a construction for pi. The curve may be described in the following manner. If a circle of unit radius (Figure 3) has two perpendicular radii, OA and OB, and if two points, M and L, move with constant velocity, one upon the radius OB, the other upon the arc AB, starting at the same time at O and A, they arrive simultaneously at B. The point of intersection $P(x, y)$ of OL and the parallel to OA through M describes the quadratrix.[39] From this definition, it follows that y is proportional to θ. Furthermore, since if $y = 1$, $\theta = \pi/2$; therefore, $\theta = \pi/2\ y$. And from $\theta = \text{arc tan } y/x$, the equation of the curve becomes $y/x = \tan \pi/2\ y$. The curve meets the axis of x at the point whose abscissa is

$$x = \lim_{y \to 0} \left(\frac{y}{\tan \pi/2\ y} \right); \text{ hence, } x = 2/\pi.$$

According to this formula, the radius of the circle is the mean proportional between the abscissa of the intersection of the quadratrix with the axis of X and the length of the quadrant. It is necessary that the measurement of the quadrant be known before an approximation of pi can be determined, however. The formula is $2\pi/1 = 1/1.5707963$. Solving for pi, one obtains $\pi = 3.1415926$, a value that is correct to the indicated decimal places.

A Discovery by Archimedes. The circumference of a circle has a value between the perimeter of any circumscribed polygon and that of any inscribed polygon. Since it is a simple matter to compute the perimeters of the regular circumscribed and inscribed six-sided polygons, it is easy to obtain bounds for pi. By doubling the sides of the regular polygons, one may obtain the perimeters of the regular circumscribed and inscribed polygons. If the process is continued, it will further yield closer bounds for pi.

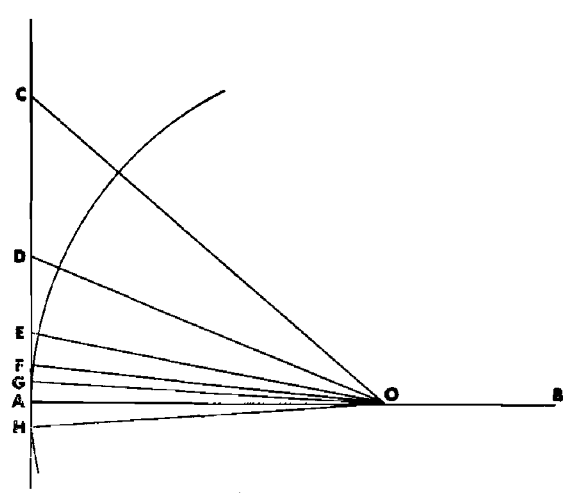

Figure 4

Sides of Regular Circumscribed Polygons

Archimedes (c. 240 B.C.) discovered that pi is less than 3 1/7 and greater than 3 10/71.[40] He established this value by circumscribing about a circle, and inscribing in it, regular polygons of 96 sides. T. L. Heath in *The Works of Archimedes* shows the procedure supposedly used by Archimedes in establishing this value for pi. Heath's explanation is hereafter paraphrased.[41] In Figure 4, GH is one side of a regular polygon of 96 sides, circumscribed to a given circle. Since OA : AG>4673½ : 153, while AB=2OA, GH=2AG, it necessarily follows that

$$AB : (\text{perimeter of polygon of 96 sides}) > 4673\tfrac{1}{2} : (153 \cdot 96) = 4673\tfrac{1}{2} : 14688.$$

However,

$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} = 3\tfrac{1}{7}$$

In Figure 5, *BG* is a side of a regular inscribed polygon of 96 sides. Therefore, AB : BG < 2017¼ : 66, and BG : AB > 66 : 2017¼. It follows that

$$(\text{perimeter of polygon}) : AB > (96 \quad 66) : 2017\tfrac{1}{4} = 6336 : 2017\tfrac{1}{4}$$

But,

$$\frac{6336}{2017\frac{1}{4}} = 3\ 10/71.$$

Thus, the ratio of the circumference to the diameter lies between 3 10/71 and 3 1/7.

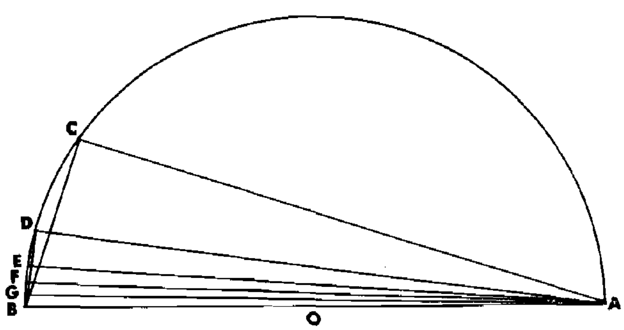

Figure 5

Sides of Regular Inscribed Polygons

A Value by Ptolemy. A value of pi was given by Claudius Ptolemy of Alexandria (c. 150 A.D.) in his astronomical treatise, *Syntaxis* (*Almagest* in Arabic).[42] In this work, he gives pi, in sexagesimal notation, as 3° 8′ 30″. It was probably derived from his table of chords, in which the length of the chord of one degree is given as 1° 2′ 50″. Ptolemy stated that the circumference is divided into 360 equal parts or degrees, and the diameter into 120 equal parts.[43] Since a chord of one degree is found to be 1° 2′ 50″, the circumference of a circle equals very nearly 360 times (1° 2′ 50″); and since the length of the diameter is 120 equal parts, it follows that pi equals (1° + 2/60 + 50/3600), or 3° 8′ 30″. In decimal notation, the value of pi is found to be

$$3+\frac{8}{60}+\frac{30}{60^2}$$
$$=\frac{3(3600)+8(60)+30}{60^2}$$
$$=\frac{10800+480+30}{3600}=3.14166.$$

This value is 0.00007 greater than 3.14159, the value of pi correct to the fifth decimal place.

A Value by Tsu Ch'ung-chih. About 480 A.D., Tsu Ch'ung-chih, an early Chinese worker in mechanics, gave the rational approximation of 355/113 for pi. His method was probably similar to that outlined below.

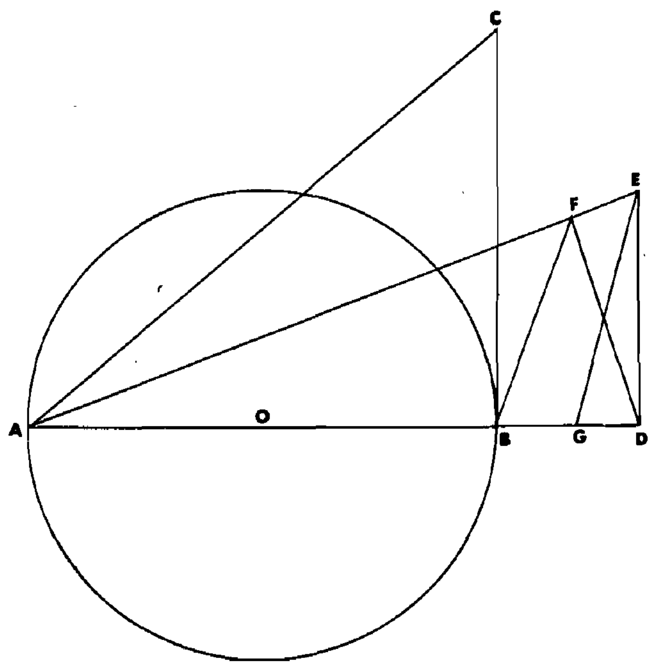

Figure 6
Tsu Ch'ung-Chih's Approximation of Pi

In Figure 6, *AOB* is taken equal to unity and is the diameter of the indicated circle.[44] One should draw *BC* equal to 7/8, perpendicular to *AB* at *B*, and mark off *AD* equal to *AC* on *AB* produced. Next, one should draw *DE* perpendicular to *AD* at *D* and equal to ½, and let *F* be the foot of the perpendicular from *D* on *AE*. He should then draw *EG* parallel to *FB*, cutting *BD* in *G*. Thus, GB/BA = EF/FA. Since, by similar right triangles, EF/FD = DE/AD and FA/FD = AD/DE, then GB/BA = DE^2/DA^2. Since *ABC* is a right triangle, $DA^2 = BA^2 + BC^2$, and GB/BA = $DE^2/BA^2 + BC^2$. In solving for *GB*, one obtains

$$\frac{GB}{1}=\frac{(\frac{1}{2})^2}{(1)^2+(\frac{7}{8})^2}$$
$$GB=\frac{4^2}{7^2+8^2}=\frac{16}{113}.$$

GB is approximately equal to the fractional part of pi. Thus,

$$3 + \frac{4^2}{7^2 + 8^2} = \frac{355}{113},$$

or very nearly equal to pi. The decimal equivalent of the rational fraction, 355/113, is approximately 3.1415929, in agreement with the value of pi to six decimal places.

A Value by Kochansky. In 1685, Father Kochansky, a librarian of the Polish King John III, gave the following approximate geometrical construction for pi.[45] In Figure 7, *AOB* is the diameter of the indicated circle.

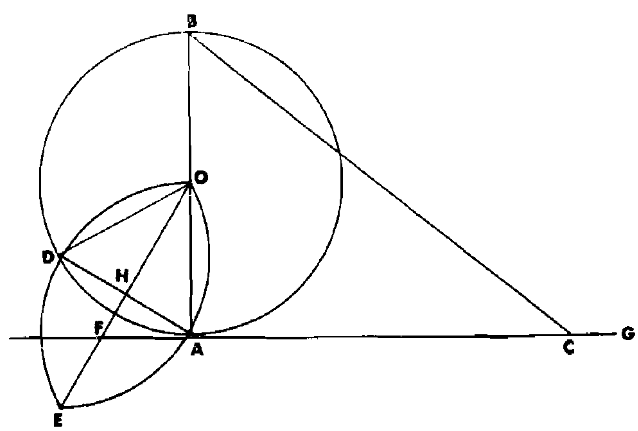

Figure 7

Kochansky's Approximation of Pi

The radius of the circle is equal to unity. One may now draw *AG* tangent to circle *O* at *A*. Taking point *A* of the circle, he may next draw a circle that has the same radius as circle *O* and obtain point *D*. With point *D*, he must trace a second circle with the same radius to obtain point *E*. The line joining *O* and *E* intersects the tangent drawn at *A* in point *F*. By measuring off a triple radius upon line *AG* from point *F*, one may obtain point *C*. Segment *BC* is approximately equal to one-half of the circumference. The triangles *OAH* and *FAH* are similar; therefore, $FA/1 = \frac{1}{2}/(\frac{1}{2}\sqrt{3})$. Solving for *FA*, one obtains $FA = \sqrt{3}/3$. The line, $AC = 3 - \sqrt{3}/3$, equals $9 - \sqrt{3}/3$. Since *ABC* is a right triangle, $BC^2 = AB^2 + AC^2$, and

$$\begin{aligned} BC^2 &= 4 + 9 - 2\sqrt{3} + 1/3 \\ &= 40/3 - 2\sqrt{3} \\ &= 9.869231 \cdots, \\ BC &= 3.141533, \text{ approximately.} \end{aligned}$$

This value of *BC* is 0.000059 less than 3.141592, the value of pi correct to the sixth decimal place.

III

CALCULATION OF PI BY INFINITE PRODUCTS AND SERIES

The second period, which commenced in the latter half of the seventeenth century, was characterized by the application of powerful analytical methods. With such assistance, pi could be expressed by convergent products and convergent series. The older geometrical forms of investigation, therefore, gave way to the analytical processes, or the new methods of systematic representation that stimulated fresh activity in the calculation of pi. Key formulas were applied and re-applied to obtain numerical approximations of pi to more, and still more, significant digits.[46] In this period, covered by the span of about a century, mathematicians were interested in calculating pi by employing convergent products and convergent series. Hence, an infinite product was that which contained an unlimited number of factors; and an infinite product, having a definite limit for the product of its factors as the number of factors is allowed to increase without limit, was called a convergent product. A series, on the other hand, was a sum of terms which progressed according to a given law. If the number of terms were limited, the series was said to be *finite*. If the number of terms were unlimited, the series was referred to as being *infinite*. Therefore, an infinite series which had a definite limit for the sum of its terms, as the number of terms was allowed to increase without limit, was called a convergent series.

A Formula by Vieta. About 1593, Francois Vieta expressed the value of pi in a regular mathematical pattern for the first time.[47] He proved that, if two regular polygons were inscribed in a circle, the first having half the number of sides of the second, the area of the first would be to that of the second as the side of the first polygon, drawn to the extremity of the diameter, was to the diameter of the circle. Taking a square, an octagon, and then polygons of 16, 32, etc., sides, he expressed the side of each polygon drawn to the extremity of the diameter, thus obtaining the ratio of the area of each polygon to that of the next. He discovered that if the diameter be taken as unity, the area of the circle would be

$$\frac{2}{\sqrt{\tfrac{1}{2}}\,\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\,\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots}$$

from which was obtained

$$\pi = \frac{2}{\sqrt{\tfrac{1}{2}}\,\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\,\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}}}}\cdots}$$

Here, the denominator is an infinite product of expressions of square roots with a regular pattern. The accuracy of the approximation of pi obtained by the use of this formula, however, depends upon the number of factors in the denominator of the right member used.

A Formula by Wallis. John Wallis, in 1590, gave an expression for pi as an infinte product.[48] His expression for $\pi/2$ may be derived in the following manner. Applying the method of integration by parts to the indefinite integral, $\int \sin^n x\,dx$, where $n > 1$,

$$\text{(VII)} \quad \int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x\,dx.$$

Therefore,

$$\text{(VIII)} \quad \int_O^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \int_O^{\pi/2} \sin^{n-2} x\,dx,\ n > 1.$$

The latter is a recurrence formula which, in successive applications, yields diminishing positive powers of the *sin x* factor of the integrand of the right member. Two cases need to be distinguished, according as *n* is even or odd. If $n = 2m$,

$$\text{(VIII)} \quad \int_O^{\pi/2} \sin^{2m} x\,dx = \frac{2m-1}{2m} \bullet \frac{2m-3}{2m-2} \cdots \tfrac{1}{2} \bullet \int_O^{\pi/2} dx.$$

If $n = 2m + 1$,

$$\text{(IX)} \quad \int_O^{\pi/2} \sin^{2m+1} x\,dx = \frac{2m}{2m+1} \bullet \frac{2m-2}{2m-1} \cdots \tfrac{2}{3} \bullet \int_O^{\pi/2} \sin x\,dx$$

Hence,

$$\text{(X)} \quad \int_O^{\pi/2} \sin^{2m} x\,dx = \frac{2m-1}{2m} \bullet \frac{2m-3}{2m-2} \cdots \tfrac{1}{2} \bullet \pi/2,$$

and,

$$\text{(XI)} \quad \int_O^{\pi/2} \sin^{2m+1} x\,dx = \frac{2m}{2m+1} \bullet \frac{2m-2}{2m-1} \cdots \tfrac{2}{3} \bullet 1.$$

By division of corresponding members of formulas X and XI above,

$$\text{(XII)} \quad \pi/2 = \frac{2 \bullet 2}{1 \bullet 3} \bullet \frac{4 \bullet 4}{3 \bullet 5} \quad \frac{6 \bullet 6}{5 \bullet 7} \cdots \frac{2m \bullet 2n}{(2m-1 \bullet (2m+1)} \quad \frac{\int_O^{\pi/2} \sin^{2m} x\,dx}{\int_O^{\pi/2} \sin^{2m-1} x\,dx}.$$

The quotient of the two integrals on the right-hand side converges to 1 as *m* increases. This may be established by the following considerations. In the interval $O < x \leqslant \pi/2$,

(XIII) $0 < \sin^{2m+1} x \leqslant \sin^{2m} x \leqslant \sin^{2m-1} x, \ m > 0.$

Consequently,

(XIV) $$0 < \int_0^{\pi/2} \sin^{2m+1} x \, dx \leqslant \int_0^{\pi/2} \sin^{2m} x \, dx \leqslant \int_0^{\pi/2} \sin^{2m-1} x \, dx.$$

If each term of the inequality (XIV) be divided by

$$\int_0^{\pi/2} \sin^{2m+1} x \, dx,$$

(XV) $$1 \leqslant \frac{\int_0^{\pi/2} \sin^{2m} x \, dx}{\int_0^{\pi/2} \sin^{2m+1} x \, dx} = \frac{\int_0^{\pi/2} \sin^{2m-1} x \, dx}{\int_0^{\pi/2} \sin^{2m+1} x \, dx}.$$

Since

(XVI) $$\frac{\int_0^{\pi/2} \sin^{2m-1} x \, dx}{\int_0^{\pi/2} \sin^{2m+1} x \, dx} = \frac{2m+1}{2m} = 1 + \frac{1}{2m},$$

one obtains

(XVII) $$1 \leqslant \frac{\int_0^{\pi/2} \sin^{2m} x \, dx}{\int_0^{\pi/2} \sin^{2m+1} x \, dx} \leqslant 1 + \frac{1}{2m},$$

and

(XVIII) $$\lim_{m \to \infty} \frac{\int_0^{\pi/2} \sin^{2m} x \, dx}{\int_0^{\pi/2} \sin^{2m+1} x \, dx} = 1.$$

The relation (XII) may then be written in the limiting form,

(XIX) $$\pi/2 = \lim_{m \to \infty} \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \frac{6}{5} \frac{6}{7} \cdots \frac{2m}{2-1} \frac{2m}{2m+1}.$$

In the numerator, one finds the even numbers; in the denominator, the odd. Both appear in pairs, with the exception of the first factor in the denominator. Wallis showed that the approximation obtained by stopping at any fraction in the expression on the right is in defect or in excess of the value of $\pi/2$, accordingly if the fraction is proper or improper.[49] This is illustrated in the successive products of Figure 8:

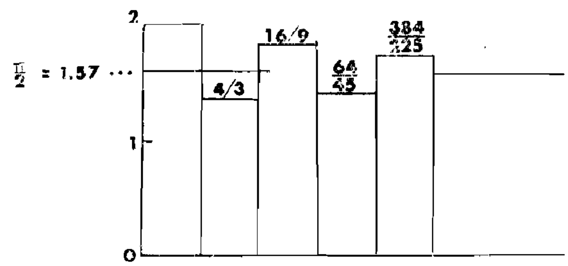

Figure 8
Wallis' Product

$$2/1 = 2 = 2.0000,$$
$$2/1 \bullet 2/3 = 4/3 = 1.3333,$$
$$2/1 \bullet 2/3 \bullet 4/3 = 16/9 = 1.7777,$$
$$2/1 \bullet 2/3 \bullet 4/3 \bullet 4/5 = 64/45 = 1.4222,$$
$$2/1 \bullet 2/3 \bullet 4/3 \bullet 4/5 \bullet 6/5 = 384/225 = 1.7066$$
$$\pi/2 = 1.5707.$$

As it is seen, the convergence of the infinite product, here, is very slow, and the method is not well suited to the calculation of pi.

A Series by Gregory. In 1668, James Gregory derived an infinite series which proved important for the calculation of pi.[50]

$$\text{(XX)} \quad \frac{1}{1+t^2} = 1 - t^2 + t^4 - + \cdot \cdot \cdot + (-1)^{n-1}\, t^{2n-2} + r_n,$$

where

$$r_n = (-1)^n \frac{t^{2n}}{1+t^2},$$

where *n* is positive. Integration of both members of the equality,

$$\text{(XXI)} \quad \int_0^x \frac{dt}{1+t^2} = \text{arc tan } x, \text{ yields}$$

$$\text{(XXII)} \quad \text{arc tan } x = x - \frac{x^3}{3} + \frac{x^5}{5} - + \cdot \cdot \cdot + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_n,$$

where $R_n = (-1)^n \displaystyle\int_0^x \frac{t^{2n}}{1+t^2}\, dt$. In the interval, $-1 \leqslant x \leqslant 1$,

$$\text{(XXIII)} \quad |R_n| \leqslant \int_0^{|x|} t^{2n}\, dt = \frac{|x|^{2n+1}}{2n+1} \leqslant \frac{1}{2n+1}.$$

Therefore, it is evident that R_n tends to zero as n increases. For $|x| > 1$, the absolute value of the remainder increases beyond all bounds as n increases. Gregory's infinite series,

$$\text{(XXIV)} \quad \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$$

is thus valid for $-1 \leqslant x \leqslant 1$. It was by means of this expression and with the help of other relationships that most of the practical methods of calculating pi have been obtained.

A Formula by Leibniz. Gottfried Wilhelm Leibniz, a German mathematician, obtained a formula in 1673 that could be used in calculating pi.[31] He took Gregory's series, formula XXIII, and let x equal unity. Then, since *arc tan* $1 = \pi/4$,

$$\text{(XXV)} \quad \pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots.$$

The successive sums of the terms of this series yield a value of pi which may, theoretically, be found as accurately as desired. This process, typical of the powerful methods of approximation used in mathematics, still entails a great deal of calculation. Figure 9 illustrates some the successive sums of this series:

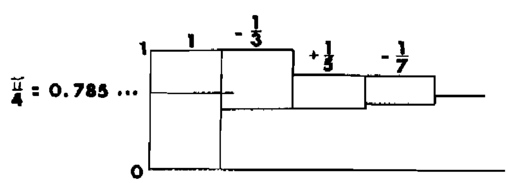

Figure 9

Leibniz' Series

$$1 = 1 = 1.000,$$
$$1 - 1/3 = 2/3 = 0.6666,$$
$$1 - 1/3 + 1/5 = 13/15 = 0.8666,$$
$$1 - 1/3 + 1/5 - 1/7 = 76/105 = 0.7238,$$
$$\pi/4 = 0.7853$$

After an approximation for $\pi/4$ has been obtained by taking the first 50 terms of this series, the next 50 will not yield an approximation which is sufficiently more accurate to justify the additional computation, since the series converges very slowly.

A Value by Sharp. Abraham Sharp, an English mathematician, calculated the value of pi in 1699 by making the substitution, $x = \sqrt{1/3}$, in the Gregory series, formula XXIV.[41] Since *arc tan* $\sqrt{1/3} = \pi/6$,

$$\pi/6 = \sqrt{1/3}\left(1 - \frac{1}{3 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \frac{1}{3^4 \cdot 9} - \frac{1}{3^5 \cdot 11} + \cdots\right).$$ [52]

Replacing each term in the right member by its decimal equivalent,

$$\pi/6 = 0.5773502691 \quad (1.0000000000 - 0.1111111111 + 0.2222222222 - 0.0052910052 + 0.0013717421 - 0.0003741114 + \cdots).$$

Taking the first six terms of the right member, $\pi/6 = 0.523551$, from which, $\pi = 3.141306$, correct to three decimal places. This series converged more rapidly than did the series XXV, and it was used to calculate pi correct to seventy-one decimal places.

A Formula by Machin. In 1706, John Machin, Englishman, developed a convenient method for calculating pi. He applied the Gregory series, formula XXIV, to a trigonometric identity and obtained a good value of pi.[53] For example, if one lets A=arc tan 1/5, B=arc tan 1/239, then

$$\tan(4A - B) = \frac{\dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A} - \tan B}{1 + \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A} \cdot \tan B}$$

$$= \frac{\dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} - 1/239}{1 + \dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} \cdot 1/239}$$

$$= \frac{\dfrac{28561}{28441}}{\dfrac{28561}{28441}} = 1.$$

Since $\tan \pi/4 = 1$, one has (XXVI) $\pi/4 = 4 \text{ arc tan } 1/5 - \text{arc tan } 1/239$. By using formulas XXIV and XXVI, the process of determining a value for pi is established in the following manner. By using formula XXIV, $\text{arc tan } b = b - b^3/3 + b^5/5 - b^7 + \cdot \cdot \cdot$ (XXVII). Since $A = \text{arc tan } 1/5$, then, (XXVIII) $A = 1/5 - 1/3(5)^3 + 1/5(5)^5 - 1/7(7)^7 + \cdot \cdot \cdot \cdot$ Since $b = 1/5$, the decimal equivalents of the odd powers of *b* are

b = 0.2,
b^{3} = 0.008,
b^{5} = 0.000,32,
b^{7} = 0. ,012,8,
b^{9} = 0. , ,512,
b^{11} = 0. , ,020,48,
b^{13} = 0. , , ,819,2,
b^{15} = 0. , , ,032,768,
b^{17} = 0. , , ,001,310,72,
b^{19} = 0. , , , ,052,428,8,
b^{21} = 0. , , , ,002,097,152,
b^{23} = 0. , , , , ,083,886,08,
b^{25} = 0. , , , , ,003,355,44,
b^{27} = 0. , , , , , ,134,21,
b^{29} = 0. , , , , , ,005,36,
b^{31} = 0. , , , , , , ,21,
. . . .

Hence, the positive terms of the series of formula (XXVII) are

$b = 0.2$
$b^5/5 = 0.000,064,$
$b^9/9 = 0. \quad , \quad ,056,888,888,888,888,8,$
$b^{13}/13 = 0. \quad , \quad , \quad ,063,015,384,615,3,$
$b^{17}/17 = 0. \quad , \quad , \quad , \quad ,077,101,176,4,$
$b^{21}/21 = 0. \quad , \quad , \quad , \quad , \quad ,099,864,3,$
$b^{25}/25 = 0. \quad , \quad , \quad , \quad , \quad , \quad ,134,2,$
$b^{29}/29 = 0. \quad , \quad , \quad , \quad , \quad , \quad ,1,$
. . .

The sum of the positive terms of the series of formula XXVIII is (XXIX) 0.200,064,056,951,981,474,679,1.

The negative terms of the series of formula (XXVII) are

$-b^3/3 = -0.002,666,666,666,666,666,666,6,$
$-b^7/7 = -0.000,001,828,571,428,571,428,5,$
$-b^{11}/11 = -0. \quad , \quad ,001,861,818,181,818,1,$
$-b^{15}/15 = -0. \quad , \quad , \quad ,002,184,533,333,3,$
$-b^{19}/19 = -0. \quad , \quad , \quad , \quad ,002,759,410,5,$
$-b^{23}/23 = -0. \quad , \quad , \quad , \quad , \quad ,003,647,2,$
$-b^{27}/27 = -0. \quad , \quad , \quad , \quad , \quad , \quad ,004,9,$
$-b^{31}/31 = -0. \quad , \quad , \quad , \quad , \quad , \quad , \quad ,0,$
. . . .

The sum of the negative terms of the series of formula XXIX is (XXX) −0.002,668,497,102,100,716,309,1. Adding the numbers (XXIX) and (XXX), the value of arc tan 1/5 is found to be 0.187,395,559,849, 880,758,370,0; and $\pi/4 = 4(0.197,395,559,849,880,758,370,0) -$ arc tan 1/239. By formula XXIV, one establishes (XXXI): arc tan $e = e - e^3/3 + e^5/5 - e^7/7 + \cdots$ Since B = arc tan 1/239, then

$$\text{(XXXII)}\quad B = \frac{1}{239} - \frac{1}{3(239)^3} + \frac{1}{5(239)^5} - \frac{1}{7(239)^7} + \cdots$$

Since $e = 1/239$, the decimal equivalents of the odd powers of *e* are

$e = 0.004,184,100,418,410,041,841,0,$
$e^3 = 0.000,000,073,249,775,361,251,4,$
$e^5 = 0. \quad , \quad , \quad ,001,282,361,572,1,$
$e^7 = 0. \quad , \quad , \quad , \quad , \quad ,022,449,9,$
$e^9 = 0. \quad , \quad , \quad , \quad , \quad , \quad , \quad ,3, \cdots$

Hence, the positive terms of the series of formula XXXI are

$e = 0.004,184,100,418,410,041,841,0,$
$e^5/5 = 0.000,000,000,000,256,472,314,4,$
$e^9/9 = 0. \quad , \quad , \quad , \quad , \quad , \quad , \quad ,0, \cdots$

The sum of the positive terms of the series of formula XXXII is (XXXIII) 0.004,184,100,418,666,514,155,4. The negative terms of the series of formula XXXI are

— e3/3 = — 0.000,000,024,416,591,787,083,8,
— e7/7 = — 0. , , , , ,003,207,1, · · · ·

The sum of the negative terms of the series of formula XXXII is (XXXIV) – 0.000,000,024,416,591,790,290,9. Adding the numbers XXXIII and XXXIV, the value of arc tan 1/239 is found to be 0.004, 184,076,002,074,723,864,5. Therefore,

$$\pi/4=4(0.197,395,559,849,880,758,370,0) - 0.004,184,076,002,074,723,864,5;$$

or,

$$\pi/4=0.785,398,163,397,448,309,615,5.$$

Multiplying each side by 4, one obtains

$$\pi=3.141,592,653,589,793,238,462,0.$$

This sum is correct to the twenty-first decimal place, the error occuring in the twenty-second, which in the correct value would be a 6 instead of a 0.

A Formula by Euler. Leonhard Euler, a Swiss mathematician, in 1779 applied the series XXIV to a simple trigonometric identity and obtained a fairly good value of pi.[54] Following his method, if one lets A = arc tan 1/2, B = arc tan 1/3, then

$$\tan\ (A - B) = \frac{\tan\ A + \tan\ B}{1 - \tan\ A \bullet \tan\ B}$$

$$\frac{1/2 + 1/3}{1-(1/2)\ (1/3)} = \frac{5/6}{5/6} = 1.$$

Since $\tan \pi/4 = 1$, it follows that $\pi/4 = \text{arc tan } 1/2 + \text{arc tan } 1/3$.

Applying the series of formula XXIV to each term of the right member,

$$\begin{aligned}\pi/4 = \ & 1/2 && + \ 1/3 \\ & - (1/3)\ (1/2)^3 && - (1/3)\ (1/3)^3 \\ & + (1/5)\ (1/2)^5 && + (1/5)\ (1/3)^5 \\ & - (1/7)\ (1/2)^7 && - (1/7)\ (1/3)^7 \\ & + (1/9)\ (1/2)^9 && + (1/9)\ (1/3)^9 \\ & - (1/11)\ (1/2)^{11} && - (1/11)\ (1/3)^{11} \\ & + \cdots && + \cdots\end{aligned}$$

Replacing each term in the right member by its decimal equivalent,

$$\begin{aligned}\pi/4 = & + 0.500,000,000,0 + 0.333,333,333,3 \\ & - 0.041,666,666,6 - 0.012,345,679,0 \\ & + 0.006,250,000,0 + 0.000,823,045,2 \\ & - 0.001,116,071,4 - 0.000,065,321,0 \\ & + 0.000,217,013,8 + 0.000,005,645,0 \\ & - 0.000,044,389,2 - 0.000,000,513,1 \\ & + \cdots\end{aligned}$$

Simplifying the right member, $\pi/4 = 0.785,390,397,0$. Multiplying each side by 4, one obtains $\pi = 3.141561$. This value is correct to four decimal places. It would take more than twenty-two terms of the combined series to give pi correct to seven decimal places.

A Formula by Dase. In 1844, Zacharias Dase, a German mathematician, replaced the trigonometric identity by one somewhat more involved and obtained a value of pi correct to 200 decimal places, a feat which he completed in two months.[55] If one lets A = arc tan 1/2, B = arc tan 1/5, and C = arc tan 1/8, then

$$\tan\ (A + B) = \frac{\tan A + \tan B}{1 - \tan A \bullet \tan B}$$

$$= \frac{1/2 + 1/5}{1 - (1/2)\ (1/5)} = \frac{7/10}{9/10} = \frac{7}{9}.$$

Since tan (A + B) equals 7/9, one has

$$\tan\ (\overline{A + B} + C) = \frac{\tan\ (A + B) + \tan C}{1 - \tan\ (A + B) \bullet \tan C}$$

$$= \frac{7/9 + 1/8}{1 - (7/9)\ (1/8)} = \frac{65/72}{65/72} = 1.$$

Since tan $\pi/4$ = 1, one has

$$\pi/4 = \text{arc tan } 1/2 + \text{arc tan } 1/5 + \text{arc tan } 1/8.$$

With the aid of the formula (XXIV),

$$\begin{array}{lll}
\pi/4 = \quad 1/2 & + 1/5 & + 1/8 \\
- (1/3)\ (1/2)^3 & - (1/3)\ (1/5)^3 & - (1/3)\ (1/8)^3 \\
+ (1/5)\ (1/2)^5 & + (1/5)\ (1/5)^5 & + (1/5)\ (1/8)^5 \\
- (1/7)\ (1/2)^7 & - (1/7)\ (1/5)^7 & - (1/7)\ (1/8)^7 \\
+ (1/9)\ (1/2)^9 & + (1/9)\ (1/5)^9 & + (1/9)\ (1/8)^9 \\
- (1/11)\ (1/2)^{11} & - (1/11)\ (1/5)^{11} & - \cdot\ \cdot\ \cdot\ \cdot \\
+ (1/13)\ (1/2)^{13} & + \cdot\ \cdot\ \cdot\ \cdot & \\
- (1/15)\ (1/2)^{15} & & \\
+ (1/17)\ (1/2)^{17} & & \\
- (1/19)\ (1/2)^{19} & & \\
+ (1/21)\ (1/2)^{21} & & \\
- (1/23)\ (1/2)^{23} & & \\
+ (1/25)\ (1/2)^{25} & & \\
(1/27)\ (1/2)^{27} & & \\
+ \cdot\ \cdot\ \cdot\ \cdot & &
\end{array}$$

Replacing each term in the right member by its decimal equivalent,

$$
\begin{aligned}
\pi/4 = &+ 0.500{,}000{,}000{,}0 + 0.200{,}000{,}000{,}0 + 0.125{,}000{,}000{,}0 \\
&- 0.041{,}666{,}666{,}6 - 0.002{,}666{,}666{,}6 - 0.000{,}651{,}041{,}6 \\
&+ 0.006{,}250{,}000{,}0 + 0.000{,}064{,}000{,}0 + 0.000{,}006{,}103{,}5 \\
&- 0.001{,}116{,}071{,}4 - 0.000{,}001{,}828{,}5 - 0.000{,}000{,}068{,}1 \\
&+ 0.000{,}217{,}013{,}8 + 0.000{,}000{,}056{,}8 + 0.000{,}000{,}000{,}8 \\
&- 0.000{,}044{,}389{,}2 - 0.000{,}000{,}001{,}8 \\
&+ 0.000{,}009{,}390{,}0 \\
&- 0.000{,}002{,}034{,}5 \\
&+ 0.000{,}000{,}448{,}7 \\
&- 0.000{,}000{,}100{,}3 \\
&+ 0.000{,}000{,}022{,}7 \\
&- 0.000{,}000{,}005{,}1 \\
&+ 0.000{,}000{,}001{,}1 \\
&- 0.000{,}000{,}000{,}2 \\
&+ \cdot\cdot\cdot\cdot
\end{aligned}
$$

Simplifying the right member, $\pi/4 = 0.785{,}398{,}163{,}5$. Multiplying each side by 4, one obtains $\pi = 3.141{,}592{,}654{,}0$. This value is correct to nine decimal places.

A Formula by Rutherford. In 1841, William Rutherford, an English mathematician, applied the series XXIV to a more involved triginometric identity and obtained a value of pi, correct to 152 decimal places.[56] He returned to the problem in 1853 and obtained a value of pi to 440 decimal places. Following his method, one lets A=arc tan 1/5, B=arc tan 1/70, and C=arc tan 1/99. Then,

$$
\tan(4A - B) = \frac{\dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A} - \tan B}{1 + \dfrac{4\tan A - 4\tan^3 A}{1 - 6\tan^2 A + \tan^4 A}\cdot\tan B}
$$

$$
= \frac{\dfrac{4/5 - 4/125}{1 - 6/25 + 1/625} - 1/70}{1 + \dfrac{4/5 - 4/125}{1 - 6/25 + 1/625}\cdot 1/70}
$$

$$
= \frac{\dfrac{8281}{8330}}{\dfrac{8450}{8330}} = \frac{8281}{8450}.
$$

Since tan (4A − B) equals 8281/8450,

$$\tan \overline{(4A - B - C)} = \frac{\tan (4A - B) + \tan C}{1 - \tan (4A - B) \bullet \tan C}$$

$$= \frac{8281/8450 + 1/99}{1 - (8281/8450)(1/99)}$$

$$= \frac{\dfrac{828269}{836550}}{\dfrac{828269}{836550}} = 1.$$

Since tan $\pi/4 = 1$, one has $\pi/4 = 4 \text{ arc tan } 1/5 - \text{arc tan } 1/70 + \text{arc tan } 1/99$.

Applying the formula (XXIV),

$$\pi/4 = [4\ 1/5 - (1/3)(1/5)^3 + (1/5)(1/5)^5 - (1/7)(1/5)^7 + (1/9)(1/5)^9 - (1/11)(1/5)^{11} + \cdots] - [1/70 - (1/3)(1/70)^3 + (1/5)(1/70)^5 - \cdots] + [1/99 - (1/3)(1/99)^3 + \cdots].$$

Replacing each term in the right member by its decimal equivalent,

$$\pi/4 = 4\,(0.200{,}000{,}000{,}0 - 0.002{,}666{,}666{,}6 + 0.000{,}064{,}000{,}0 - 0.000{,}001{,}828{,}5 + 0.000{,}000{,}056{,}8 - 0.000{,}000{,}001{,}8 + \cdots) - (0.014{,}285{,}714{,}2 - 0.000{,}000{,}971{,}8 + 0.000{,}000{,}000{,}1 - \cdots) + (0.010{,}101{,}010{,}1 - 0.000{,}000{,}343{,}5 + \cdots).$$

Simplifying the right member, $\pi/4 = 0.785{,}398{,}163{,}7$. Multiplying each side by 4, one obtains $\pi = 3.141{,}592{,}654{,}8$. This value is correct to eight decimal places.

A Value by Shanks. William Shanks, an English mathematician, in 1873 obtained a value of pi to 707 decimal places by using the formula XXVI.[57] In 1946, D. F. Ferguson of England discovered errors in Shanks' work, starting with the 528th place. The value of pi, correct to 527 decimal places is

3.141,592,653,589,793,238,462,643,383,279,502,884,197,169
399,375,105,820,974,944,592,307,816,406,286,208,998,628,
034,825,342,117,067,982,148,086,513,282,306,647,093,844,
609,550,582,231,725,359,408,128,481,117,450,284,102,701,
938,521,105,559,644,622,948,954,930,381,964,428,810,975,
665,933,446,128,475,648,233,786,783,165,271,201,909,145,
648,566,923,460,348,610,454,326,648,213,393,607,260,249,
141,273,724,587,006,606,315,588,174,881,530,920,962,829,
254,091,715,364,367,892,590,360,011,330,530,548,820,466,
521,384,146,951,941,511,609,433,057,270,365,759,591,953,
092,186,117,381,932,611,793,105,118,548,074,462,379,962,
749,567,351,885,752,724,891,227,938,183,011,949,129,833,
673,362,440,656,643,086,021,39.

IV

PI AND PROBABILITY

It is, perhaps, correct to say that the mathematical treatment of probability came into being in the late fifteenth century or early sixteenth, and it is true that certain Italian writers, notably Pacioli (1494), Tartaglia (1556), and Cardan (1545), had discussed the problem of the division of a stake between two players, a situation involving a probability theme.[58] However, it is generally conceded that the one problem to which the origin of the science of probability can be safely credited was the so-called "problem of the points," a puzzle, so to speak, requiring the determination of the division of stakes of an interrupted game of chance between two supposedly equally skilled players, knowing the scores of the players at the time of interruption, and recognizing the number of points needed to win the game. This famous problem was proposed to Blaise Pascal and Pierre de Fermat, probably in 1654, by the Chevalier de Mere, an able and most experienced gambler. A remarkable correspondence between Pascal and Fermat ensued as a result, in which the problem was correctly, but differently solved by each man. It was in their correspondence over this problem that Pascal and Fermat laid the foundations for the science of probability. Since that time, there has been a number of experiments in connection with pi and probability, fundamental among which are the tossing of a needle onto a ruled surface, the pitching of a coin onto a cross-ruled board, and the writing of two numbers at random.

Buffon's Experimental Method. Comte de Buffon in 1760 presented to the world his famous Needle Problem, in which an approximation of pi may be found by an experiment involving the random tossing of a needle onto a ruled surface. Upon a plane, Buffon ruled off parallel lines, equally spaced and $2a$ units apart. He let the needle be tossed N times at random

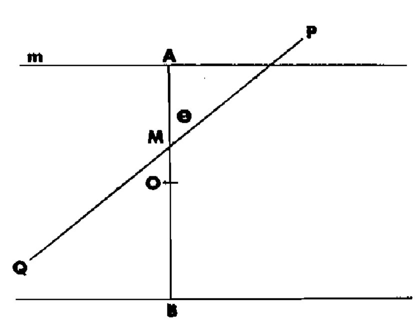

Figure 10

Buffon's Needle Problem

upon the ruled surface, and also let the number of intersections be K. He then concluded that the probability of the needle's intersecting the line in a single random toss of the needle would be K/H. One may also deduce this same probability in the following manner.[59] In Figure 10, one may let M denote the midpoint of the needle, PQ, in a random position, and let O

be the midpoint of that segment, *AB*, perpendicular to the parallel lines which pass through *M*. In the segment *AB*, *A* is the endpoint nearer *M*, being the intersection of the perpendicular with the parallel *m*. One next denotes *MA* by *x*, and the angle *AMP* by θ. For a given value x_0 of x, the probability that *MP* will intersect line *m* when $x=x_0$ equals

$$\frac{\theta_0}{\pi/2} = 2/\pi \text{ arc cos } \frac{x_0}{b}.$$

The probability that *x* will have the value of x_0 is dx/a. The probability that *MP* will intersect the line *m* is

$$p = \int_O^a 2/\pi \text{ arc cos } x/b \bullet dx/a$$

$$= 2b/\pi a$$

Combining the two expressions for the probability, *p*, that the needle will intersect one of the parallel lines,

$$K/N = 2b/\pi a,$$

or,

$$\pi = 2bN/aK.$$

Experiments for the Determination of Pi by Random Tosses. By using the experimental method presented by Comte de Buffon, the present author obtained a reasonably close approximation of pi. The distance between the parallel lines was 1 1/8 inches, and the length of the needle was 9/16 of an inch. In 788 random tosses, there were 251 intersections. Dividing the number of tosses by the number of intersections, the value obtained was 3.13944. This value is 0.00215 less than 3.14159, the value of pi correct to the fifth decimal place. In 1855, Mr. A. Smith of Aberdeen, Scotland, made 3204 random tosses and obtained an approximation of 3.1553, a value that is approximately 0.0138 greater than the true value of pi.[60] A pupil of Professor DeMorgan, from 600 trials, obtained an approximation of 3.137, less than the true value of pi by 0.004. Of the many experiments that have been performed, perhaps the most accurate approximation of pi was obtained by Lazzerini, an Italian mathematician, in 1901.[61] He executed 3,408 tosses and obtained an approximation of 3.1415929, in error only by 0.000,000,3.

V

IRRATIONALITY AND TRANSCENDENCE OF PI

The first period in the history of pi, from 3000 B.C. to the middle of the seventeenth century, was known as the geometrical period, in which the main interest, as it has been shown, was the approximate determination of pi by calculation of the sides of regular polygons inscribed and circumscribed to a circle.[62] The second period in the history of pi, dating from the middle of the seventeenth century and lasting for about a century, was characterized by the application of the powerful analytical methods, then available. At this time, the value of pi was determined by convergent series, products, and continued fractions. In the third period, from the middle of the eighteenth century until the late nineteenth century, man's attention was turned to critical investigations of the true nature of the number *pi*. The number was first investigated to determine its rationality or irrationality, and it was eventually proved to be irrational. When the discovery was made of the fundamental distinction between algebraic and transcendental numbers,–*i.e.*, between those numbers which can be, and those numbers which cannot be, roots of an algebraical equation with rational coefficients,–the question then arose as to which of these categories the number *pi* belonged. Subsequent experiments eventually established that the number *pi* is transcendental.

Irrationality of Pi. Johann Heinrich Lambert, a German mathematician, in 1761 proved that pi was irrational; *i.e.*, pi cannot be expressed as an integer or as the quotient of two integers.[63] Proof of the irrationality of pi was shown by Lambert to depend upon the contained fraction (XXXIV),

$$\tan x = \cfrac{x}{1 - \cfrac{x^2}{3 - \cfrac{x^2}{5 - \cfrac{x^2}{7 - \cdots \cfrac{- x^2}{2n+1 - \cdots}}}}}.$$

The derivation of this continued fraction is given by E. W. Hobson in *A Treatise on Plane and Advanced Geometry*. If π were rational, $\pi/4$ would be rational. If one puts $x = \pi/4$, and, if possible, lets $\pi/4 = p/q$, he then has, by XXXIV,

(XXXIV)

$$1 = \cfrac{p/q}{1 - \cfrac{p^2/q^2}{3 - \cfrac{p^2/q^2}{5 - \cfrac{p^2/q^2}{7 - \cdots \cfrac{- p^2/q^2}{2n+1 - \cdots}}}}},$$

(XXXV)

$$1 = \cfrac{p}{q - \cfrac{p^2}{3q - \cfrac{p^2}{5q - \cfrac{p^2}{7q - \cdots - \cfrac{p^2}{(2n+1)q \cdots}}}}}.$$

Since p and q are fixed finite integers, if one takes n large enough, he shall have $(2n + 1)q \; p^2 + 1$. If $a_2, a_3, \cdots, a_n, b_2, b_3, \cdots, b_n$ all be positive integers, then it follows that the infinite continued fraction,

$$\cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cdots - \cfrac{b_n}{a_n - \cdots}}},$$

converges to an irrational limit, provided that, after some finite value of n, the condition, $a_n \geqslant b_n + 1$, is always satisfied, where the sign [>] need not always occur, but must occur infinitely often. If π is rational, the continued fraction in the right member of (XXXVI) converges to an irrational limit.[64] But, the right member actually converges to the rational value 1. Hence, $\pi/4$ cannot be equal to a fraction p/q in which p and q are integers, and, therefore, π is irrational.

Transcendence of Pi. In 1882, Ferdinand Lindemann, a German mathematician, proved that the number *pi* was transcendental. Previously, in 1873, Charles Hermite, a French mathematician, had proved that the number e was transcendental; *i.e.*, that no equation of the form,

$$ae^m + be^n + ce^r + \cdots + ke^y = O,$$

can exist, if $m, n, r, \cdots y, a, b, c, \cdots k$ are whole numbers.[65]. Lindemann, then, in 1882, proved that such an equation cannot hold, when $m, n, r, \cdots y, a, b, c, \cdots k$ are algebraic numbers, not necesarily real. In the particular case, $e^{ix} + 1 = O$, x cannot be an algebraic number. But, $e^{i\pi} + 1 = O$ is known to be true, and, therefore, pi is not algebraic but transcendental.

VI

FORMULAS INVOLVING PI

Pi represents the ratio of the two most significant measurements associated with the circle: the distance around it to the distance across it. This ratio, c/d or c/2r, is only one of several ratios equivalent to pi. For instance, pi is also the ratio of the area of a circle to the area of the square erected on its radius. It further appears in many other formulas expressing geometric relations. Pi is the ratio of the volume of a circular cylinder to the radius squared of the base times the altitude; also, it is the ratio of three times the volume of a circular cone to the altitude times the square of the radius of the base. It appears in the formulas, as well, for finding the surface area of a sphere and the volume of a sphere. The influence of pi also extends beyond circular structures and may be found in the area of an ellipse and the volume of an ellipsoid. In using the formulas dealing with the circle, sphere, cylinder, and cone in finding an approximation of pi, it is necessary that the measurement of the radius diameter, circumference, area, altitude, slant height, and volume in such formulas be known before an approximation of pi may be determined. All measurements used in the following problems are approximate.

Circle. The circumference of a circle is 62.5 inches, and the radius is 10 inches. Find the approximation of pi. The formula is $\pi = C/2r$. Substituting the known values in the formula, one has $\pi = 62.5/2(10) = 3.12500$. This approximation of pi is 0.01659 less than the value 3.14159.

The circumference of a circle is 154 inches, and the diameter is 49 inches. Find the approximation of pi. The formula is $\pi = C/d$. Substituting the known values in the formula, one has $\pi = 154/49 = 3.14285$. This approximation of pi is 0.00126 greater than the value 3.14159.

The area of a circular base of a cylinder is 38.5 square inches, and the radium is 3.5 inches. Find the approximation of pi. The formula is $\pi = A/r^2$. Substituting the known values in the formula, one has $\pi = 38.5/(3.5)^2 = 3.14285$. This approximation of pi is 0.00126 greater than the value 3.14159.

Sphere. The surface area of a sphere is 452 square inches, and the radius is 6 inches. Find the approximation of pi. The formula is $\pi = S/4r^2$. Substituting the known values in the formula, one has $\pi = 452/4(6)^2 = 3.13888$. This approximation of pi is 0.00271 less than the value 3.14159.

The volume of a sphere is 904 cubic inches, and the radius is 6 inches. Find the approximation of pi. The formula is $\pi = 3V/4r^3$. Substituting the known values in the formula, one has $\pi = 3(904)/4(6)^3 = 3.13888$. This approximation of pi is 0.00271 less than the value of 3.14159.

Cylinder. The total surface area of a right circular cylinder is 376 square inches. The altitude of the cylinder is 7 inches, and the radius of the base is 5 inches. Find the approximation of pi. The formula is $\pi = T/2r(r+h)$. Substituting the known values in the formula, one has $\pi = 376/2(5)(5+7) = 3.13333$. This approximation of pi is 0.00826 less than the value of 3.14159.

The lateral surface area of a cylinder is 377 square inches The cylinder is 12 inches deep, and the radius of the base is 5 inches. Find the *approximation of pi.* The formula is $\pi = S/2rh$. Substituting the known values in the formula, one has $\pi = 377/2(5)(12) = 3.14166$. This approximation of pi is 0.00007 greater than the value 3.14159.

The volume of a cylinder is 942 cubic inches. The cylinder is 12 inches deep, and the radius of its base is 5 inches. Find the approximation of pi. The formula is $\pi = V/r^2h$. Substituting the known values in the formula, one has $\pi = 942/(5^2)(12) = 3.14000$. This approximation is 0.00159 less than the value 3.14159.

Cone. The total area of a right circular cone is 283 square inches. The slant height of the cone is 13 inches, and the radius of the base is 5 inches. Find the approximation of pi. The formula is $\pi = T/r(r+s)$. Substituting the known values in the formula, one has $\pi = 283/5(5+13) = 3.14444$. This approximation of pi is 0.00285 greater than the value 3.14159.

The lateral area of a right circular cone is 204 square inches. The slant height of the cone is 13 inches, and the radius of the base is 5 inches. Find the approximation of pi. The formula is $\pi = S/rs$. Substituting the known values in the formula, one has $\pi = 204/5(13) = 3.13846$. This approximation of pi is 0.00313 less than the value 3.14159.

The volume of a right circular cone is 3284 cubic inches. The radius of the base of the cone is 14 inches, and the altitude is 16 inches. Find the approximation of pi. The formula is $\pi = 3V/r^2h$. Substituting the known values in the formula, one has $\pi = 3(3284)/(14)^2(16) = 3.14158$. This approximation of pi is 0.00001 less than the value 3.14159. These problems demonstrate that the accuracy of pi, as calculated here, depends upon the accuracy of the values in the formula.

While the number of pi was, originally, and for a long time, directly associated with the measurements of circles, it is now regarded as an important and fundamental constant appearing in a wide variety of mathematical and physical situations having no evident involvement with circles. It remains to present the many varied applications of pi in diverse fields and to investigate the numerous situations, mathematical or physical, in which pi plays a critical role, today.

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FOOTNOTES

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