NUMERALS FOR RATIONAL NUMBERS:
A BRIEF HISTORY

A THESIS

SUBMITTED TO THE DEPARTMENT OF
MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE
TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

by
KAREN ALLISON

May 1961

Thesis
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ACKNOWLEDGMENT

[illegible] M. Burger, Head of the Department of [illegible] Kansas State Teachers College of Emporia, [illegible] rected this thesis, the writer wishes to [illegible] re gratitude. The writer also wishes to [illegible] indebtedness to Mr. Tom Bonner and Mr. [illegible] rofessors of Mathematics, for encouragement and helpful suggestions on the thesis, and to Mr. Charles Tucker, Professor of Mathematics, who gave the writer her first interest in the history of numerals for numbers.

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CHAPTER I

INTRODUCTION

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[illegible] Numeral systems are often studie[illegible]ory of mathematics. Seldo[illegible]ected the various numeral syst[illegible] to study their origin, for[illegible] the intricacies of co[illegible] occasionally wonder how n[illegible] just how standardized [illegible]ods of time.

[illegible]he problem. The objective of this study is to p[illegible]f historical background of the numerals for rational n[illegible]bers, with emphasis on the many variations through the years, and without considering their use in computation.

1.3. Purpose. The purpose of this thesis is to provide a handbook showing at least some of the highlights in the world-wide evolution of numeral systems.

1.4. Scope. To speak of numbers is to speak of a concept infinite in scope. The author will, therefore, limit discussions to the numerals for rational numbers, explicitly, the non-negative integers and the "fractions". In view of the fact that volumes have been written about the early development of numerals in specific countries, obviously only highlig ... ion throughout the world can be pres...

1.5. ... The material for this thesis is ... rious books and magazines. ... Florian Cajori's Notation ... Volume I of A History of Mathemat... ne Smith's History of Mathema... ics of Elementary Mathema... aking" section of The Mathematics Teacher ... special value.

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Where differences of interpretation existed in the various presentations, the writer usually found the theories of Cajori and Smith more consistently in agreement. However, the variations in theories offered sometimes necessitated a lengthy discussion in order to incorporate more than one of the many viewpoints in the thesis.

1.6. Organization of the thesis. The organization of the thesis is designed to enable the reader to trace separately the development of numerals for integers and of numerals for fractions. Chapter II presents the development of the numerals for integers, beginning with the earliest known examples of them, then tracing the Hindu-Arabic numerals to ... resent somewhat standardized form. In a like man ... numerals for fractions is discussed ... summerizes the material ... art of numerals through ...

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[illegible]HAPTER II

[illegible]S FOR INTEGERS

[illegible] The concept of number and the p[illegible] long before the time of re[illegible]y primitive man probably had son[illegible] of recognizing more or less obje[illegible] first began to use numbers, they [illegible]ing with them, counting. Employ[illegible]one correspondence, countin[illegible]ved gradually with society. Perhaps the earl[illegible] manner of keeping count was to use a tally of pebbles, sticks, scratches on dirt, notches on wood, or knots in string. More extensive counts necessitated systemizing the counting process, usually by arranging the numbers into convenient basic groups. As we have five fingers on each hand, some people began, after several centuries, to count by fives; then they used the fingers of both hands and counted by tens. These, of course, were not the only primitive number bases. Other men found it more convenient to use groups of two, three, four, six, or other number.

2.2. Vocal numerals. Later, an assortment of vocal sounds developed as a numerical tally. Number names are believed to be among the first words used when people began to talk.[1] [illegible] ystems developed slowly and were p[illegible] he need for them arose. In the ea[illegible] ng, dissimilar sounds were used fo[illegible] t objects. The abstraction of the [illegible] epresented by a sound considered [illegible] e association, was probably a lon[illegible]

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[illegible] ook much more time after people [illegible] efore they used signs for them. F[illegible] he expression of numbers by various positions [illegible] the fingers and hands, were used at one time. Evidence of the prevalence of these finger symbolisms is found among the ancient Egyptians, Babylonians, Greeks, and Romans.[3] However, both finger and vocal numerals lacked permanence and were not suitable for performing calculations. Thousands of years passed before people learned how to use an assortment of written symbols to set down a tally for permanent records, the beginning of written numerals. While

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[1] David Smith and Jekuthiel Ginsburg, Numbers and Numerals (Washington, D.C.: The National Council of Teachers of Mathematics, 1956), p. 4.

[2] Howard Eves, An Introduction to the History of Mathematics (New York: Rinehart and Company, Inc., 1958), p. 7.

[3] Florian Cajori, A History of Elementary Mathematics (New York: The Macmillan Company, 1917), p. 1.

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m[illegible] numerals prior to the time of [illegible] r of conjecture, the ideas [illegible] nthropological reports on [illegible].[4]

[illegible]RAL SYSTEMS

[illegible] O B.C., mathematical records exi[illegible] d numeral systems. These will [illegible] g order: Babylonian, Egypt[illegible] ew, Greek, Gothic, Roman, Aztec, Maya, [illegible] se and Japanese, Early Hindu, and Early Arabic.

2.4. Babylonian. The Sumerians probably invented the cuneiform symbols Y , (, and Y≻ , which stand for one, ten, and one hundred, respectively, in Babylonian notation.[5] Early Sumerian clay tablets also contain the circle, ○, for ten and a semicircle, D , for one. Both sets of numerals were made with a stylus, the former with the pointed end, the latter with the blunt circular end. These Sumerian numerals were then passed on to the Babylonians as they took over power in Mesopotamia.

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[4] Eves, loc. cit.

[5] Cajori, op. cit., p. 6.

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... ation commonly employed two principles, ... ative, and made limited use of the ... n. Following the first principle, ... rs below two hundred were to be ... of a small numeral in front of a ... hat the two were to be multiplied ... plicative principle, the numeral ... thousand. Further, ‹‹Y≻ represented ... usand. No numbers as large as a million have been found in this notation.[6] Babylonians apparently had an ideogram, translated as Lal, for minus or subtraction. Using this, some numbers were expressed via the principle of subtraction. For example, ‹‹Y ≻Y represented nineteen. The ‹‹ to the left signified twenty, the Y one and ≻Y minus. To avoid confusion, some early Babylonian documents contained symbols for 1, 60, 3600, 216000, and also for 10, 600, and 3600; nevertheless, numbers were usually expressed using the "principle of position" in the sexagesimal or perhaps decimal system.[7] The sexagesimal system, employing the principle of position, evolved sometime between 3000 and 2000 B.C. It was really a mixed system, as numbers less than sixty were

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6 Ibid.

7 Florian Cajori, A History of Mathematical Notations (Chicago: The Open Court Publishing Company, 1928), I, p. 2.

written by a simple grouping system to the base ten and numbers exceeding sixty were written according to the positional principle.[8]

Use of place value as we know it today requires a symbol for zero; however, Babylonians apparently had no such symbol. In some cases, a horizontal line was drawn after symbols in the sexagesimal system, apparently indicating the absence of units of lower denominator, but this was not done in a manner to indicate the number of places included. About the second century B.C., the symbol ⟨ was used to designate the absence of a number in astronomical data, although it was not used in computations.[9] Another symbol for zero used about 200 B.C. was the ⟪.[10] There is apparently no evidence that the ancient Babylonians at any time regarded zero as a number by itself which could enter into operations with other numbers.

In addition to Lal for minus, the Babylonians used the following translated ideograms in writing numbers: Igi-Gal,

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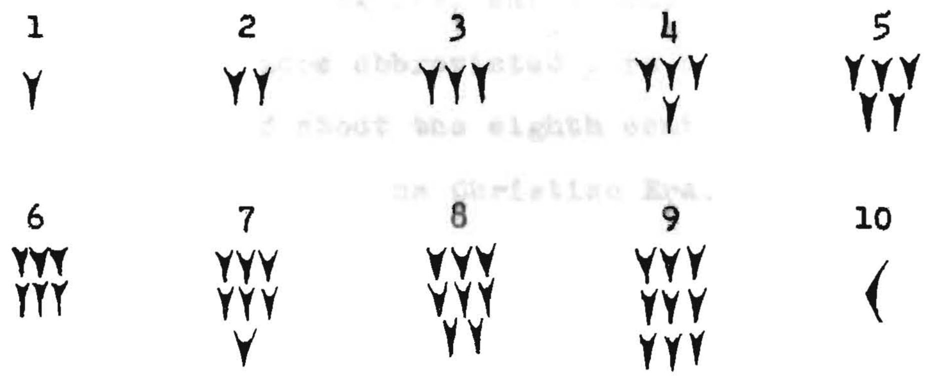
8 Eves, *op. cit.*, p. 15.

9 Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 7.

10 Cajori, *A History of Elementary Mathematics*, *op. cit.*, p. 10.

denominator or division; *Igi-Dua*, division; and *A-Du* and *Ara*, times.[11]

The following figures represent Babylonian numerals and their modern Hindu-Arabic equivalents:



2.5. *Egyptian*. Four forms of characters, the hieroglyphic, hieratic, demotic, and coptic, and various bases, five, ten, twelve, twenty, and sixty, were used in Egyptian numeral systems.[12] Hieroglyphics, the oldest numerals, date back to about 3300 B.C. and were chiseled on monuments of stone, wood, or metal. The hieroglyphic characters were written both from right to left and from left to right.[13] Earlier hieroglyphics were often written from the

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[11] Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 11.

[12] *Ibid*.

[13] David Smith, *History of Mathematics* (Boston: Ginn and Company, 1925), II, p. 45.

top down.[14] When the reed-pen came into use, more rounded forms known as hieratic numerals evolved and were used in many important ancient Egyptian mathematical documents written on papyrus. Hieratic writing usually proceeded from right to left, with occasional early numerals running from the top down.[15] A more abbreviated form of cursive writing, the demotic, evolved about the eighth century B.C., and was used to the beginning of the Christian Era.[16] After the demotic forms came into general use, the hieratic was reserved for religious purposes.[17] The Egyptian Coptic numerals, derived from the demotic and Greek writing, were used by Christians in Egypt after the third century B.C., and were later adopted by the Mohammedans after their conquest of Egypt.[18]

The Egyptians used the principles of addition and multiplication in writing numbers. The oldest principle, the additive, was applied by placing not more than four symbols of the same kind in one group. In case of two unequal

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[14] Ibid., p. 46.

[15] Ibid., p. 47.

[16] Cajori, A History of Mathematical Notations, loc. cit.

[17] Smith, loc. cit.

[18] Cajori, A History of Mathematical Notations, op. cit., p. 17.

groups, the larger appeared before or above the smaller; for example, seven was written $\overset{||||}{|||}$.[19] In the multiplicative principle, which came into use about 2000 to 1600 B.C., a numeral representing a smaller unit written before the numeral of a larger unit designated multiplication of the larger by the smaller.[20]

Following are hieroglyphic symbols and their modern Hindu-Arabic equivalents:

1 10 100 1000

10,000 100,000 1,000,000 10,000,000

Due to the fact that numbers were written either from right to left or from left to right, a numeral often faced different directions than shown above.[21] Further, the numerals for one and ten sometimes occurred in a horizontal position.

The hieroglyphic symbols were taken from common objects which in some way suggested the idea to be conveyed. For instance, the symbol for one represented a vertical

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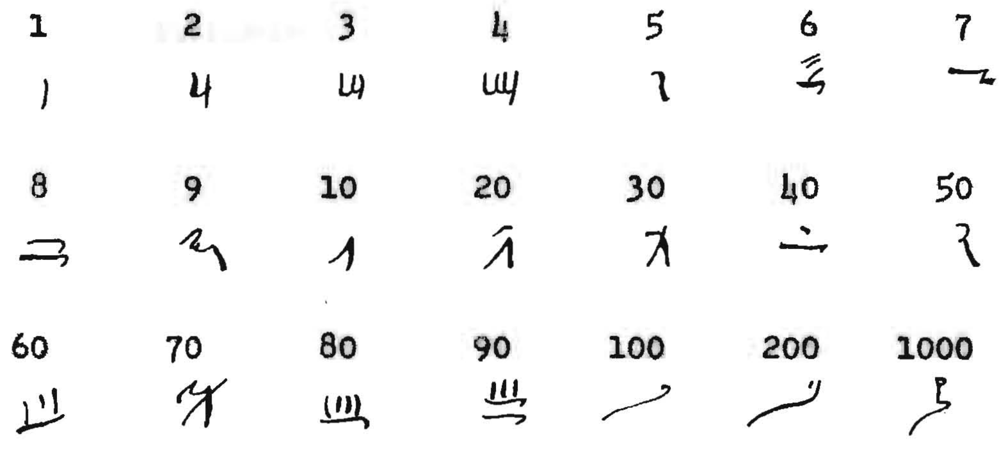
[19] Ibid., p. 13.

[20] Ibid., p. 14.

[21] Smith, op. cit., p. 46.

staff; for one hundred, a tadpole; for one thousand, a lotus plant; for ten thousand, a pointing finger; for one hundred thousand, a burbot bird; and for one million, a man in astonishment, or according to some authors, a picture of the cosmic deity Hh.[22] In older hieroglyphs, lotus plants were grouped as if grown on one bush. For example, two thousand was designated as ; seven thousand as . Later, this changed to simply placing lotus plants together without the appearance of springing from one and the same bush.[23]

The following is a comparison of hieratic signs to modern Hindu-Arabic numerals:[24]



Since there are more hieratic symbols than hieroglyphic, numbers could be written more concisely with hieratic numerals.

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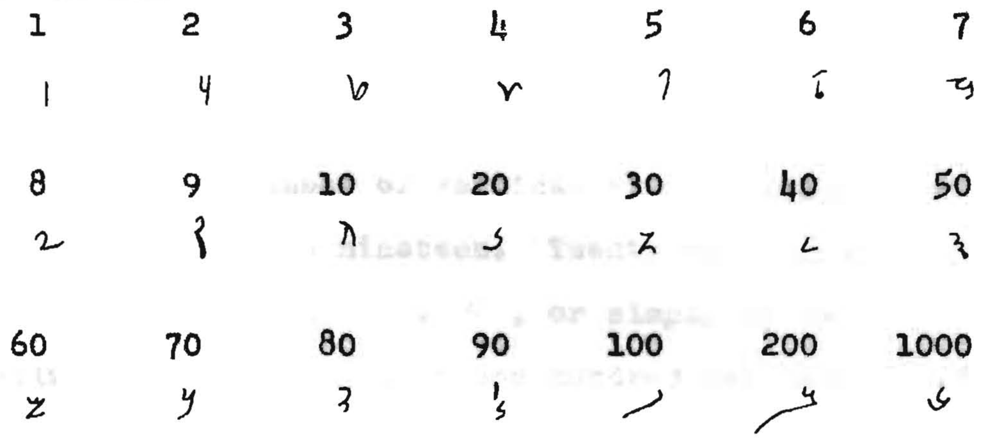
[22] Cajori, A History of Mathematical Notations, op. cit., p. 12.

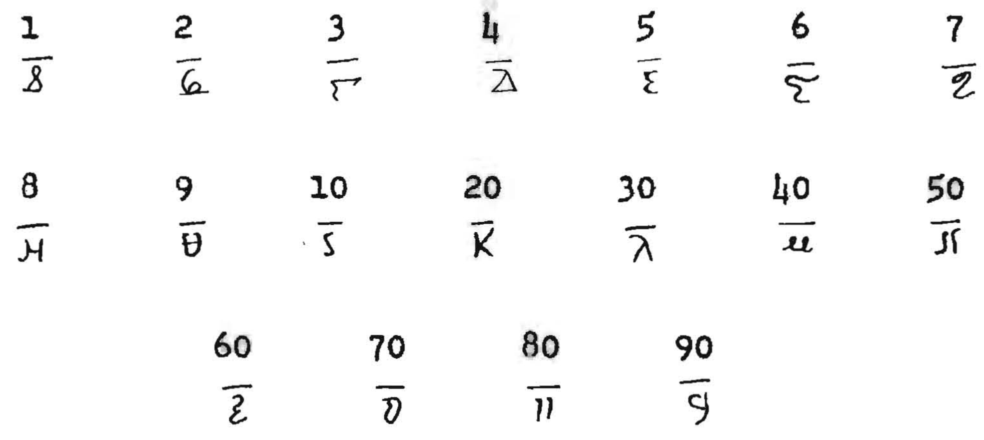
[23] Ibid., p. 14.

[24] Ibid., p. 12.

Next, demotic numerals are compared to current Hindu-Arabic equivalents.[25]



Finally, coptic numerals are compared to current Hindu-Arabic numerals.[26]



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25 <u>Ibid</u>.

26 <u>Ibid</u>., p. 17.

2.6. Phoenician. Phoenician inscriptions date back several centuries before Christ. The Phoenicians used the respective number of vertical strokes to represent the numbers one through nine and a horizontal bar to represent ten. The juxtaposition of a horizontal bar farthest to the right plus the required number of vertical strokes represented the numbers eleven through nineteen. Twenty was represented in various ways, including ∧ , *H* , or simply by two horizontal or inclined parallel strokes; one hundred was designated by |<| or />/ .[27]

These are examples of Phoenician numerals and their current Hindu-Arabic equivalents:

1	2	3	4	5	6	7
|	||	|||	////	/////	//////	///////
8	9	10	11	12	13	14
////////	/////////	—	/—	//—	///—	////—
15	16	17	18	19	20	100
/////—	//////—	///////—	////////—	/////////—	∧	|<|

Somewhat similar to Phoenician numerals were those found in Palmyra. One variation below one hundred is the use

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[27] Ibid., p. 18.

of ⊢ for five; new numeral forms started with one hundred. One vertical stroke placed to the right of the symbol for ten signified multiplication of ten by ten for one hundred; two vertical strokes indicated two hundred; three, three hundred; and so on.[28] The preceding numerals for one, two, and three hundred would appear as -| , -||, and -||| , respectively.

Syrian numerals of the sixth and seventh centuries A.D. were related to Phoenician numerals. The twenty-two letters of the Syrian alphabet represented the numbers one through nine, ten through ninety, and the hundreds one through four. Five hundred through nine hundred was indicated either by juxtaposition; 500 = 400 + 100, 600 = 400 + 200, ..., 900 = 400 + 400 + 100, or by placing a dot over the letters for fifty through ninety. Thousands, ten thousands and millions were indicated by modifying the letters for units and tens. A stroke annexed as a subscript on the letters for units signified thousands, two strokes annexed in the same manner signified millions. A small dash below the letter for units and tens expressed ten thousands.[29]

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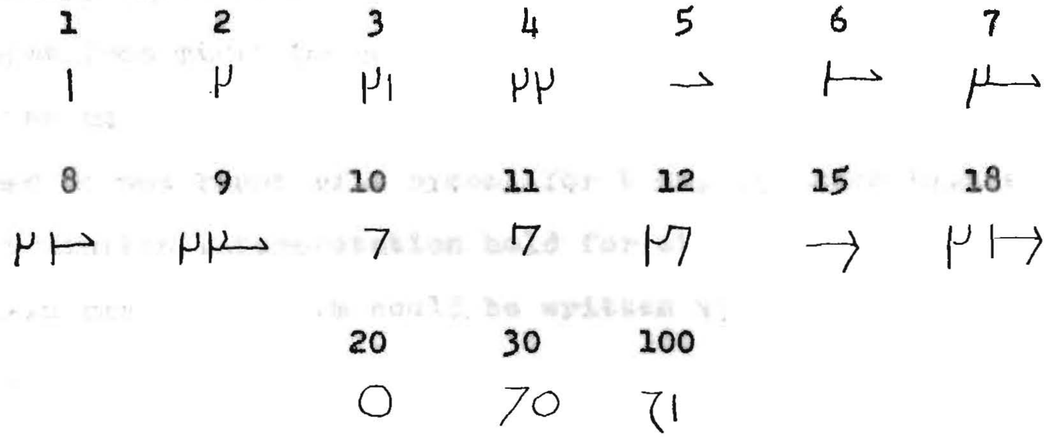
28 *Ibid.*

29 *Ibid.*, p. 19.

Syrian numerals are compared to current Hindu-Arabic counterparts in the following table:[30]



2.7. <u>Hebrew</u>. The earliest use of an entire alphabet for representing numbers has been attributed to the Hebrews. Three forms of characters, the Samaritan, Hebrew, and Rabbinic or cursive, and the decimal scale designated Hebrew numbers up to four hundred.[31] The hundreds from five hundred through eight hundred were at first represented by juxtaposition of the sign for four hundred and a second sign for one hundred through three hundred. Later, end forms of the five Hebrew letters representing twenty, forty, fifty, eighty, and ninety were used to represent five hundred through nine hundred. Two dots placed over each letter increased its value a thousand fold and, thus, numbers up to

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[30] <u>Ibid</u>.

[31] <u>Ibid</u>.

a million were represented.[32] In addition, the beginning of an imperfect application of place value appears in Hebrew notation. As were most old notations, Hebrew numerals were written from right to left, with the numeral of highest value written on the right. If, however, any symbol of units was placed to the right of a symbol for tens, it meant thousands, and a similar interpretation held for other signs. Thus, in certain cases, numbers could be written without the use of dots.

The following represent Hebrew numerals and current Hindu-Arabic equivalents:[33]

1	2	3	4	5	6	7
א	ב	ג	ד	ה	ו	ז
8	9	10	20	30	40	50
ח	ט	י	כ	ל	מ	נ
60	70	80	90	100	200	300
ס	ע	פ	צ	ק	ר	ש
400	500	600	700	800	900	1000
ת	ך	ם	ן	ף	ץ	א׳

32 Ibid.

33 Smith, op. cit., p. 53.

2.8. __Greek.__ Herodianic signs, named after a Byzantine grammarian of 200 A.D. who described them, were the oldest strictly Greek numerals. Generally called Attic numerals, also called the acrophonic system,[34] these initial letters of numeral adjectives were used as early as 600 B.C. and included the following symbols, as compared to current Hindu-Arabic forms:[35]

1	5	10	100	1,000	10,000
I	Γ	Δ	H	X	M

Combinations of the symbol for five with the symbols for ten, one hundred, and one thousand represented fifty, five hundred, and five thousand. From about 470 to 350 B.C., this system competed with a system based on alphabetic numerals, the Ionic. The Greeks finally adopted the latter, which used the twenty-four letters of the Greek alphabet plus three antique letters to represent numbers below one thousand. Of the twenty-seven letters, the first nine represented the first nine numbers; the second nine represented multiples of ten; and the third nine represented multiples of one hundred. Further, the first nine letters written with a stroke

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[34] __Ibid.__, p. 49.

[35] __Ibid.__, p. 50. Cf. Cajori, __A History of Mathematical Notations__, __op. cit.__, p. 22.

indicated multiples of one thousand. Greeks often placed a mark (/ or ') by or a bar over each letter to show that it stood for a number.[36] In the Middle Ages, a numeral was occasionally written as if lying on its side.[37]

Early Greek Ionic numerals are compared to present-day Hindu-Arabic numerals in the following table.[38]

1	2	3	4	5	6	7	8	9
Α	Β	Γ	Δ	Ε	Ϝ	Ζ	Η	Θ
10	20	30	40	50	60	70	80	90
Ι	Κ	Λ	Μ	Ν	Ξ	Ο	Π	Ϙ
100	200	300	400	500	600	700	800	900
Ρ	Σ	Τ	Υ	Φ	Χ	Ψ	Ω	ϡ

The capital letters were first used for numerals, with the small letters being substituted much later. The following comparison with modern Hindu-Arabic numerals will illustrate the system with the small letters.[39]

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[36] Smith and Ginsburg, op. cit., p. 12. Smith, op. cit., p. 52.

[37] Smith, loc. cit.

[38] Smith and Ginsburg, loc. cit.

[39] Eves, op. cit., p. 14. Cf. Cajori, A History of Mathematical Notations, op. cit., p. 25.

1	2	3	4	5	6	7	8	9
α	β	γ	δ	ε	ϛ	ζ	η	θ

10	20	30	40	50	60	70	80	90
ι	κ	λ	μ	ν	ξ	ο	π	ϙ

100	200	300	400	500	600	700	800	900
ρ	σ	τ	υ	φ	χ	ψ	ω	ϡ

A numeral system based on the additive principle developed on the island of Crete as early as 1500 B.C. under Egyptian influence. The comparison of symbols with current Hindu-Arabic numerals is as follows:[40]

1	5	10	100	1000
/	/////	·	\	◇
or	or		or	or
)	)))))		/	⌖
	or		or	
	/// //		○	

[40] Cajori, A History of Mathematical Notations, op. cit., p. 21.

2.9. Gothic. Another set of alphabetic numerals, for the most part of Greek origin, were used by the Goths. Their comparison to current Hindu-Arabic numerals follows.[41]

1	2	3	4	5	6	7	8	9
A	B	Γ	d	ϵ	u	z	h	ϕ
10	20	30	40	50	60	70	80	90
ï	K	λ	M	N	G	n	π	ч
100	200	300	400	500	600	700	800	900
R	S	T	Y	F	X	θ	ℓ	↑

2.10. Roman. The origin of the Roman numeral system is somewhat uncertain. The Etruscans, who ruled Rome until 500 B.C., used numerals which resembled their alphabet and the numerals used later by the Romans, but the Romans never used successive letters of their alphabet as numerals.[42] The well-known symbols are included in the following comparison.

Modern Hindu-Arabic	1	10	50	100
Roman	I	X	L	C

Roman numerals generally represented numbers by means of the

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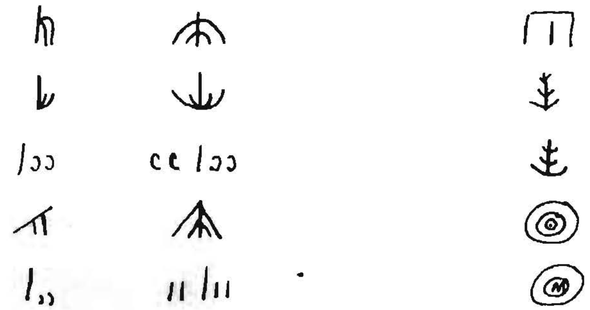
[41] Smith and Ginsburg, op. cit., p. 18.

[42] Cajori, A History of Mathematical Notations, op. cit., p. 30.

additive principle. The rule "that a smaller numeral placed to the left of a larger shall be subtracted from the latter" was rarely used by the old Romans.[43] A few cases of IX appear in the middle of the fifteenth century; it was an even later date before the subtraction principle was commonly used. As the Romans had relatively little need for writing large numbers, they developed no general system of numerals for them.[44] For one thousand, the Romans had various symbols: M, ∞, and CIↃ. The symbol CIↃ (a one enclosed in *apostrophos*), dating to perhaps 500 A.D., was not widely used before the Middle Ages. The number of *apostrophos* and proper placement was sometimes used to represent other numbers. Roman numerals for large numbers were often written as in the table which follows:[45]

Hindu-Arabic	500	5,000	10,000	50,000	100,000
Roman	IↃ	IↃↃ	CCIↃↃ	IↃↃↃ	CCCIↃↃↃ



43 *Ibid.*, p. 31.

44 Smith, *op. cit.*, p. 60.

45 *Ibid.* Cf. Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 32.

Sometimes the preceding forms were repeated the proper number of times to represent larger numbers.[46] The thousand-fold value of a number was indicated in some instances by a horizontal line placed above it. Lines placed on the top and sides indicated hundred thousands.[47] Confusion frequently resulted from this practice, as Romans commonly placed a bar over a number to distinguish it from a word.[48]

The Roman numerals held a commanding place long after Hindu-Arabic numerals became generally known. A Florence banking house used Roman numerals in writing small sums of money in account books as late as the fourteenth century.[49]

2.11. Aztec. An illustration of North American Indian numerals is given by the Aztec system. The Aztec Indians, with a system based on twenty, used the dot to mark units up to ten, which was represented by a lozenge. A flag represented twenty and was repeated up to five times in numerals. Parts of a feather represented the numbers one hundred, two hundred, three hundred, and four hundred, with

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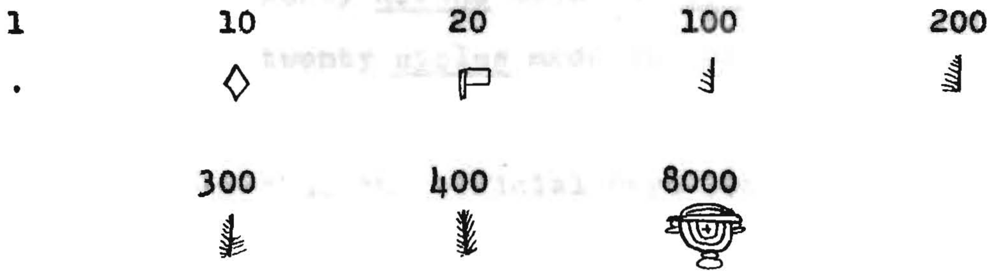
[46] Smith, loc. cit.

[47] Cajori, A History of Mathematical Notations, op. cit., p. 32.

[48] Smith, loc. cit.

[49] Cajori, A History of Mathematical Notations, op. cit., p. 34.

one quarter of the feather barbs equivalent to one hundred, half to two hundred, three-fourths to three hundred, and all to four hundred. A purse represented eight thousand. A chart of the symbols, as compared to present-day Hindu-Arabic numerals, follows.[50]



The additive principle and the juxtaposition of symbols enabled Aztecs to represent other numbers. As was common among North American Indian tribes, the Aztecs had no symbol for zero, not did they use the principle of place value.[51]

2.12. Maya. The Maya of Central America and Southern Mexico differed from other American Indian tribes in their use of a remarkable number system and chronology, apparently from about the beginning of the Christian Era. Their numeral system disclosed a symbol for zero five or six centuries

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50 Ibid., p. 41.

51 Ibid.

before the Hindus developed their zero.[52] In addition, Maya numerals showed application of the principle of place value.

The system was, with the exception of one step, vigesimal. Twenty units, *kins*, made one *uinal* (20 days); eighteen *uinals* made one *tun* (360 days); twenty *tuns* made one *Katun* (7200 days); twenty *Katuns* made one *cycle* (144,000 days); and finally, twenty *cycles* made up one *great cycle* (2,880,000 days).[53] One explanation for the discrepancy in the *tun* is the fact that the official Maya year consisted of 360 days.[54]

A symbol resembling a half-closed eye ,[55] or ,[56] represented zero, and the numbers one through nineteen were expressed by dots (one unit) and bars (five units). Large numbers were written vertically, beginning with the lowest order in the lowest position.

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52 Florian Cajori, *A History of Mathematics* (New York: The Macmillan Company, 1893), p. 69.

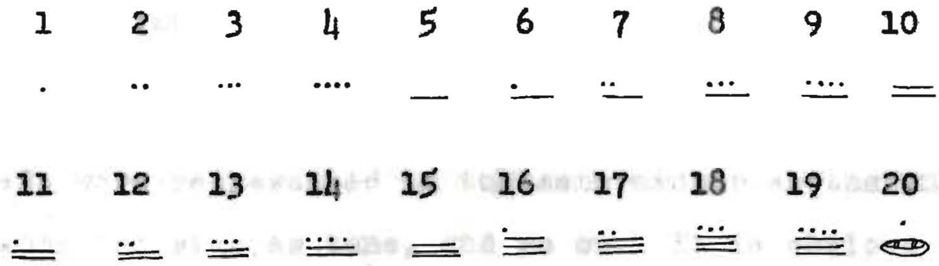
53 *Ibid.* Cf. Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 43.

54 Eves, *op. cit.*, p. 15.

55 *Ibid.*

56 Smith, *op. cit.*, p. 44.

The following is a comparison of Maya numerals and their current Hindu-Arabic counterparts.[57]



2.13. <u>Chinese and Japanese</u>. The oldest representatives of Chinese numbers were knots tied in strings. Odd numbers were designated by white knots, even numbers by black knots. Three other systems, Old Chinese numerals, mercantile numerals, and scientific numerals were also used by the Chinese in writing numbers. Again, as in Roman numerals, the origin of each system is uncertain.

Vertical and horizontal rods made up the Chinese scientific numerals. The numbers one through nine were represented as in the following comparison with modern Hindu-Arabic numerals:

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[57] Eves, <u>op</u>. <u>cit</u>., p. 16.

Using the same comparison, the following illustrates the numbers ten through ninety:

10	20	30	40	50	60	70	80	90
𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

Hundreds were represented in the same manner as the units, thousands the same as tens, and so on. It is obvious that a system of place value was necessary.[58] A circle, ○ , was used for zero in the Sung Dynasty (950-1280) and later.[59]

The traditional Chinese-Japanese numeral system was a multiplicative system written vertically. The base, ten, had symbols for ten, one hundred, and one thousand, in addition to the symbols for one through nine. They are now compared to the current Hindu-Arabic numerals.[60]

1	2	3	4	5	6
一	二	三	四	五	六
7	**8**	**9**	**10**	**100**	**1000**
七	八	九	十	百	千

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58 Ibid., p. 24. Cf. Cajori, A History of Mathematical Notations, op. cit., p. 44.

59 Eves, op. cit., p. 24. Cf. Smith, op. cit., p. 42.

60 Eves, op. cit., p. 13.

To indicate the number of tens, hundreds, or thousands in large numbers, the numerals for units were followed by the numeral for ten, a hundred, or a thousand. For instance, 7653 would be written (top to bottom) 七千六百五十三 .

Another numeral system used by Chinese merchants wrote the numbers one through ten as follows.[61]

In addition, a circle was used for zero. There were also special symbols for one hundred, one thousand, and ten thousand. These were 百 , 千 , and 万 , respectively.[62] Knowledge of the ancient forms from which these numerals came is limited; however, variations used by Ch'in Kiu-shao (1247) of  for five and  or  for nine are known.[63]

Japanese cursive numerals, quite similar to Chinese numerals, are as follows in the comparison to their modern Hindu-Arabic counterparts:[64]

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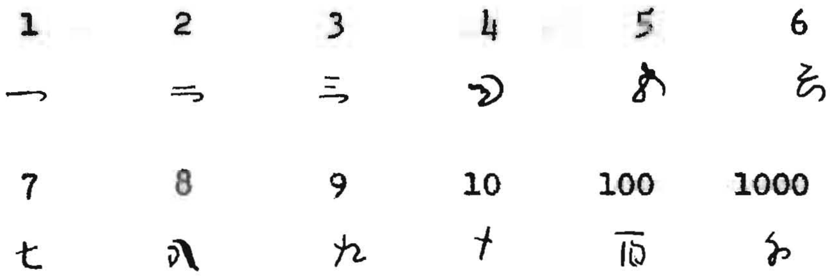
61 Smith, op. cit., p. 40.

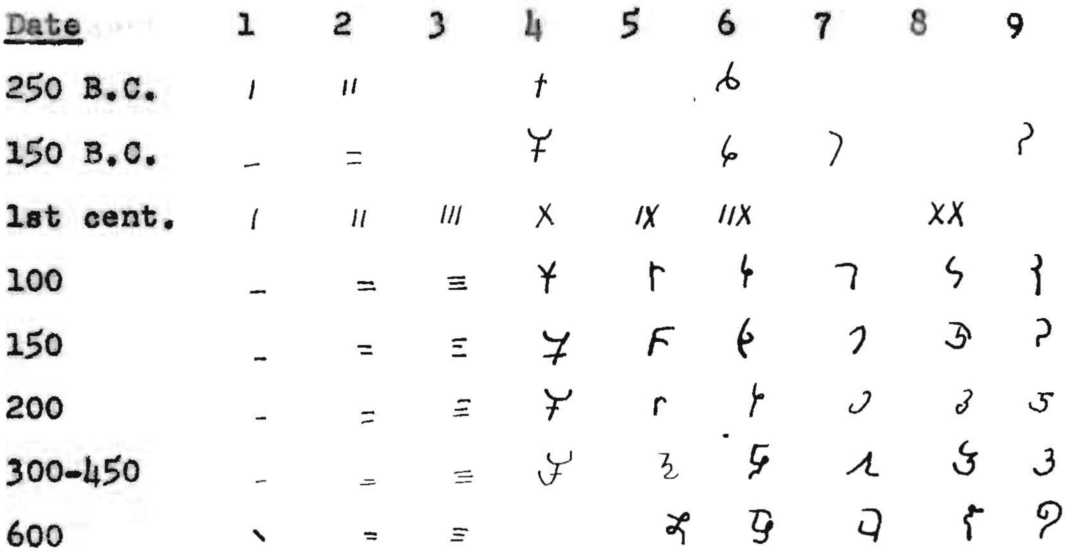
62 Cajori, A History of Mathematical Notations, op. cit., p. 45.

63 Smith, loc. cit.

64 Cajori, A History of Mathematical Notations, loc. cit.



2.14. Early Hindu. Early numerals of India were not uniform, usually varying to meet linguistic conditions in different areas. The earliest known forms appeared in the inscriptions of King Asoka in the third century B.C.[65] Inscriptions at Nana Shat and Nasik were later interpreted by some authorities as numerals, but this was not universally accepted. Variants of the Hindu forms preceding the invention of zero seem to be as in the following tables:[66]



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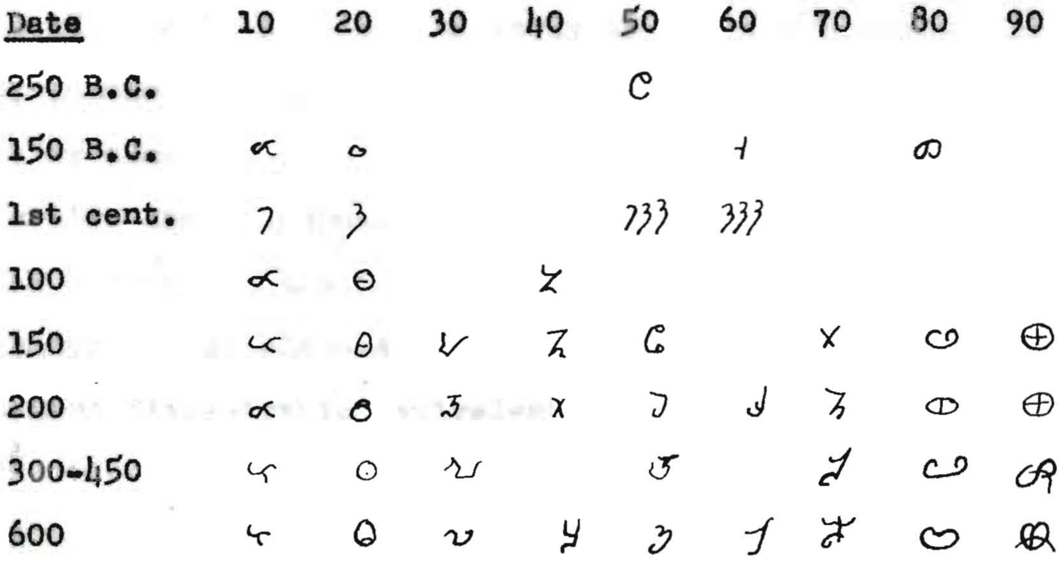
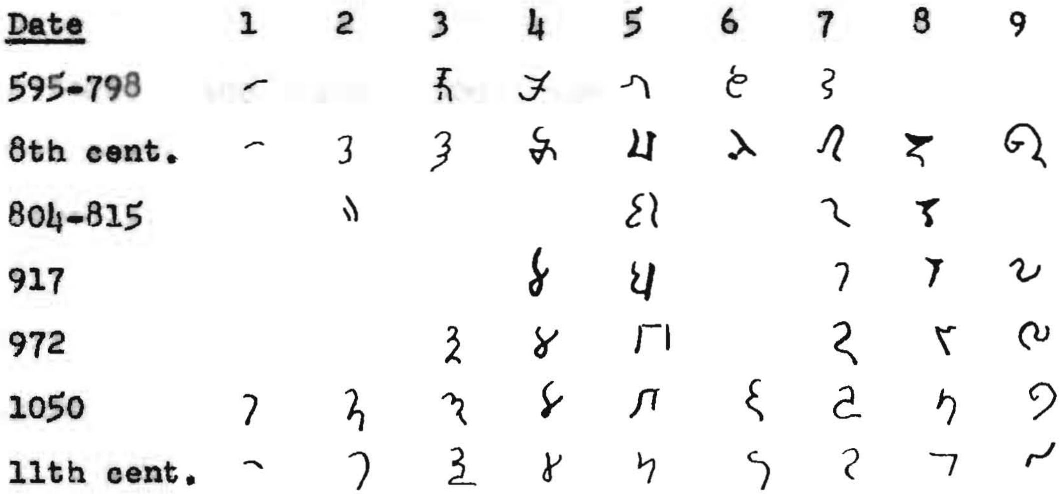
65 Smith, op. cit., p. 65. 66 Ibid., p. 67.



The various forms of the numerals used in India after the zero appeared may be judged from the following table.[67]



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[67] *Ibid.*, p. 70.

2.15. Early Arabic. As early as the sixth century of our era, the Arabs used the letters of the early Arabic alphabet as numerals. However, after the time of Mohammed, Moslem armies came in contact with Greek culture and acquired the Greek numerals. Arabic alphabetic numerals used before the introduction of Hindu-Arabic numerals are compared to their modern Hindu-Arabic equivalents in the following table.[68]

1	2	3	4	5	6	7	8	9
ا	ب	ج	د	ه	و	ز	ح	ط
10	20	30	40	50	60	70	80	90
ى	ك	ل	م	ن	س	ع	ف	ص
100	200	300	400	500	600	700	800	900
ق	ر	ش	ت	ث	خ	ذ	ض	ظ
1000	2000	3000	4000	5000	6000	7000	8000	9000
غ	بغ	جغ	دغ	هغ	وغ	زغ	حغ	طغ
10000	20000	30000	40000	50000	60000	70000	80000	90000
يغ	كغ	لغ	مغ	نغ	سغ	عغ	فغ	صغ

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68 Cajori, A History of Mathematical Notations, op. cit., p. 29.

Arabic numerals used beginning one thousand years ago and continuing to the present time represent the numbers one through ten and zero, respectively, as follows:[69]

١ ٢ ٣ ٤ ٥ ٦ ٧ ٨ ٩ ٠

The zero, as one would observe, is merely a dot, and the five is quite like the Hindu-Arabic zero.

HINDU-ARABIC NUMERALS

The time and place of the origin of Hindu-Arabic numerals is still an unsettled question. Various theories confront historians, and several investigators, working independently of one another, have at one time or another denied the Hindu origin.

As was previously shown, early numerals of India took various forms. The early Hindu mathematicians, Arybhata (476 A.D.) and Brahmagupta (598 A.D.) do not give the information deemed necessary to trace the Hindu-Arabic numerals to India. Aryabhata's Aryabhaliya used an alphabetical numeral system, and the manuscript of Brahmagupta, Pulverizer, which used our notation and a zero, belonged to a late century and, according to some authorities, cannot be accepted as evidence that he himself used the principle of place value and zero in

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[69] Smith and Ginsburg, op. cit., p. 21.

a decimal system.[70] However, Brahmagupta gave rules for computing with zero and thus indicated the use of some kind of numerals with zero.[71] Further, Brahmagupta discussed division of zero by zero, but the use of a decimal system was not in evidence. Further doubt of Hindu origin is caused by the fact that the Babylonians had, at an earlier date, used the principle of place value in their own numeral system well ahead of the Hindu use of the zero.[72] Another way in which historians sought to prove the Hindu origin of our numerals was to connect them with the Sanskrit letters of the second century A.D. However, careful comparison of forms indicated that the resemblance was no closer than to many other alphabets. Most of the arguments presented to deny the Hindu origin of our numerals were inconclusive and the hypothesis of the Hindu origin is commonly accepted. Many authors do, however, feel that place value was not recognized in India from the beginning and, therefore, perfected Hindu-Arabic notation was the result of gradual evolution.[73]

The earliest-known valid reference to Hindu numerals outside of India is the one due to Bishop Severus Sebokht of

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70 Cajori, A History of Mathematical Notations, op. cit., p. 47.

71 Ibid., p. 48.

72 Ibid., p. 47.

73 Ibid., p. 54.

Nisibis, who, living in the convent of Kenneshre on the Euphrates, refers to them in a fragment of a manuscript of the year 662 A.D.[74] Sebokht refers to the Hindus, "their valuable methods of calculation, and their computing that surpasses description. I wish only to say that this computation is done by means of nine signs."[75]

Since almost no commercial records have been preserved from the so-called Dark Ages of Europe, and since the number of scientific works that have been passed down to the present time is also limited, the date when the Hindu-Arabic numerals first came to the west is uncertain.[76] However, Hindu-Arabic numerals were generally known in Europe as early as the year 1000 A.D.[77] Despite this fact, their use was forbidden in commercial documents and in some European banks as late as 1300 A.D., the reason being that authorities felt Hindu-Arabic numerals were more easily falsified or forged than Roman numerals currently used at that time.[78] Hindu-Arabic numerals made their most rapid progress in Europe with the beginning of printing of books.[79]

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74 Ibid., p. 48.

75 Ibid. Cf. Smith, op. cit., p. 65.

76 Smith, op. cit., p. 73.

77 Smith and Ginsburg, op. cit., p. 17.

78 Ibid.

79 Ibid.

2.16. *Early numeral variations*. After the Hindu-Arabic notation was accepted, numerals varied considerably in form. G. F. Hill, of the British Museum,[80] has devoted a book of 125 pages to early European forms of the Hindu-Arabic numerals. Some of the more common variations are now noted.

Zero perhaps took on the most distinct variations of form. The Hindus first used a dot for zero. Twelfth-century numerals for zero included T, θ, ϕ, and $\bar{o}$.[81] Arithmetic manuscripts of the fifteenth century in England represented the zero by a circle crossed by a vertical stroke and resembling the Greek phi,[82] while the French mathematician Michael Rolle (1652-1719) habitually used θ for zero.[83]

However, zero was not the only numeral to vary. In earlier centuries, an upright seven was a rarity, and a five was written in almost any manner.[84] The numerals for one, six, eight, and nine varied the least among medieval Arabs and Christians, while the three and the five were originally

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80 Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 45.

81 *Ibid.*, p. 51.

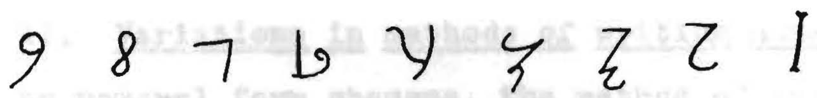
82 *Ibid.*, p. 52.

83 *Ibid.*, p. 51.

84 *Ibid.*

reversed among both the Christians and the Arabs of the Occident.[85] The numerals for five, six, seven, and eight differed distinctly, depending on whether the writer was an Arab of the Orient or of the Occident.[86] Not until the fifteenth century were the numerals for four and five changed to their present forms.[87] An interesting variation was the use of i for one in the first printed arithmetic, the <u>Treviso</u>, printed in 1478.[88] This particular variation was probably due to the fact that there was no type for one.

The oldest known example of the Hindu-Arabic numerals in any European manuscript was written in Spain in the year 976 A.D.[89] The numerals appear as follows:



The following table shows the changes in our numerals from the time of their first use in Europe to the beginning of printing.[90]

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[85] <u>Ibid</u>., p. 53.

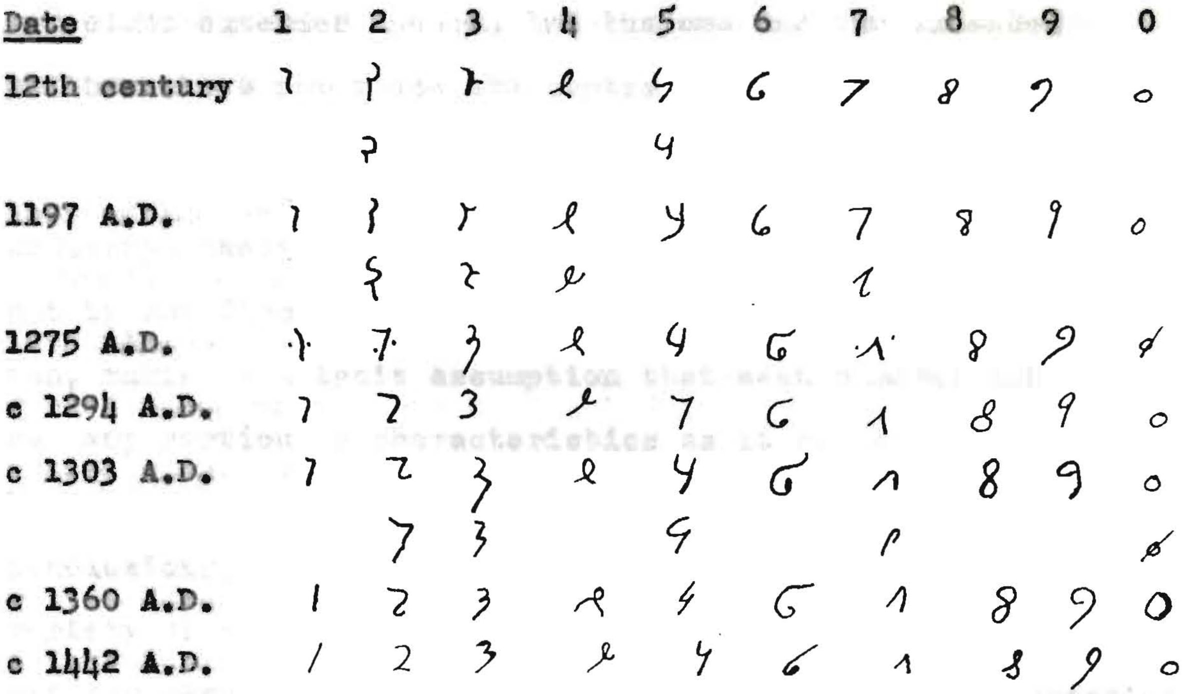
[86] <u>Ibid</u>.

[87] Smith, <u>op</u>. <u>cit</u>., p. 77.

[88] Cajori, <u>A History of Mathematical Notations</u>, <u>loc</u>. <u>cit</u>.

[89] Smith and Ginsburg, <u>op</u>. <u>cit</u>., p. 22.

[90] <u>Ibid</u>.



2.17. Variations in methods of writing numerals. In addition to numeral form changes, the method of writing numerals varied. In some instances, a dot appeared before and after each numeral, as .3. Numerals having several digits were written in various ways. In the fourteenth century, the current year, 1384, was written by one author as 1000. 300. 80. 4; 5782 appeared from another writer as 5. 7. 8. 2.[91] As late as the 1900's, numerals varied in terms of relative size. Some numerical tables gave all numerals as the same height, while in others the seven and the nine were prolonged downward, the three, four, five, six,

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[91] Smith, loc. cit.

and eight extended upward, and the one and two extended neither above nor below the central body of the writing.

2.18. Numeral Origin Hypotheses. Many authors concerned their work with the origin of the numeral forms, not by who first used them, but by how they were first written, making the tacit assumption that each numeral contained as many particular characteristics as it represented units. Despite the fact that their hypotheses have given no valid conclusions, since the authors did not explain the great variety of numeral forms and only endeavored to explain modern European numerals, a few of the theories were interesting to mathematicians. The oldest numeral theory was given by Aben Ragel, an Arabic astrologer of the tenth or eleventh century. He hypothesized that a circle and two of its diameters contained the required forms. The hypothesis of origin from dots was the earliest on the European Continent. These dots which originally formed the numerals, according to the theory, became dashes in cursive writing, and finally were joined by slanting lines. These authors did not believe that they were indulging in fanciful pastimes and were generally convinced of the correctness of their explanations. However, they did not co-ordinate known facts, nor did they suggest new inquiries likely to advance knowledge of the origin of numeral forms.

2.19. <u>Methods of writing numerals for large numbers</u>. The rarity of large numbers is a striking feature of ancient arithmetic. A few exceptions include the Hindu tradition of Buddha's skill with numbers, records on certain Babylonian tablets, and Archimedes' <u>Sand Reckoner</u>.[92] The people, and even most substantial mathematicians, ordinarily had little need for or interest in very large numbers. However, as they came into common use in later centuries, it was desirable to have some symbol separating digits into groups when writing the numerals of large numbers. The most frequently-occurring signs of separation through the ages have been dots, vertical bars, arcs, colons, and commas. The groups have been called various names, including periods, regions, and ternaries.[93] While most authors grouped by threes, a few instances were found of grouping by sixes with vertical bars.[94]

In Liber algorismi, a manuscript of 1200 A.D., dots mark periods of three. In his <u>Liber Abbaci</u> (1202), Leonardo of Pisa, also known as Fibonacci, directs that the hundreds, hundred thousands, hundred millions, and so on, be marked with an accent above and the thousands, millions, thousands of millions, and so on, with an accent below.[95]

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[92] Smith, <u>op</u>. <u>cit</u>., p. 80.

[93] <u>Ibid</u>.

[94] <u>Ibid</u>.

[95] Cajori, <u>A History of Mathematical Notations</u>, <u>op</u>. <u>cit</u>., p. 58.

The following are other methods of grouping digits:[96]

Author	Date	Notation
Fibonacci	1228	678 935 784 105 296
Sacrobasco	1256	9123456789
Pellos	1492	7.538.275.136
Pacioli	1494	8 659 421 635 894 676
Reisch	1503	4.5.9.3.6.2.9.0.2.2
Reisch	1503	a c b a c b a c b a 4593629022
Peurbach	1505	3790528614
Riese	1535	86·7·89·3·25·178
Frisius	1540	24 456 345 678
Riese	1544	86789325178
Stifel	1544	2329089562800
Tonstall	1544	3210987654321
Creszfeldt	1557	58 7 49 3 62 5 34

96 *Ibid.*, pp. 58-59. Cf. Smith, *op. cit.*, p. 87.

<table>
<tr><th>Author</th><th>Date</th><th>Notation</th></tr>
<tr><td>Barozzi</td><td>1585</td><td>25783910627512346894352</td></tr>
<tr><td>Santa-Cruz</td><td>1594</td><td>23.456.207.840.000.305.321</td></tr>
<tr><td>Blundevilli</td><td>1636</td><td>5|936|649</td></tr>
<tr><td>Oughtred</td><td>1652</td><td>Intergri Parti
9|876|543|210|12|345|678|9</td></tr>
<tr><td>Schott</td><td>1661</td><td>7697432329089562436</td></tr>
<tr><td>Rudolff</td><td>1670</td><td>2340563956 7</td></tr>
<tr><td>Greenwood</td><td>1729</td><td>1,234,567</td></tr>
<tr><td>Greenwood</td><td>1729</td><td>1.234.567</td></tr>
<tr><td>Barreme</td><td>1732</td><td>254.567.804.652</td></tr>
<tr><td>Karsten</td><td>1760</td><td>872 III 094,826 152 II,870 I 364,008</td></tr>
<tr><td>deSegner</td><td>1767</td><td>5|329|870|325|743|297|174</td></tr>
<tr><td>Blassiere</td><td>1769</td><td>68|765|432|189|716|789|132</td></tr>
<tr><td>Dilworth</td><td>1784</td><td>789 789 789</td></tr>
<tr><td>Pike</td><td>1788</td><td>356;809,379;120,406;129,763</td></tr>
<tr><td>Hutton</td><td>1795</td><td>281,427,307</td></tr>
<tr><td>Bezout</td><td>1797</td><td>23,456,789,234,565,456</td></tr>
</table>

A perhaps amazing observation is that although it is nearly a thousand years since the appearance of the Hindu-Arabic numerals, we have not even yet decided on a universal method for writing large numbers.

CHAPTER III

NUMERALS FOR FRACTIONS

Natural numbers apparently served the purposes of the world until about the beginning of the historic period.[1] Then, when man discovered that everything could not be represented in terms of whole numbers, he sought ways of conveying the idea of parts of the whole object. Just as numerals for integers varied, so did fractional numerals vary, from country to country, writer to writer, and even from manuscript to manuscript by the same author. This use of fractions goes back to the very early mathematical records.

OLD NUMERALS FOR FRACTIONS

3.1. Babylonian. Sexagesimal fractions are usually accredited to the Babylonians. However, supporting evidence is not so great as in other systems, since early Babylonians, having little access to suitable stone and no papyrus, resorted to clay as a principal writing medium. Cuneiform tablets dating from 200 B.C. seem to reveal operations with sexagesimal fractions which some historians feel resemble

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[1] Smith, op. cit., p. 208.

operations with decimal fractions. Cajori[2] advances a clay tablet described by A. Ungnad in support of Babylonian use of sexagesimal fractions. In it the diagonal of a rectangle with sides ten and forty units is computed by approximations. Cajori states, "These computations are difficult to explain, except on the assumption that they involve sexagesimal fractions."[3] Except in very simple cases, the idea of a fraction with numerator greater than unity probably arose in Babylon.[4] The cuneiform records which seem to show sexagesimal and unit fractions also include symbols for 2/3, 2/18, 4/18, 5/6, and other cases of a like degree of difficulty.[5] However, an elaborate treatment of the subject is not given.

3.2. Egyptian. Ancient Egyptians favored unit fractions, 1/n, based on the nth part of the whole. In hieroglyphic writing, the symbol ⬭ over an integer indicated a unit fraction with the integer as the denominator. The fractions one-half and two-thirds were exceptions to this mode of writing, as they were used frequently. The

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[2] Cajori, A History of Mathematical Notations, op. cit., p. 10.

[3] Ibid.

[4] Smith, op. cit., p. 213.

[5] Ibid.

hieroglyphs and represented one-half at different times; was a more common of the varying hieroglyphic forms for two-thirds.[6] The first important treatment of fractions as such was found in the Rhind Papyrus (c. 1550 B.C.).[7] Hieratic writing, as exemplified by Ahmes in the Rhind Papyrus, employed the same hieroglyphic principle for writing fractions, with a dot replacing the symbol . Ahmes' numerals deviated somewhat from other hieratic writing and he added a few symbols of his own. This deviation was common. He used special numerals for one-third and one-fourth, in addition to the and for one-half and two-thirds, respectively.[8] An accepted hieratic symbol for one-fourth was the X . Cajori felt it might have signified a part obtainable from two sections of a body through the center.[9] The following is a table of some Egyptian symbolism for simple fractions.[10]

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[6] Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 14.

[7] Smith, *op. cit.*, p. 214.

[8] Cajori, *A History of Mathematical Notations*, *loc. cit.*

[9] *Ibid.*, p. 13.

[10] *Ibid.*

The Rhind Papyrus contained extensive tables of unit fractions. For example, 2/43 was expressed as

$$1/42 + 1/86 + 1/229 + 1/301$$

Why 1/43 + 1/43 was not used is unknown. Further, a table of sums of unit fractions extended to 1/101 + 1/101.

3.3. Greek. By using submultiples, the Greeks avoided the difficulty of computing with fractions for a period of time.[11] However, as the need for a fraction symbolism became more apparent, they, too, developed a system to serve their purpose. Greek writers considered fractions

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[11] Smith, loc. cit.

as the ratio of two numbers. Ancient Greek Attic numerals contained the signs $|$, $\sqsubset$, T , and X for the fractions 1/6, 1/12, 1/24, and 1/48.[12] While late Greek writers often expressed fractional values in words, they would also denote a fraction by first writing the numerator marked with an accent, then the denominator marked with two accents, and written twice. For example, three-fourths might be written $\gamma' \, \delta'' \, \delta''$. This, however, was not the only fractional notation. The sign $\sim$ sometimes took the place of the accent. Aristarchus wrote the word numeral for the numerator of a fraction and the figure numeral for the denominator, as in "ten 71sts".[13] Archimedes and Diophantus, among other Greek writers, placed the denominator in the position of the modern exponent.[14] In some early manuscripts, Diophantus placed the denominator above the numerator, as did Heron.[15] The one idea upon which Greek writers did seem to agree, even as late as the Middle Ages, was their preference for unit fractions. Unit fraction numerals omitted the accent for the numerator, indicated the denominator only once,

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12 Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 22.

13 Smith, *op. cit.*, p. 24.

14 Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 26.

15 Smith, *loc. cit.*

and generally marked the denominator by a double accent.[16] Other denominator markings included the single accent, and single accent modifications in the form of $\wedge$ and χ .[17] Again, as with the Egyptians, one-half had symbols of its own, h , $\llcorner$, h' , $\llcorner'$, c'' , c , $\angle'$, and S ; two-thirds was represented by a symbol similar to the small omega.[18] Rather than use a fraction other than a unit fraction, the Greeks, like the Egyptians, expressed it as the sum of unit fractions. A Greek papyrus from Egypt, supposedly intermediate between the Rhind and Akhmim gives a table for expressing ordinary fractions as the sum of unit fractions.[19] Here, no distinction is made between integers and the corresponding unit fractions.

3.4. Roman. The most popular Roman method of avoiding common fractions was the use of fractional sub-multiples. To avoid using five-twelfths of a unit of measure, these ancients invented a unit one-twelfth as large and used five of these.[20]

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[16] Cajori, A History of Mathematical Notations, op. cit., p. 27.

[17] Ibid., p. 74.

[18] Ibid., p. 27. Cf. Smith, op. cit., p. 214.

[19] Cajori, A History of Mathematical Notations, op. cit., pp. 27-28.

[20] J. Houston Banks, Elements of Mathematics (Boston: Allyn and Bacon, Inc., 1956), p. 153.

Historians know nothing definite regarding the time, place, or manner of origin of Roman fractions, which were, to a great extent, duodecimal. Each duodecimal subdivision had a name and symbol, even though not all names and signs were used to the same degree. One-half, for instance, ordinarily appeared as S, but sometimes during the Middle Ages took a form almost identical with one.[21]

3.5. Chinese. The Chinese apparently made use of fractions of considerable difficulty at a very early date.[22] These fractions were not stated in numerical symbols, but given in words. The Chou-pei of 1105 B.C. or earlier states problems involving such numbers as $247\frac{933}{1460}$.[23] As in all earlier civilizations, the unit fraction played an important role in Chinese mathematics.

HINDU-ARABIC FRACTIONS

It is probable that the present method of writing common fractions is due essentially to the Hindus.[24] Hindu writers usually wrote the denominator of the fraction beneath

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[21] Cajori, A History of Mathematical Notations, op. cit., p. 311.

[22] Smith, op. cit., p. 215.

[23] Ibid.

[24] Ibid.

the numerator, but without a separating line. An arithmetic written on leaves of birch-bark, the Bakshali M.S.,[25] used a notation similar to that of Brahmagupta and Bhaskara. In it, a whole number was indicated by placing the numeral for one beneath the other integer numeral; for example, $\begin{matrix}3\\1\end{matrix}$ was three. Other numerals beneath one another indicated fractions; $\begin{matrix}1\\3\end{matrix}$ was one-third. Three numerals under one another indicated mixed numbers; $\begin{matrix}1\\1\\3\end{matrix}$ meant one plus one-third, or one and one-third. However, a plus behind the lowest numeral indicated that the fraction was to be subtracted from the integer. Therefore, $\begin{matrix}1\\1\\3+\end{matrix}$ meant one minus one-third, or two-thirds.[26] The arithmetic of al-Khowarizmi was elaborated in a tract of Alnasavi (1030 A.D.). Again, the fractional part of a mixed number appeared below the integral part; further, Alnasavi wrote a zero when there was no integral part. For example, one-eleventh was written $\begin{matrix}0\\1\\11\end{matrix}$.[27]

3.6. *The fractional line.* The fractional line or bar was probably first used around the twelfth century. Greeks or Romans never employed the fractional line in the manner that it is currently used to indicate a fraction.[28]

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25 Cajori, *A History of Mathematical Notations*, *op. cit.*, p. 77.

26 *Ibid.*

27 *Ibid.*, p. 310.

28 Smith, *op. cit.*, p. 217.

During the Renaissance, when Arabic and classical forms mingled, Roman numerals occasionally appeared in cases such as $\frac{I\ X}{X\ I}$, as did the Greek numerals, but this was not a common practice.[29] There is confusion concerning the date of Leonardo of Pisa, an Italian, but he is believed to be of a later period than the Arabic author, al-Hassar, who used the fractional line in his writings. If this is the case, the first appearance of a fractional line is in al-Hassar's direction, "Write the denominators below a line and over each of them the parts belonging to it."[30] He wrote "three-fifths and a third of a fifth" in this manner $\frac{3\ \ 1}{5\ \ 3}$.[31] Leonardo of Pisa, probably influenced by Arabic authors, used the fractional line with regularity. He invariably wrote fractions before the integers in mixed numbers, theoretically following Arabic script, which proceeded from right to left.[32] In his Liber abbaci (1202) he wrote, as translated by Cajori,[33]

> When above any number a line is drawn, and above that is written any other number, the superior number stands for the part or parts of the inferior number. The inferior is called the denominator, the superior the numerator.

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29 Ibid.

30 Cajori, A History of Mathematical Notations, op. cit., p. 268.

31 Ibid.

32 Ibid.

33 Ibid.

This is not to infer that the fractional line was an accepted symbol at that time. The fractional line was not used in a twelfth-century Murich manuscript, the thirteenth-century works of Jordanus Nemorarues, or the Bamberger arithmetic of 1483.[34] The notation $\overline{3}\ \overline{5}$ stood for three-fifths and $\overline{4}\ \overline{7}$ for four-sevenths in a fourteenth century manuscript.[35] Further, __Gernardus algorithmis demonstratus__, edited by Schones in 1534 represents fractions in the form ab (a numerator, b denominator).[36] Among the interesting early fractional forms were those in Ciacchi's __Regole generali d'abbaco__ (Florence, 1675), where every fraction in the book was set up like $\frac{2}{3}$, for 2/3.[37] The fractional line came into general use during the sixteenth century; however, omissions of it occurred as late as the seventeenth century.

When used, the fractional line did not necessarily standardize fractional representation. A translation of Abu Kamil contained the fractional line, but the form would be unfamiliar to many mathematicians today. The continued fraction $\frac{1}{9}\frac{1}{9}$ stood for 1/9 plus 1/81.[38] The notations

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[34] __Ibid__., p. 310.

[35] __Ibid__.

[36] __Ibid__.

[37] Smith, __op__. __cit__., p. 216.

[38] Cajori, __A History of Mathematical Notations__, __op__. __cit__., p. 311.

$\frac{2}{9}\,\frac{1}{2}$ and $\frac{0}{9}\,\frac{1}{2}$ represented one-eighteenth.[39] Leonardo of Pisa represented 5/64 thusly: $\frac{5}{8}\,\frac{2}{8}$.[40]

3.7. Special forms. As was common in earlier mathematics, often-used fractions took on special forms. One-half was represented as a line between two points, $\div$, a small cross to the right of the number in the place of today's exponent, $^{+}$, or $1/1$ in some fifteenth-century Arabic numerals.[41] In England, $\sim$ stood for one-half, $\cdot$ for one-fourth, and $\dot{\sim}$ for three-fourths around the time of Recorde (c. 1542).[42] Sixteenth and seventeenth century English archives gave one-half in the form $\frac{1}{\sim}$.[43] The symbol $\underline{o}$ represented one-half in the earliest arithmetic printed in America (Mexico, 1623) and was found along with the form $\overset{o}{m}$ in Arithmetica practica (1784) by the Spaniard Juan Perez de Moya.[44]

3.8. The solidus. The fractional line, when finally accepted, was objected to by printers as it required three

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39 Ibid.

40 Ibid.

41 Ibid.

42 Ibid., p. 312.

43 Ibid.

44 Ibid.

terraces of type and the solidus, as in a/b, was introduced as a typographical improvement. In the Gazetas de Mexico (1784), Manuel Valdes wrote fractions with a curved line, $\int$, similar to the sign of integration, separating the numerator and denominator.[45] Another arithmetic written in 1843 featured the curved line in fractions.[46] While DeMorgan[47] recommended the solidus in 1843 in an article published in the Encyclopedia Metropolitana, he used a;b in his subsequent works. Leibniz frequently used the colon, :, in writing fractions; while some authors felt that lifting the pen from the paper three times in the process of writing $\div$ was too cumbersome, others used this symbol for designating fractions.[48] Along with the solidus, one author proposed that the terms immediately preceding and following be welded into one, with a period arresting the welding action.[49] Following this system, which was adopted in England in 1915, the expression a/bc.d would mean $\frac{a}{bc}d$.

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[45] Ibid., p. 313.

[46] Ibid.

[47] Ibid.

[48] Ibid.

[49] Ibid.

DECIMAL FRACTIONS

Decimal notation is a comparatively recent development in mathematics. It had been in use for thousands of years before man realized that its usefulness and simplicity could be increased by the adoption of the principle of position.[50] It is difficult to say where decimal fractions were first used systematically. One author gives the priority to the Chinese, who used the decimal place value notation in the fourteenth century B.C. and decimal fractions in meterology as early as the third century A.D.[51] Contact with the Orient may or may not have instigated decimal notation in Europe. In any case, European computers began to use the Hindu-Arabic numeral system during the fifteenth and sixteenth centuries with ever-increasing efficiency. The Nuremberg astronomer and mathematician, Regiomontanus, used the decimal system for computation of trigonometric tables, but these tables contained no decimal fractions in the strict sense of the word. Early notations which one might be tempted to look upon as

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[50] Cajori, A History of Elementary Mathematics, op. cit., p. 11.

[51] D. J. Struik, "Simon Stevin and the Decimal Fractions," The Mathematics Teacher, LII (October, 1959), p. 474.

decimal notations appear in works whose author had no real comprehension of decimal fractions and their importance.[52] One mathematician, Bartholomaeus Pitiscus, made extended use of decimal fractions but failed to use the dot as the separatrix between units and tenths.[53] Pellos (1492) used a decimal point where others had used a bar, but he did not develop the idea of a decimal fraction.[54] Rudolff (1530) worked intelligently with decimal fractions, using a vertical stroke for a separatrix, but did not write upon the theory, as he failed to appreciate the importance and generality of his procedure.[55] The invention of decimal fractions is usually ascribed to Simon Stevin, a Belgian engineer and mathematician. Stevin wrote upon decimal fraction theory, but used clumsy circle notation. The unit was designated by zero, the tenth by a one, and so on. For inexperienced people, the notation may have been advantageous, since it allowed intermediate steps, as seen by 7⓪5①8② plus 4⓪7①5②. This could be written first as 11⓪12①13②, then 11⓪13①3②, and finally 12⓪3①3②. Zero as a placeholder

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[52] Cajori, <u>A History of Mathematical Notations</u>, <u>op</u>. <u>cit</u>., p. 315.

[53] <u>Ibid</u>., p. 322.

[54] Smith, <u>op</u>. <u>cit</u>., p. 247.

[55] Cajori, <u>A History of Mathematical Notations</u>, <u>loc</u>. <u>cit</u>.

was unnecessary, as 3③8⑤ meant .00308. Several writers around 1600 tried to improve the symbolism. Bürgi used a comma for the decimal point and Vieta used a vertical stroke in addition to smaller numerals for the fractional part. Both comprehended the nature and advantages of decimal fractions. Clavious, in his *Astrolabe* published in 1593, used a dot in the position of a decimal separatrix.[56] John Napier was another early mathematician who used decimal notation. In his *Constructio* published in 1619 he says, "Whatever is written after the period is a fraction."[57] The use of decimal fractions by the afore-mentioned mathematicians did not mean that notation and use became standardized. A table follows of various seventeenth century decimal notations.[58]

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[56] *Ibid*.

[57] *Ibid*., p. 324.

[58] *Ibid*., pp. 325-326. Cf. Smith, *op. cit*., p. 246.

Author	Date	Notation
Johnson	1623	$3\,|\,\overset{1}{2}\overset{2}{2}\overset{3}{9}\overset{4}{1}\overset{5}{6}$
Briggs	1624	$5^{\underline{9321}|}$
Gerard	1629	347
Stegman	1630	39(063
Oughtred	1631	$2\underline{|5}$
Kersey	1650	.0025
Jager	1651	$16\ \dot{7}24\dot{9}$
Balam	1653	3:04
Capra	1655	198.½
Schooten	1657	17579625 ③
Schooten	1657	58,5
Foster	1659	$31.\overline{\underline{008}}$
Tacquet	1665	$25.\overset{i}{8}\overset{ii}{0}\overset{iii}{0}\overset{iv}{7}\overset{v}{9}$
Caramuel	1670	22=3
Casati	1685	$\begin{array}{c}438\\100000\end{array}$
Prestet	1689	$27209779\overset{vi}{2}$
Buetel	1690	$645.\frac{879}{1000}$
Molyneux	1692	30˛24
Mercator	1668	12[345

During the eighteenth century a semblance of order replaced the chaos in decimal fraction notation. Of course, many devious notations were still present, as 0;9985 for

.9985; 35 345 for 35.345; 732, 569 for 732.569; $4.\overset{\text{I}}{2}\ \overset{\text{IV}}{5}$ for 4.2005; and (4)2677 for .00002677.[59] The main trials of strength for separatrices were, however, between the comma and the dot. There were complicated on the European continent by the fact that Leibniz had proposed the dot as a multiplication symbol, a proposal which met with favorable reception. English mathematicians preferred the $\times$ as a multiplication symbol, however, and the dot was used as a separatrix. The comma as a separatrix had the lead over the dot in Germany, France, and Spain, while the most determined continental stand in favor of the dot was made in Belgium and Italy.[60]

The nineteenth century found the dot serving in England in a double capacity, as both the decimal separatrix and multiplication symbol. The two dots introduced no confusion, as a raised dot indicated a decimal and a dot at the bottom of the figures indicated multiplication. However, the use of the comma prevailed; it was usually, but not always, placed high. The elevated and inverted comma, ‘, never became popular, yet was used as late as 1924. Spain and the Spanish-American countries frequently used the elevated but not inverted comma,'. Another Spanish separatrix was the

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[59] Cajori, A History of Mathematical Notations, op. cit., p. 327.

[60] Ibid., p. 328.

inverted, wedge-shaped comma, in a lower position, $_\wedge$.[61] The people of Scandinavia and Denmark preferred slightly the comma and also printed the decimal part of a number in smaller type than the integral part, in this manner, $2,_5$ or $2._5$.[62] The decimal point separatrix has always been more popular than the comma in the United States, but the comma was used quite extensively between 1750 and 1850. Since about 1850, the dot, in either elevated or period position, has been used almost exclusively.[63] For a while the dot in America had a double meaning, but when the dot became frequently used in multiplication, the need for a distinction arose. About 1880,[64] the present American distinction was thoroughly established when the dot for multiplication was elevated to a central position and the decimal separatrix was lowered to period position. The present English notation was established with the dot positions in exactly opposite roles--that is, the period position indicated multiplication and the central position indicated a decimal separatrix.

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61 Ibid., p. 331.

62 Ibid.

63 Ibid., p. 332.

64 Ibid.

3.9. <u>Repeating decimals</u>. John Marsh, in 1742, was perhaps the earliest writer to use a special notation for the designation of repeating decimals. His notation distinguishes a repeating decimal by "placing a Period over the first Figure, or over the first and last Figures of the given Repetend".[65] Other early notations of repeating decimals include .6 for .666... or .642 for .642642642..., and .2 for .222... or 2.4 18 for 2.4181818...[66]

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[65] <u>Ibid</u>., p. 325.

[66] <u>Ibid</u>.

CHAPTER IV

SUMMARY

Recording the corresponding number of strokes or bars was apparently one of the earliest representations of small numbers from one to five. Larger numbers called for new devices, such as symbols to be associated with the primitive strokes by means of the additive, multiplicative, or subtractive principle, or perhaps by placing primitive symbols in different positions to represent higher values. The diversity among old numerals indicated that each nation of the ancient world probably invented its own numeral system. After alphabets were introduced and a fixed letter sequence established, letter numerals were written by the Syrians, Greeks, Hebrews, and early Arabs. Only primitive powers of invention were required for these alphabetic numeral systems, which gave little aid in computation processes and made heavy demands on the memory. Inherent in the Babylonian, Maya, and Hindu-Arabic numeral systems was the important principle of place value, which utilized relative positions of the primitive symbols in representing large numbers. These systems adapted easily to the processes of computation, even though present records do not give sufficient evidence to indicate in what manner the Maya or

Babylonians carried out computations with their system. Despite intensive study, little is really known about the origin of the Hindu-Arabic notation.

In the study of numeral systems, authorities often categorize them as ciphered, positional, simple grouping, or multiplicative.[67] Ciphered numeral systems include the Egyptian hieratic and demotic, Hebrew, Hindu Brahmi, Syrian, Ionic Greek, and early Arabian. The Egyptian hieroglyphic, Greek Attic, and Roman numerals are simple grouping systems. The best representative of a multiplicative numeral system is the traditional Chinese-Japanese system, and of the positional system, our familiar Hindu-Arabic numerals. The differences in these types of systems, as well as a summary of the numeral systems in this thesis are illustrated in the tables that follow.

One last interesting note is that no uniformity in numeral shapes has been reached, even in the twentieth century.

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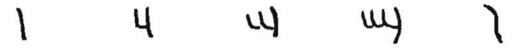
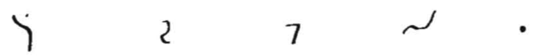
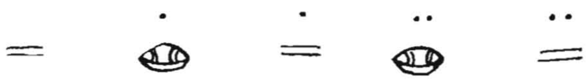
67 Eves, _op_. _cit_., pp. 10-14.

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	1	2	3	4	5
Babylonian	𒁹	𒈫	𒐈	𒐉	𒐊
Egyptian Hieroglyphic	|	||	|||	||||	|||||
Hieratic					
Demotic					
Coptic	$\overline{Ⲁ}$	$\overline{Ⲃ}$	$\overline{Ⲅ}$	$\overline{Ⲇ}$	$\overline{Ⲉ}$
Phoenician	|	||	|||	||||	|||||
Hebrew	א	ב	ג	ד	ה
Greek Attic	Ι	ΙΙ	ΙΙΙ	ΙΙΙΙ	Γ
Ionic	α	β	γ	δ	ε
Gothic	𐌰	𐌱	𐌲	𐌳	𐌴
Roman	I	II	III	IV	V
Aztec	.	..	...	....	.....
Maya	.	..	...	....	—

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	1	2	3	4	5
Chinese					
Scientific	I	II	III	IIII	IIIII
Traditional	一	二	三	四	五
Mercantile	〡	〢	〣	〤	〥
Japanese	一	二	三	四	五
Early Hindu					
150 A.D.	𑁒	𑁓	𑁔	𑁕	𑁖
11th century	१	२	३	४	५
Arabic					
Early Arabic	ا	ب	ج	د	ه
Modern Arabic	١	٢	٣	٤	٥
Hindu-Arabic (1442 A.D.)	1	2	3	4	5

<u>Numeral System</u>	<u>Numerals for Integers</u>				
Modern Hindu-Arabic	6	7	8	9	0
Babylonian					
Egyptian Hieroglyphic					
Hieratic					
Demotic					
Coptic					
Phoenician					
Hebrew	ו	ז	ח	ט	
Greek Attic					
Ionic	ϛ	ζ	η	θ	
Gothic					
Roman	VI	VII	VIII	IX	
Aztec					
Maya					

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	6	7	8	9	0
Chinese					
Scientific	𝍥	𝍦	𝍧	𝍨	〇
Traditional	六	七	八	九	
Mercantile	〦	〧	〨	〩	
Japanese	六	七	八	九	
Early Hindu 150 A.D.					
11th century					
Arabic					
Early Arabic					
Modern Arabic	٦	٧	٨	٩	٠
Hindu-Arabic (1442 A.D.)					

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	10	20	30	40	50
Babylonian					
Egyptian Hieroglyphic					
Hieratic					
Demotic					
Coptic					
Phoenician					
Hebrew	י	כ	ל	מ	נ
Greek Attic	Δ	ΔΔ	ΔΔΔ	ΔΔΔΔ	𐅄
Ionic	ι	κ	λ	μ	ν
Gothic	𐌹	𐌺	𐌻	𐌼	𐌽
Roman	X	XX	XXX	XL	L
Aztec					
Maya					

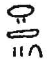
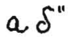
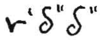
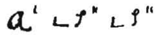
Numeral System	Numerals for Integers				
Modern Hindu-Arabic	10	20	30	40	50
Chinese					
Scientific	𝍩	𝍪	𝍫	𝍬	𝍭
Traditional	十	二十	三十	四十	五十
Mercantile	〸	〢〸	〣〸	〤〸	〥〸
Japanese	十	二十	三十	四十	五十
Early Hindu 150 A.D.	𑁛	𑁜	𑁝	𑁞	𑁟
11th century	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Arabic					
Early Arabic	ى	ك	ل	م	ن
Modern Arabic	١٠	٢٠	٣٠	٤٠	٥٠
Hindu-Arabic (1442 A.D.)	10	20	30	40	50

<u>Numeral System</u>	<u>Numerals for Integers</u>				
Modern Hindu-Arabic	60	70	80	90	100
Babylonian	𒌋𒌋𒌋𒌋𒌋𒌋	𒌋𒌋𒌋𒌋𒌋𒌋𒌋	𒌋𒌋𒌋𒌋𒌋𒌋𒌋𒌋	𒌋𒌋𒌋𒌋𒌋𒌋𒌋𒌋𒌋	𒈨
Egyptian Hieroglyphic	𓎆𓎆𓎆𓎆𓎆𓎆	𓎆𓎆𓎆𓎆𓎆𓎆𓎆	𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆	𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓎆	𓍢
Hieratic					
Demotic					
Coptic	$\overline{ξ}$	$\overline{ο}$	$\overline{π}$	$\overline{ϥ}$	
Phoenician					|<|
Hebrew	ס	ע	פ	צ	ק
Greek Attic	𐅄Δ	𐅄ΔΔ	𐅄ΔΔΔ	𐅄ΔΔΔΔ	Η
Greek Ionic	ξ	ο	π	ϙ	ρ
Gothic	𐌾	𐌿	𐍀	𐍁	𐍂
Roman	LX	LXX	LXXX	XC	C
Aztec					
Maya					

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	60	70	80	90	100
Chinese					
Scientific	𝍮	𝍯	𝍰	𝍱	|
Traditional	六十	七十	八十	九十	百
Mercantile	〦十	〧十	〨十	〩十	θ
Japanese	六十	七十	八十	九十	百
Early Hindu 150 A.D.		x	ω	⊕	
11th century	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
Arabic					
Early Arabic	س	ع	ف	ص	ق
Modern Arabic	٦٠	٧٠	٨٠	٩٠	١٠٠
Hindu-Arabic (1442 A.D.)	6o	7o	8o	9o	1oo

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	300	500	900	1000	1961
Babylonian					
Egyptian Hieroglyphic					
Hieratic					
Demotic					
Coptic					
Phoenician					
Hebrew	ש	ך	ץ	א׳	א׳ץסא
Greek Attic	HHH	𐅅	𐅅HHHH	Χ	Χ𐅅HHHH𐅄ΔI
Ionic	Τ	φ	ϡ	͵α	͵αϡξα
Gothic	𐍄	𐍆	𐍊		
Roman	CCC	IↃ	CM	M	MCMLXI
Aztec					
Maya					

Numeral System	Numerals for Integers				
Modern Hindu-Arabic	300	500	900	1000	1961
Chinese					
Scientific	|||	|||||	𝍨	—	— 𝍨 ⊥	
Traditional	三百	五百	九百	一千	一千九百六十一
Mercantile	〣百	〤百	〩百	〡千	〡千〩百〦十〡
Japanese	三百	五百	九百	千	千九百六十一
Early Hindu 150 A.D.					
11th century	३००	५००	९००	१०००	१९६१
Arabic					
Early Arabic	ش	ن	ظ	غ	غظسا
Modern Arabic	٣٠٠	٥٠٠	٩٠٠	١٠٠٠	١٩٦١
Hindu-Arabic (1442 A.D.)	300	500	900	1000	1961

Numeral System	Numerals for Fractions			
Modern Hindu-Arabic	1/4	1/2	3/4	1/17
Egyptian Hieroglyphic				
Hieratic				
Demotic				
Greek				
Roman	=-	S	S=-	

Note: Blank spaces in the preceding tables, with the exception of zero, indicate that the particular numeral was not in sources available to the writer. In the case of zero, the absence of a numeral indicates that the system had no numeral for zero.

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