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THE EMPORIA STATE
Research Studies

V. 8 No. 1

THE GRADUATE PUBLICATION OF THE KANSAS STATE TEACHERS COLLEGE, EMPORIA

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The Emporia State Research Studies

KANSAS STATE TEACHERS COLLEGE
EMPORIA, KANSAS

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VOLUME 8 **September 1959** **NUMBER 1**

THE EMPORIA STATE RESEARCH STUDIES are published in September, December, March and June of each year by the Graduate Division of the Kansas State Teachers College, Emporia, Kansas. Entered as second-class matter September 16, 1952, at the post office at Emporia, Kansas, under the act of August 24, 1912.

KANSAS STATE TEACHERS COLLEGE

EMPORIA · KANSAS

This publication is a continuation of *Studies in Education* published by the Graduate Division from 1930 to 1945.

Papers published in this periodical are written by faculty members of the Kansas State Teachers College of Emporia and by either undergraduate or graduate students whose studies are conducted in residence under the supervision of a faculty member of the college.

The History and Calculation of Pi

by Herman H. Harris, Jr.*

I

A BRIEF HISTORY OF PI

The number *pi* occupies an unique place in the history of mathematics. While academicians, students, and laymen generally recognize the importance of pi and are aware of some of its many implications, there exists today a vagueness as to its historical significance, and an uncertainty concerning the numerous methods which, in the past, have been employed in its calculation. Whether defined as the ratio of the circumference of a circle to the diameter, or as the ratio of the area of a circle to the square on half the diameter, it has been the object of an intensive search by all nations from the earliest of times to the present day. It is significant that the number *pi* has wound itself through the structure of mathematics into the fabric of modern civilization.

The computation of pi is closely associated with the quadrature of the circle, one of three famous mathematical problems.[1] Its earliest recorded approximation is the number *three*, used by the Hebrews, Egyptians, and Babylonians, an approximation which was accepted for many centuries. One discovers references to this figure in the Old Testament, 1 *Kings* 7:23, and 2 *Chronicles* 4:22. In the latter source, one finds the following statement:

> Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits compass it round about.

However, the earliest traces of pi are to be noted in the Rhind Papyrus (1700 B.C.), now preserved in the British Museum, translated and explicated by Eisenlohr, in which is recorded ". . . the area of a circle is equal to that of a square whose side is the diameter diminished by one-ninth."[2] Such a qualification gives an approximation for pi as 3.1604, greater than 3.1416 by about 0.6%. Later, the so-called geometrical method of computing the value of pi was frequently employed, some mathematicians obtaining a closer approximation than others. This particular method consisted of inscribing in a circle, and circumscribing about the circle, a number of regular polygons. Next, by doubling in succession the number of sides and determing the perimeters or areas of the polygons, mathematicians were subsequently enabled to calculate an approximation. Furthermore, if this process were performed for a number of successive times, individuals discovered that they were able to obtain an even closer approximation. Among many who computed pi by this method were Archimedes,

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Francois Vieta, Adriaen Van Roomen, Ludolph Van Ceulen, Willebrord Snell, and Grienberger.

Today, it is generally conceded that the first truly scientific attempt to compute pi was undertaken by Archimedes (c. 240 B.C.), who discovered an approximation by means of the aforementioned method of circumscribing and inscribing regular polygons in a circle. In his efforts to obtain an approximation for pi, he recorded an upper and lower limit, and eventually concluded that pi was located between these limitations. He first used a regular hexagon circumscribed about a circle. Each time, thereafter, by determining the perimeter of the regular polygon (up to 96 sides), he resolved that pi was less than 3 1/7. Next, he determined a lower limit by inscribing regular polygons of six, twelve, twenty-four, forty-eight, and ninety-six sides into the circle, establishing for each successive polygon its subsequent perimeter, which he discovered to be always less than the circumference of the circle. From such an experiment, he concluded that the circumference exceeded three times its diameter by a part of which was less than 1/7, but more than 10/71 of the diameter.[3] Since 3 1/7 is greater than 3.1416 by about 0.04%, and is a smple number for ordinary computations, Archimedes' value for pi is still in common use, today. His approximation for pi is considerably closer, also, than that described in the biblical references.[4]

Claudius Ptolemy (c. 150 A.D.), a teacher in Athens and Alexandria, gave the world its next notable value for pi:

> His value for pi is given, in sexagesimal notation, as 3° 8′ 30″, which is equal to approximately 3.1416. This value was probably derived from the table of chords, which appears in his treatise. This table gives the lengths of the chords of a circle subtended by central angles of each degree and half degree. If the length of the chord of one degree central angle is multiplied by three-hundred-sixty, and the result divided by the length of the diameter of the circle, the value of pi is obtained.[5]

In the Eastern culture, the Chinese values of pi were 3 and the $\sqrt{10}$.[6] The most interesting extant record from the Chinese is, however, that of Tsu Ch'ung-chih in the fifth century, who found 31.415927 and 31.415926 for the limits of ten pi, from which he inferred, by a reasoning process unexplained in his work, that 22/7 and 355/113 were approximate values.[7]

An early Hindu mathematician, Aryabhata (c. 530), gave 62,832/20,000 as an approximate value, a figure that is equal to 3.1416:

> He showed that, if **a** is the side of a regular polyon of **n** sides inscribed in a circle of unit diameter, and if **b** is the side of a regular inscribed polygon of **2n** sides, then $b^2 = \frac{1}{2} - \frac{1}{2}(1 - a^2)^{\frac{1}{2}}$. From the side of an inscribed hexagon, he found successively the sides of polygons of twelve, twenty-four, forty-eight, ninety-six, one hundred ninety-two, and three hundred eighty-four sides. The perimeter of the last is given as equal to $\sqrt{9.8684}$, from which his result was obtained by approximation.[8]

The most prominent Hindu mathematician of the seventh century was Brahmagupta (c. 650), who gave the $\sqrt{10}$ as a value for pi, equal to 3.1622, approximately 0.66% greater than the value, 3.1416.

> He obtained this value by inscribing in a circle of unit diameter regular polygons of twelve, twenty-four, forty-eight, and ninety-six sides, and calculating successively their perimeters, which he found to be $\sqrt{9.65}$, $\sqrt{9.81}$, $\sqrt{9.86}$, and $\sqrt{9.87}$, respectively; and to have [sic] assumed that as the number of sides is increased indefinitely the perimeter would approximate to $\sqrt{10}$.[9]

Bhaskara, also a Hindu mathematician (c. 1150), gave 3927/1205 equal to 3.14160. Furthermore, he cited 754/240, equivalent to 3.14166 for pi, but scholars question whether this last figure was cited only as an approximate value.[10]

In 1579, Francois Vieta, a French mathematician, determined pi correct to nine decimal places, by showing that ". . . pi was greater than $31415926535/10^{10}$, and less than $31415926537/10^{10}$."[11] He deduced this information from ". . . the perimeters of the incribed and circumscribed polygons of 6×2^{16} sides, obtained by repeated use of the formula, $2\sin^2 \frac{1}{2}\theta = 1 - \cos\theta$"[12] In 1585, another Frenchman, Adriaen Anthonisz, produced the ratio 355/113, which is equal to 3.14159292, correct to six decimal places. It was apparently a lucky accident for Anthonisz, since all he had demonstrated was that pi was located between 377/120 and 333/106. He averaged the numerators and the denominators, thereafter, to obtain the approximate value for pi.[13]

Adriaen Van Roomen, a Dutch mathematician, in 1593 calculated the perimeter of the inscribed regular polygon of a 2^{30} sides from which he, in turn, determined the value of pi correct to fifteen decimal places.[14] Ludolph Van Ceulen, a German, computed pi to thirty-five decimal places by calculating the perimeter of a polygon having 2^{82} sides.[15] It was a Dutch physicist, Willebrord Snell, however, who, in 1621, devised a trigonometrical improvement over the so-called "classical method" of computing pi. From each pair of bounds on pi, established by the classical method of computation, he was able to obtain considerably closer bounds, even to duplicating Van Ceulen's earlier thirty-five decimal places. He accomplished this result by using polygons ". . . having only 2^{30} sides."[16] It is further significant to realize that the classical method, making use of such polygons, yielded only fifteen decimal places, and for polygons of ninety-six sides, two decimal places; whereas, ". . . Snell's improvement gives seven places."[17] Grienberger, in 1630, using Snell's refinement of method, carried the approximation to thirty-nine decimal places.[18]

While the geometric method of computation had been popularly relied upon for many centuries, it now became possible to calculate pi by other methods introduced into mathematics. One such means was the analytical method of computing by the convergent series, products, and continued fractions. Vieta (c. 1593), for example, discovered an interesting approximation for pi, using continued products for the result. His value may be obtained from the following formula:

$$2/\pi = \sqrt{\tfrac{1}{2}}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}}}} \cdot \cdot \cdot$$

In 1650, John Wallis, an English mathematician, proved that

$$\pi/2 = 2/1 \cdot 2/3 \cdot 4/3 \cdot 4/5 \cdot 6/5 \cdot 6/7 \cdot \cdot \cdot \cdot$$ [30]

He quoted, in addition, a proposition which had been given a few years earlier by Viscount Brouncker, to the effect that

$$4/\pi = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cdot \cdot \cdot}}}$$ [21]

However, neither of these theorems was used to any large extent for future calculation. In 1668, James Gregory, a Scotch mathematician, derived a series which was used subsequently in connection with other relationships in calculating the value of pi. His series was as follows:

$$\text{arc tan } x = x - x^3/3 + x^5/5 - x^7/7 + \cdot \cdot \cdot \cdot$$ [22]

Later, in 1673, Gottfried Wilhelm Leibniz, a German mathematician, utilizing Gregory's series and permitting $x=1$, derived a subsequent series:

$$\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdot \cdot \cdot \cdot$$ [23]

It is obvious that Leibniz' series converges rather slowly for an accurate value of pi. In 1699, Abraham Sharp, Englishman, also making use of Gregory's series and letting $x = \sqrt{1/3}$, derived the following series:

$$\pi/6 = \sqrt{\tfrac{1}{3}}\left(1 - \frac{1}{3 \bullet 3} + \frac{1}{3^2 \bullet 5} - \frac{1}{3^3 \bullet 7} + \frac{1}{3^4 \bullet 9} - \cdot \cdot \cdot \cdot\right)$$ [24]

Sharp's series produces a result more usable than that for $\pi/4$, giving an approximation for pi to seventy-one decimal places. In 1706, John Machin, also an Englishman, by substituting Gregory's infinite series for *arc tan* 1/5 and *arc tan* 1/239, gave the expression,

$$\pi/4 = 4 \text{ arc tan } 1/5 - \text{arc tan } 1/239.$$ [25]

Such a convergent series is faster and more useful, and Machin succeeded in calculating pi correctly to one hundred decimal places. In 1873, another Englishman, William Shanks, computed pi to 707 decimal places, using Machin's formula; but in 1946, D. F. Ferguson found Shanks' computation to be in error in the 528th place.[26]

It was soon discovered that the value of pi might be determined experimentally by applications of the probability theory. For example, in 1760, Comte de Buffon devised his now-famous Needle Problem to determine pi by probability:

> On a plane a number of equidistant parallel straight lines, distance apart a, are ruled; and a stick of length 1, which is less than a, is dropped on to the plane. The probability that it will fall so as to lie across one of the lines is $2\ 1/\pi a$. If the experiment is repeated many hundreds of times, the ratio of the number of favorable cases to the

whole number of experiments will be very nearly equal to this fraction; hence, the value of pi can be found.[27]

In the middle of the eighteenth century, mathematicians began anew to investigate the nature of the number *pi,* to determine if it were rational or algebraic or transcendental. The first such investigation of a fundamental importance was that undertaken by Johann Heinrich Lambert, a German, in 1761. Lambert obtained the two fractions,

$$\frac{e^x - 1}{e^x - 1} + \frac{1}{2/x} + \frac{1}{6/x} + \frac{1}{10/x} + \frac{1}{14/x + \cdots},$$

and

$$\tan x = \frac{1}{1/x} - \frac{1}{3/x} - \frac{1}{5/x} - \frac{1}{7/x - \cdots},$$

both are closely related with continued fractions obtained by Euler, but the convergence of which Euler had not established. As a result of an investigation of the properties of these continued fractions, Lambert established the following theorems: (1) if x is a rational number, different from zero, e^x cannot be a rational number. (2) If x is a rational number, different from zero, tan x cannot be a rational number. (3) If $x=\pi/4$, we have $\tan x=1$, and therefore $\pi/4$ cannot be a rational number, hence π cannot be a rational number.[28]

Following the discovery of the distinction between algebraic and transcendentals, man questioned to which of these two categories the number *pi* belonged. An algebraic irrational, first of all, is one which is a root of the equation of the form,

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0,$$

where *n* is a rational number and $a_0, a_1, a_2, \cdots, a_n$ are rational. A transcendental number, on the other hand, is one for which such an equation is not satisfied. In 1882, Ferdinand Lindemann, a German mathematician, proved that the above equation could not hold, when x=e; n and $a_0, a_1, a_2, \cdots, a_n$ are algebraic numbers, not necessarily real. Euler had previously shown that $e^{i\pi} + 1 = 0$. Now, if π is algebraic, then $i\pi$ is algebraic, and, thus, $e^n + 1 = 0$ is satisfied by $n = i\pi$, contradicting the theorem of Lindemann; hence it is proved that π is not algebraic, but transcendental.[29]

In addition, there are occasions today when it is desired to express pi to more than the well-remembered four decimal places. When this occurs, memory can be aided by the use of mnemonics, in which the number of letters in a word is the key to the appropriate digit involved. One example, taken from *School Science and Mathematics,* gives pi correct to twelve digits:

3 1 4 1 5 9

See, I have a rhyme assisting

2 6 5 3 5 9

My feeble brain its tasks resisting.[30]

Another mnemonics system, giving pi to thirty-one significant digits, ap-