INNOVATIONS IN MATHEMATICS
FOR THE NINTH GRADE

A Thesis

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by

Roy A. Englert

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Approved for the Major Department

John M. Burger

Approved for the Graduate Council

Laurence C. Boylan

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Roy A. Englert

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CHAPTER I

THE PROBLEM

1.1 Introduction. The teacher of mathematics in a secondary school need not search beyond the realm of his own experience to recognize the existence of a situation probably unique to American education. All children who have a potential for some kind of learning will enter the mathematics classroom for a period of at least one year. The mathematics teacher is therefore faced with undertaking to provide adequate instruction and materials commensurate to the wide range of intellectual potential of his students. As a result the mathematics curriculum which has evolved in most secondary schools consists of either a sequence of courses beginning with algebra in the ninth grade or, as an alternative, a more general course in mathematics, usually terminal. In either case all the concepts, principles, and procedures of ninth grade mathematics carry over into the work of later years and in fact form the very foundation of that work. Thus, the ninth grade is a most critical grade so far as mathematics is concerned. It is in the ninth grade that the serious study of mathematics begins for most students and, unfortunately, it is with this grade that it ends for many of them. Here the student's interest is either kindled and nourished or allowed to die.

1.2 Statement of the problem. To create and maintain interest is one of the most important tasks of the teacher of secondary-school mathematics. It is also one of the most difficult problems the teacher encounters. In general the mathematics textbook is not organized so as to initiate the recognized power of sheer intellectual curiosity as a motive for the highest type of work in mathematics. Thus, the responsibility of creating and perpetuating interest in mathematics lies squarely with the teacher. The purpose of this thesis is to provide the ninth grade teacher of mathematics with selected materials, situations, and problems designed to stimulate the student's interest in mathematics through a challenge to his curiosity.

1.3 Background of the problem. As a rule, students become interested in things which are new or exciting, in things for which they can perceive practical values for themselves or applications to situations in which they are already interested, and in things which involve puzzle elements or elements of mystery. Other things being equal, the possession of a background of related information tends to intensify interest in new work, but this is neither a necessary condition nor a sufficient guarantee for the awakening of interest. Novelty is sometimes more compelling than familiarity. The elements of novelty, of usefulness, and of sheer intellectual curiosity are the primary stimuli for the awakening of interest.[1]

Students tend to remain interested in those things which they can do most successfully and which they understand most completely. Inability to understand or to perform satisfactorily usually creates a condition of listlessness, inattention, and general loss of interest which often ripens into open disaffection. This is not to say that the work should be made easy and should never present difficulties to the students. Nothing could cause interest to lag more quickly than this, and nothing could be more undesirable from the educational point of view. The work must present a continual challenge, but it must not be merely drudgery at tasks devoid of meaning or unreasonably difficult. Consequently, it is important that work in mathematics be so organized and conducted as to emphasize the values and the inherent intellectual challenge of the subject. Equally important, understanding and a reasonable degree of competence should be ensured by keeping the subject matter and the activities at a level of difficulty appropriate to the intellectual maturity of the students.

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[1]Charles H. Butler and F. Lynwood Wren, _The Teaching of Secondary Mathematics_ (second edition; New York: McGraw-Hill Book Company, Inc., 1951), p. 126.

Within these conditions are to be found the motives basic to hard and effective work in mathematics.[2]

Genuine interest in mathematics probably depends basically upon the problem-solving aspect of the subject.[3] Mathematical situations and problems need not lack essential curiosity provoking possibilities. Puzzle problems, often popular with the layman, may well have a mathematical basis somewhat obscured, perhaps, by a screen of mysticism which only serves to stimulate curiosity. People are interested in seeing how numbers behave, and one aspect of mathematics is the science of the behavior of numbers. Many problems in mathematics are often criticized as being unreal or having no genuine application to life situations. Experience in teaching mathematics, however, will convince the most skeptical critic that problems do not need to represent "real" situations in order to be interesting to students. As a matter of fact, it is quite possible that the presence of the mystery element in problems is often a greater stimulus to interest than those elements of so-called "reality"

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[2] Ibid., pp. 126-27.

[3] Maurice L. Hartung, "Motivation for Education in Mathematics," The Learning of Mathematics, Its Theory and Practice, Twenty-first Yearbook of the National Council of Teachers of Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1953), p. 51.

which are usually incorporated in the problems.[4] It should also be said that even the lowliest and most unimportant looking problem may possibly lead to general and important considerations. Some apparently simple problems have led to mathematics of such difficulty that they are still unsolved. In any case, much energy and ingenuity has been expended by professional mathematicians on what may be called mathematical amusements or recreations.[5]

The student must be given the opportunity to develop a taste for mathematics. The opportunity can be lost even if the student has some natural talent for mathematics because he, as everyone else, must discover his talents and tastes. He may find that a mathematics problem is as much fun as a crossword puzzle and that vigorous mental effort is not all drudgery. Having tasted pleasure in mathematics he will not forget it easily and there is a good chance that mathematics will come to mean something to him.[6]

Mathematics, of course, must not be regarded as nothing but a collection of tricks or frivolous and trivial recreations and pastimes. Nevertheless, if mathematics is

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[4]Butler and Wren, op. cit., p. 128.

[5]Moses Richardson, Fundamentals of Mathematics (rev. ed.; New York: The MacMillan Company, 1958), p. 223.

[6]G. Polya, How to Solve It (Garden City, New York: Doubleday and Company, Inc., 1957), p. VI.

properly taught it presents the student with an abundance of problems designed expressly to initiate curiosity and interest. With each successful solution he feels a sense of satisfaction and as a result he seeks more experiences of the same kind. As the student grows in mathematical maturity he obtains satisfaction also from realization of the power of his methods. This behavior is relevant to interest, however, because it leads the student to seek more experiences with mathematics, to discuss it favorably with other people, and to value it for what it does for him personally.[7]

Students can be helped to creativity and problem solving ability only if their teachers repeatedly lead them to and through problem solving situations and encourage them to strike out mentally for themselves into areas new to them.[8]

Thus, a teacher of mathematics has a great opportunity. If he fills his alloted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. If he sets problems before them that challenge

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[7]Hartung, loc. cit.

[8]The Committee, "Preface," The Growth of Mathematical Ideas; Grades K-12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (Washington, D. C.: National Council of Teachers of Mathematics, 1959.), p. V.

the curiosity of his students, and helps them solve their problems with stimulating questions, he may give them a basis for some appreciation of mathematics, and some means of independent thinking. Such problems should, of course, be proportionate to their knowledge.[9]

Obviously there must be system and organization in mathematics. Arithmetic and algebra cannot and should not consist entirely of special problems, situations or recreations. Courses in mathematics must be developed in sequential form. Haphazard or piecemeal work will achieve nothing of value. But within the framework of the systematic organization of a course in mathematics at any level of secondary instruction there are many opportunities for motivating the work by deliberate stimulation of the curiosity of the students along the lines indicated. The greater the extent to which this is done, the greater will be the interest, understanding, and diligence with which the students will work and the more meaningful and worth while will the work become to them.[10]

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[9] Polya, _op_. _cit_., p. V.

[10] Butler and Wren, _op_. _cit_., p. 128.

1.4 Limitations of the study. The first limitation in the selection of materials for this thesis has been based on the author's evaluation of the material in terms of necessary related mathematical knowledge required of the student on the ninth grade level. Further, the materials selected are expected to be commensurate to the maturity level of the student and stimulating to his intellectual curiosity. A wealth of books is available which treat mathematics in the form of a recreational activity. It is from such books that the material of this thesis has been selected and organized subject to the limitations stated above.

An assumption that the material selected should prove interesting to the student is based on the fact that such materials do appear in books which purport to lend fascination and intrigue to mathematics, and the fact that such books have gained acceptance by mathematicians, educators, and the general public.

In the organization of the thesis the problems and situations presented are merely representative of a kind; they are not intended to indicate the total scope of the subject under consideration.

Though the primary purpose here is to organize and present a mathematical approach to situations which involve the curious and the unusual, it does not mean that mathematical principles will be sacrificed. On the contrary, the selection of material has been based on the premise that either new mathematical principles will be discovered or principles already familiar to the student will be strengthened.

1.5 Sources of material. The discussion of the background of the problem in Chapter I is based on the writing of men in the fields of education and mathematics but who are not primarily concerned with the area of recreational mathematics.

The material in Chapters II through VII has been selected from numerous books in the area of mathematical recreations. Some of the material has also been collected from such other books as Fundamental Mathematics by Harkin, Introduction to Finite Mathematics by Kemeny, The Education of T. C. Mits by Lieber, Making Sure of Arithmetic by Morton, and Algebra--Book Two by Welchons and Krickenberger.

1.6 Organization of the thesis. Chapter II presents the positional notation concept of the familiar decimal system of numeration and demonstrates the identical structure of other systems using different bases. Translation from the decimal system of numeration to other systems, and conversely, is presented in considerable detail.

Representative problems are suggested for use in the classroom.

Chapter III presents the operations of addition, subtraction, multiplication, and division in systems of numeration other than the decimal. The processes are treated primarily by example and representative practice problems are included.

In Chapter IV a relation between certain systems of numeration is discussed and accompanied by representative problems.

Chapter V introduces the fraction and "decimal" in other systems of numeration and demonstrates methods for translating from the ten-system to other systems, and conversely.

In Chapter VI several unusual problems involving probability are presented as well as a discussion of tree diagrams.

Chapter VII presents a selection of wide and varied problem situations involving the use of one or more mathematical principles or concepts. The problems were selected because of their unusual character and are not classified as to type or kind.

Chapter VIII is devoted to a summary of the content of the thesis.

1.7 To the teacher. Though much of the content of this thesis is devoted to problem solving situations, Chapters II through V necessarily require a somewhat lengthy development of systems of numeration other than the decimal system. However, the discussion and development of other systems should in itself prove interesting as well as being relative to a problem solving situation. It is the opinion of the author that Chapters II and III should be presented in their entirety to most students whereas Chapters IV and V may well be reserved as optional units for smaller groups and individuals. The stage of instruction at which any material presented here should be introduced shall be left entirely to the discretion of the teacher.

A few words may be in order concerning the presentation of problems. A problem may be presented to the entire class at a time felt to be appropriate by the teacher, or to a smaller group or even a single student. In the case of a group or the entire class being involved, the presentation and discussion of the problem should of course be a part of the lesson plan and sufficient class time must be allotted for this work just as in any other planned lesson. After presentation of a problem there should be ample opportunity for questions and further discussion. Once the teacher is reasonably assured that the students understand the problem it may be well to allow a day or more to pass before pursuing it further. Students should be encouraged to hand in a copy of their solution as soon as possible and asked to indicate the approximate amount of time they spent in finding it. It is important to allow every student as much time outside of school as he cares to take so as to insure as many successes as possible rather than hurry on to other aspects of the problem. The urgency of moving on should not be felt here as it often is by the teacher in his attempt to "cover" a textbook.

The purpose of these problems is clearly to create interest and original thinking and hence the creation of a learning situation. Therefore the teacher must lend encouragement to the student but refrain from describing the solution to the problem. Showing a class or an individual how defeats the purpose and may result in loss of attention as well as interest. Time after time it has been observed that those students who are shown answers either try to memorize the solution or lose interest in it entirely.

Suggestions as to problem solving techniques and methods of attack may be presented by the teacher as well as well chosen directed questions which tend to lead the student into self discovered methods of approach to a problem. A book on the different aspects of problem solving by G. Polya could be very instructive and an invaluable aid to the teacher and student alike in overcoming certain difficulties in the solving of problems.[11]

A trial and error approach to a problem has always been one of the first ways of investigating a new problem. Unfortunately it takes time and often leads to disappointments. Nevertheless, students at the ninth grade level will persist in using this approach even under the guidance of a well trained and qualified mathematics teacher. Persistance, however, is one of the abilities often listed as an aim of mathematics education, hence even a trial and error approach to a problem by a student may give the teacher a measure of the students' persistence.

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[11] G. Polya, How to Solve It (Garden City, New York: Doubleday and Company, Inc., 1957, 253 pp.)

CHAPTER II

SYSTEMS OF NUMERATION

2.1 Introduction. Recreational activities with systems of numeration are not mere puzzles. The mathematical bases for such recreations are deeply rooted in fundamental properties which need be thoroughly understood. However, these fundamental properties are so analogous to the properties of the decimal system of notation that the generalization of the numerical value of the base is a procedure which is usually provided for in the fundamental objectives of mathematical instruction on the secondary school level. Thus, the transition from the base 10 to any other base should not offer many undue difficulties. If there may be a difficulty, it would be associated with the necessity of concentrating one's attention on the fact that a new numerical base is present.[1]

2.2 Base ten and five. Our numeral system is called a decimal system because the base is 10. The word "decimal" derives from the Latin word decem which means "ten."

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[1]Aaron Bakst, "Mathematical Recreations," *The Mathematics Teacher*, March, 1953, 46:pp. 185-87.

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The numeral 527, for example, means

$$500 + 20 + 7$$

or $$5(10)^2 + 2(10) + 7(1).$$

Likewise, the numeral 3809 means

$$3000 + 800 + 0 + 9$$

or $$3(10)^3 + 8(10)^2 + 0(10) + 9(1).$$

If we did not know how to express 3809 in powers of 10, as shown above, we could find how many times 3809 contains 10, 10^2, etc., by dividing. We can do this by dividing 3809 by 10, dividing the quotient by 10, etc., until the quotient is less than 10. Dividing by 10 twice is equivalent to dividing by 10^2, dividing three times is equivalent to dividing by 10^3, etc.

We can use short division, and because it is convenient, note the remainders as in the following example.

$$\begin{array}{r l} 3 & \text{R } 8 \\ 10\overline{)38} & \text{R } 0 \\ 10\overline{)380} & \text{R } 9 \\ 10\overline{)3809} & \end{array}$$

This number contains 10^3 three times, 8 extra 10^2's (shown by R 8), no extra tens, and 9 units as shown by R 9. This can be written as:

$$3(10)^3 + 8(10)^2 + 0(10) + 9.$$

Compare this with the expression originally given for 3809 in terms of powers of 10. Note the pattern established for writing 3809 once the successive short divisions by 10 are performed and the remainders indicated as shown. That is:

$$\begin{array}{ll} 3 & R\ 8 \\ & R\ 0 \\ & R\ 9 \end{array}$$

is 3809.

To generalize this and establish the process, imagine that the decimal representation of the number has five places (the process is identical for larger or smaller numbers) written as

$$A\cdot 10^4 + B\cdot 10^3 + C\cdot 10^2 + D\cdot 10 + E\cdot 1,$$

where A, B, C, D, E represent any of the digits 0, 1, 2,$\cdots$, 9 except that $A \neq 0$ for a five place numeral. Now by successive short divisions by 10, the generalized problem

$$\begin{array}{ll}
A & \\
10\overline{)A\cdot 10 + B} & R\ B \\
10\overline{)A\cdot 10^2 + B\cdot 10 + C} & R\ C \\
10\overline{)A\cdot 10^3 + B\cdot 10^2 + C\cdot 10 + D} & R\ D \\
10\overline{)A\cdot 10^4 + B\cdot 10^3 + C\cdot 10^2 + D\cdot 10 + E\cdot 1} & R\ E
\end{array}$$

shows that, in order, the final quotient, A, and the remainders, read downhill, give the proper decimal representation.[2]

Apparently, the base 10 is used because primitive man counted on his fingers. If primitive man had used the fingers of only one hand in counting, and some groups did,

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[2]Duncan Harkin, Fundamental Mathematics (New York: Prentice-Hall, Inc., 1941), p. 24.

the base would be 5. In that case, the numeral 2431 would mean

$$2(5)^3 + 4(5)^2 + 3(5) + 1.$$

With the base 10 we must have ten symbols. They are, of course, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, with all counting numbers larger than nine being expressed by means of place value as described earlier. If the base were 5, we would need only five symbols, 1, 2, 3, 4, and 0. The other familiar symbols, 5, 6, 7, 8, 9, would not exist; there would be no need for them.

At this point it becomes necessary to adopt a convenient notation in order to distinguish a numeral written in the 10 numeral system from one written in the 5 numeral system. In the above examples 3809, using the base 10 may be written $3809_{(10)}$; and 2431, using the base 5, written as $2431_{(5)}$.

Recall that $2431_{(5)}$ means

$$2(5)^3 + 4(5)^2 + 3(5) + 1.$$

When the arithmetic operations are performed, that is

$$\begin{aligned} 2(5)^3 &+ 4(5)^2 + 3(5) + 1 = \\ 2(125) &+ 4(25) + 3(5) + 1 = \\ 250 &+ 100 + 15 + 1 = 366_{(10)} \end{aligned}$$

we have $366_{(10)}$ being written in the 10 system and must be indicated as such. It is now correct to say that $2431_{(5)} = 366_{(10)}$, or in other words 2431 using the base 5 represents the same counting number as 366 using the base 10.

To further demonstrate the procedure discussed earlier, let us change the decimal numeral $366_{(10)}$ back to the numeral having the base 5. We must find how many times $366_{(10)}$ contains 5, 5^2, etc. We can do this by dividing $366_{(10)}$ by 5, dividing the quotient by 5, etc., until the quotient obtained is less than 5. Dividing by 5 twice is equivalent to dividing by 5^2; dividing by 5 three times is equivalent to dividing by 5^3; etc. We will use short division in the manner described earlier.

$$\begin{array}{rl} 2 & \text{R } 4 \\ 5\overline{)14} & \text{R } 3 \\ 5\overline{)73} & \text{R } 1 \\ 5\overline{)366_{(10)}} & \end{array}$$

This number, $366_{(10)}$, contains 5^3 two times, four extra 5^2's (shown by R 4), three fives (shown by R 3), and one unit as shown by R 1. As a check, we write

$$2431_{(5)} = 2(5)^3 + 4(5)^2 + 3(5) + 1$$
$$= 250_{(10)} + 100_{(10)} + 15_{(10)} + 1 = 366_{(10)}.$$

2.3 Summary. Numbers written with the base 5 use a units column, a fives column, a five-squared column, and so on, instead of a units column, a tens column, a ten-squared column, and so on. Such numerals may be referred to as the 5 scale or quinary system of numeration and identified by using a base subscript in parentheses such as $243_{(5)}$. The numeral $243_{(5)}$ means

$$2(5)^2 + 4(5) + 3(1)$$

and is equivalent to

$$2(25) + 4(5) + 3(1) =$$

$$50_{(10)} + 20_{(10)} + 3_{(10)} = 73_{(10)}.$$

To change a numeral in the 10 scale to its equivalent in the 5 scale, divide the decimal numeral successively by 5, noting the remainders, until the quotient is less than 5. Using the procedure described earlier, write the numeral in the 5 scale as follows.

$$\begin{array}{rl} 2 & \text{R } 4 \\ 5\overline{)14} & \text{R } 3 \\ 5\overline{)73}_{(10)} & \end{array}$$

Thus, $73_{(10)} = 243_{(5)}$.

Table I, will show representative equivalent numerals written in the ten scale and in the five scale.

The symbols 14 and 22, for example, in the last line should be read as one-four and two-two, respectively, and not fourteen and twenty-two, since they mean "one five and four units" or $1 \cdot 5 + 4$, and "two fives and two units" or $2 \cdot 5 + 2$, respectively.

The number written as $32_{(10)}$ in the 10 scale means $3 \cdot 10 + 2$ which is equal to $1(5)^2 + 1(5) + 2$ or $1(25) + 1(5) + 2$ and is therefore written as 112 in the 5 scale.

TABLE I

COMPARISON OF REPRESENTATIVE NUMERALS WRITTEN

IN THE TEN SCALE AND FIVE SCALE

	one	two	three	four	five	six	seven	eight	nine	ten	eleven	twelve	sixteen	thirty-two
In the ten scale	1	2	3	4	5	6	7	8	9	10	11	12	16	32
In the five scale	1	2	3	4	10	11	12	13	14	20	21	22	31	112

Note that in the ten scale the numeral $10_{(10)}$ is used as the base whereas the symbol itself does not occur among the original ten digits.

In the five scale the symbol $10_{(5)}$ should technically be used as the base number but it has been found more convenient and less confusing to use the symbol $5_{(10)}$ as the base. It is true that regardless of the base system used the base numeral itself does not occur among the original set of digits. The names of the numbers and the symbol used for the numbers are merely a matter of language and must not be confused with the matter of number concept.

2.4 Problems. Change the following decimal numerals to numerals having the base five. As a check, change each number back to the numeral having the base ten.

$\text{Decimal}_{(10)}$	$\text{Quinary}_{(5)}$
4	(4)
7	(12)
21	(41)
50	(200)
64	(224)
350	(2400)
512	(4022)
1000	(13,000)
5000	(130,000)
8426	(232,201)

This list of problems may of course be supplemented by as many similar problems as the teacher feels is necessary.

2.5 <u>Bases two through nine</u>. At this point it might be well to ask whether other scales, or numeral bases, can be used to write counting numbers. Other scales could be used equally well and in fact any rational number except one may be used as a base. For the purposes of this chapter, however, we shall deal only with integral number bases of ten or less in order to confine the work to reasonable limits and yet demonstrate the arithmetic principles common to different numeral base systems.

One might, for example, write a number in the 3 scale using only the three digits, 0, 1, 2. Hence twenty-four is written in the 3 scale as $220_{(3)}$ (two-two-zero) since $24_{(10)} = 2 \cdot 3^2 + 2 \cdot 3 + 0 \cdot 1$. There would be no use for the symbols 3, 4, 5, 6, 7, 8, 9 and they would not exist. Thus, $24_{(10)} = 220_{(3)}$.

One should realize now that every point of discussion concerning the 5 scale will now apply regardless of the base number being used. Thus,

$$\begin{array}{rl} 2 & \text{R } 2 \\ 3\overline{)8} & \text{R } 0 \\ 3\overline{)24}_{(10)} & \end{array}$$

and $24_{(10)} = 220_{(3)}$.

If one chooses to use eight as a base number using only the symbols 0, 1, 2, 3, 4, 5, 6, 7 then,

$$\begin{array}{rl} 4 & \text{R } 4 \\ 8\overline{)36} & \text{R } 1 \\ 8\overline{)289}_{(10)} & \end{array}$$

for example, $289_{(10)} = 441_{(8)}$.

That is,

$$
\begin{aligned}
441_{(8)} &= 4(8)^2 + 4(8) + 1(1) \\
&= 4(64) + 4(8) + 1 \\
&= 256 \quad + 32 \quad + 1 \\
&= 289_{(10)}.
\end{aligned}
$$

If the numeral base is two, the system is called "the binary system." This system has been much used in electronic computers although recently the tendency has been to change to the base four and the base eight.[3] The binary system became popular because it easily represented two conditions, such as, a hole appears at a certain place in a punch card or it does not appear, an electrical contact is made or it is not made, etc.

The binary system requires only two symbols--1 and 0. Just as 10 in the decimal system means 1 ten and 0 ones, 10 in the binary system means 1 two and 0 ones. Thus, $10_{(10)} \neq 10_{(2)}$. If the base is five, 10 means 1 five and 0 ones; if the base is eight, 10 means 1 eight and 0 ones; etc.

In Table II is given the first 25 numerals in the decimal system and for sake of comparison the equivalent

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[3]Robert Lee Morton, et. al., Making Sure of Arithmetic, Teachers Edition, (Morristown, New Jersey: Silver Burdett Company, 1958), p. 462.

TABLE II

COMPARISON OF THE FIRST TWENTY FIVE NUMERALS

IN DIFFERENT NUMERATION SYSTEMS

ten scale Decimal	2 scale Binary	3 scale Ternary	5 scale Quinary	8 scale Octonary
1	1	1	1	1
2	10	2	2	2
3	11	10	3	3
4	100	11	4	4
5	101	12	10	5
6	110	20	11	6
7	111	21	12	7
8	1000	22	13	10
9	1001	100	14	11
10	1010	101	20	12
11	1011	102	21	13
12	1100	110	22	14
13	1101	111	23	15
14	1110	112	24	16
15	1111	120	30	17
16	10000	121	31	20
17	10001	122	32	21
18	10010	200	33	22
19	10011	201	34	23
20	10100	202	40	24
21	10101	210	41	25
22	10110	211	42	26
23	10111	212	43	27
24	11000	220	44	30
25	11001	221	100	31

symbols for the base of two, three, five, and eight, respectively--binary, ternary, quinary, and octonary numerals.

The binary numeral 10 means 1(2) + 0(1); 11 means 1(2) + 1(1); 100 means $1(2)^2 + 0(2) + 0(1)$; 111 means $1(2)^2 + 1(2) + 1(1)$; 10101 means $1(2)^4 + 0(2)^3 + 1(2)^2 + 0(2) + 1(1)$; etc.

Now let us change a few decimal numerals to binary numerals by division. Divide by 2 and continue to divide by 2 until the quotient is less than 2.

			1 R 0
	1 R 0	1 R 1	2)2 R 1
1 R 0	2)2 R 0	2)3 R 1	2)5 R 0
2)2 R 1	2)4 R 0	2)7 R 1	2)10 R 0
2)5	2)8	2)15	2)20

These divisions show that $5_{(10)} = 101_{(2)}$; $8_{(10)} = 1000_{(2)}$; $15_{(10)} = 1111_{(2)}$; and $20_{(10)} = 10100_{(2)}$. Each of these numerals has already appeared in Table II.

To reinforce our thinking as to why the last quotient and the remainders, read downhill, give the proper binary representation, we will look again at the device used early in this chapter.

This time imagine that the <u>binary</u> representation of the number has, for example, five places written as

$$A \cdot 2^4 + B \cdot 2^3 + C \cdot 2^2 + D \cdot 2 + E \cdot 1,$$

where A, B, C, D, E represent 1 or 0 according as the corresponding power of 2 is or is not present in the binary representation of the number. Now, by successive halving, the generalized problem

$$\begin{array}{ll}
\underline{A\cdot 1} & R\ B \\
2\,\overline{)A\cdot 2 + B\cdot 1} & R\ C \\
2\,\overline{)A\cdot 2^2 + B\cdot 2 + C\cdot 1} & R\ D \\
2\,\overline{)A\cdot 2^3 + B\cdot 2^2 + C\cdot 2 + D\cdot 1} & R\ E \\
2\,\overline{)A\cdot 2^4 + B\cdot 2^3 + C\cdot 2^2 + D\cdot 2 + E\cdot 1} &
\end{array}$$

shows that the last quotient and the remainders, read downhill, give the proper binary representation.[4] The above verification could be further generalized by allowing the base number to be x, where x is an integer greater than 0.

Large decimal numerals can be changed to other base numerals but where the new base is less than ten, one must expect in general more digits to appear. In particular, changing a relatively large decimal numeral to the binary system will result in many figures in the binary numeral. For example,

$$\begin{array}{rl}
\underline{1} & R\ 0 \\
2\,\overline{)2} & R\ 1 \\
2\,\overline{)5} & R\ 0 \\
2\,\overline{)10} & R\ 0 \\
2\,\overline{)20} & R\ 1 \\
2\,\overline{)41} & R\ 0 \\
2\,\overline{)82} & R\ 0 \\
2\,\overline{)164} & R\ 1 \\
2\,\overline{)329} & R\ 0 \\
2\,\overline{)658} & R\ 1 \\
2\,\overline{)1317} & R\ 1 \\
2\,\overline{)2635} &
\end{array}$$

Thus, $2635_{(10)} = 101{,}001{,}001{,}011_{(2)}$.

---

[4]Hartung, loc. cit.

This numeral means $1(2)^{11} + 0(2)^{10} + 1(2)^9 + 0(2)^8 + 0(2)^7 + 1(2)^6 + 0(2)^5 + 0(2)^4 + 1(2)^3 + 0(2)^2 + 1(2) + 1 = 2048 + 0 + 512 + 0 + 0 + 64 + 0 + 0 + 8 + 0 + 2 + 1 = 2635_{(10)}$.

2.6 Summary. Changing from one scale to another is merely a problem for the packing department. When thirty-eight is written in the ten scale, it is as though thirty-eight objects were packed into 3 boxes of 10 each and 8 boxes of one each ($38 = 3 \cdot 10 + 8$). Changing to the five scale amounts to repacking the thirty-eight objects into one box of 25 ($= 5^2$), 2 boxes of 5 each and 3 boxes 1 each ($123_{(5)} = 1 \cdot 5^2 + 2 \cdot 5 + 3 \cdot 1$). In the ten scale we pack by ones, 10's, 10^2's, 10^3's, and so on. In the five scale, we pack by ones, 5's, 5^2's, 5^3's, and so on, always using the largest box that can be filled.

We are merely discussing different ways of writing the same old numbers. That is, we are discussing different systems of notation or different (written) languages. When we say that the number which is written as 24 in our usual decimal language would be written as 44 in the "5 scale" language, or as 11000 in the "binary" language, we are merely translating from one language to another.

Another point of view that must be reserved for the student with sufficient algebra preparation deserves brief mention. In the usual 10 scale, the symbol $231_{(10)}$ means

$2 \cdot 10^2 + 3 \cdot 10 + 1$, or the value of the polynomial $2x^2 + 3x + 1$ when $x = 10$, or two hundred and thirty-one. In the 5 scale, $231_{(5)}$ means $2 \cdot 5^2 + 3 \cdot 5 + 1$, or the value of the polynomial $2x^2 + 3x + 1$ when $x = 5$, or the number $66_{(10)}$.

To avoid possible confusion as to which language or scale a numeral is written in, we must either state the scale in words or use the subscript in parentheses as has been done so far. If no subscript is used nor any explicit remark made, the 10 scale is always understood.

2.7 Problems.

1. Change each of the following decimal numerals to the 2 scale, 3 scale, 4 scale, 5 scale, 6 scale, 8 scale, and 9 scale respectively. As a check change each scale back to a decimal.

Decimal	Answers			
	2 scale	3 scale	4 scale	5 scale
24	11000	220	120	44
5	101	12	11	10
32	100000	1012	200	112
96	1100000	10120	1200	341
259	100000011	100121	10002	2014

Decimal	Answers		
	6 scale	8 scale	9 scale
24	40	30	26
5	5	5	5
32	52	40	35
96	240	140	116
259	1111	403	317

2. Rewrite the entire statement "2 + 5 = 7" (now written in the 10 scale) to express it in

(a) the 5 scale Ans. $2_{(5)} + 10_{(5)} = 12_{(5)}$

(b) the 2 scale Ans. $10_{(2)} + 101_{(2)} = 111_{(2)}$

(c) the 8 scale Ans. $2_{(8)} + 5_{(8)} = 7_{(8)}$

(d) the 3 scale Ans. $2_{(3)} + 12_{(3)} = 21_{(3)}$.

3. The following numerals are already written in the binary or two scale; rewrite them in the ten scale.

(a) 1110 Ans. 14

(b) 1011 Ans. 11

(c) 11010 Ans. 26

(d) 10010 Ans. 18

(e) 100100 Ans. 36.

4. If the numeral 200 is already written in the 4 scale, rewrite it in

(a) the 8 scale Ans. 40

(b) the 2 scale Ans. 100000

(Hint, see problem 1.)

CHAPTER III

FUNDAMENTAL OPERATIONS OF ARITHMETIC

3.1 Introduction. Previous training makes simple arithmetical operations so mechanical that they must be analyzed closely in order to apply them to systems of numeration other than the decimal.

Consider the opportunities for practice in arithmetic which new systems of numeration offer. For example, the addition of 3163 and 4512, when both are written in systems other than the decimal, requires close attention to the value of the base. Such problems may call forth powers of concentration and analysis as well as develop a deeper understanding of the fundamental operations of arithmetic.[1]

3.2 Number facts. Operations with numbers in the various systems of numeration are simplified by addition and multiplication tables, similar to the decimal-system tables taught in one form or another to all grade school pupils. Following are the tables for addition and multiplication for the numeration systems, base two through base nine.

To find the sum or product of two numbers take one number in the first column and one number in the first row.

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[1]Aaron Bakst, "Mathematical Recreations," The Mathematics Teacher, March, 1953, 46:pp. 185-87.

TABLE III

ADDITION AND MULTIPLICATION FACTS FOR NUMERATION SYSTEMS, BASES TWO THROUGH NINE

Addition — Multiplication

Two-system Tables

+	1
1	10

x	1
1	1

Three-system Tables

+	1	2
1	2	10
2	10	11

x	2
2	11

Four-system Tables

+	1	2	3
1	2	3	10
2	3	10	11
3	10	11	12

x	2	3
2	10	12
3	12	21

Five-system Tables

+	1	2	3	4
1	2	3	4	10
2	3	4	10	11
3	4	10	11	12
4	10	11	12	13

x	2	3	4
2	4	11	13
3	11	14	22
4	13	22	31

TABLE III (continued)

Addition	Multiplication

Six-system Tables

+	1	2	3	4	5
1	2	3	4	5	10
2	3	4	5	10	11
3	4	5	10	11	12
4	5	10	11	12	13
5	10	11	12	13	14

x	2	3	4	5
2	4	10	12	14
3	10	13	20	23
4	12	20	24	32
5	14	23	32	41

Seven-system Tables

+	1	2	3	4	5	6
1	2	3	4	5	6	10
2	3	4	5	6	10	11
3	4	5	6	10	11	12
4	5	6	10	11	12	13
5	6	10	11	12	13	14
6	10	11	12	13	14	15

x	2	3	4	5	6
2	4	6	11	13	15
3	6	12	15	21	24
4	11	15	22	26	33
5	13	21	26	34	42
6	15	24	33	42	51

TABLE III (continued)

Addition	Multiplication

Eight-system Tables

+	1	2	3	4	5	6	7
1	2	3	4	5	6	7	10
2	3	4	5	6	7	10	11
3	4	5	6	7	10	11	12
4	5	6	7	10	11	12	13
5	6	7	10	11	12	13	14
6	7	10	11	12	13	14	15
7	10	11	12	13	14	15	16

x	2	3	4	5	6	7
2	4	6	10	12	14	16
3	6	11	14	17	22	25
4	10	14	20	24	30	34
5	12	17	24	31	36	43
6	14	22	30	36	44	52
7	16	25	34	43	52	61

Nine-system tables

+	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	10
2	3	4	5	6	7	8	10	11
3	4	5	6	7	8	10	11	12
4	5	6	7	8	10	11	12	13
5	6	7	8	10	11	12	13	14
6	7	8	10	11	12	13	14	15
7	8	10	11	12	13	14	15	16
8	10	11	12	13	14	15	16	17

x	2	3	4	5	6	7	8
2	4	6	8	11	13	15	17
3	6	10	13	16	20	23	26
4	8	13	17	22	26	31	35
5	11	16	22	27	33	38	44
6	13	20	26	33	40	46	53
7	15	23	31	38	46	54	62
8	17	26	35	44	53	62	71

Where the column and row intersect is the required number. Thus, in the four-system tables $2 + 3 = 11$ and $2 \cdot 3 = 12$.

Remember that any number plus 0 is the number itself and any number times 0 is 0. Any number times 1 is the number itself. These facts are not given in the tables but apply to all systems of numeration being considered here.

3.3 <u>Addition</u>. In totaling a column of numbers (starting at the right, of course), if a number greater than 10 is obtained, the units are recorded and the remaining digits (denoting the tens) are carried to the next column (the tens column). To add 639, 472, and 593 in the decimal system the process is as follows;

$$\begin{array}{r} 639 \\ 472 \\ \underline{593} \\ 1{,}704. \end{array}$$

That is,

$$\begin{array}{rl} 9 + 2 + 3 = 14. & \text{Record 4 and carry 1.} \\ 1 + 3 + 7 + 9 = 20. & \text{Record 0 and carry 2.} \\ 2 + 6 + 4 + 5 = 17. & \text{Record 17.} \end{array}$$

The sum is then 1,704.

The principle of carrying holds for other systems of numeration. But remember that the number that represents the system has no numeral for itself--it is always written as 10.

Here is the way to add two (or more) numbers, say 1,101 and 111, in the two-system:

$$\begin{array}{r} 1{,}101 \\ +\underline{\quad 111} \\ 10{,}100 \end{array}$$

$$\begin{array}{rl} 1 + 1 = 10. & \text{Write 0 and carry 1.} \\ 1 + 0 + 1 = 10. & \text{Write 0 and carry 1.} \\ 1 + 1 + 1 = 11. & \text{Write 1 and carry 1.} \\ 1 + 1 \quad = 10. & \text{Write 10.} \end{array}$$

Therefore the sum is 10,100.

The addition can be checked by the decimal system. Thus:

$$\begin{array}{r} 1{,}101 = 13_{(10)} \\ 111 = 7_{(10)} \\ \hline 10{,}100 = 20_{(10)} \end{array}$$

Numbers in the three-system are added in the same way. Thus:

$$\begin{array}{r} 2{,}122 \\ 212 \\ \underline{121} \\ 11{,}002 \end{array}$$

$$\begin{array}{rl} 1 + 2 + 2 = 12. & \text{Write 2 and carry 1.} \\ 1 + 2 + 1 + 2 = 20. & \text{Write 0 and carry 2.} \\ 2 + 1 + 2 + 1 = 20. & \text{Write 0 and carry 2.} \\ 2 + 2 \quad = 11. & \text{Write 11.} \end{array}$$

and the sum is 11,002.

Again the addition is checked by the decimal system.

$$\begin{array}{r} 2{,}122 = 71_{(10)} \\ 212 = 23_{(10)} \\ 121 = 16_{(10)} \\ \hline 11{,}002_{(3)} = 110_{(10)} \end{array}$$

3.4 Problems. Below are examples of addition in various numeration systems.

Two-system

$$\begin{array}{r} 101 \\ \underline{11} \\ 1000 \end{array} \qquad \begin{array}{r} 1011 \\ 1101 \\ \underline{111} \\ 11111 \end{array}$$

Three-system

$$\begin{array}{r} 112 \\ \underline{212} \\ 1101 \end{array} \qquad \begin{array}{r} 121 \\ 222 \\ \underline{112} \\ 2002 \end{array}$$

Four-system

$$\begin{array}{r} 23 \\ \underline{12} \\ 101 \end{array} \qquad \begin{array}{r} 3231 \\ 133 \\ \underline{312} \\ 11002 \end{array}$$

Five-system

$$\begin{array}{r} 234 \\ \underline{411} \\ 1200 \end{array} \qquad \begin{array}{r} 4312 \\ 432 \\ \underline{243} \\ 11042 \end{array}$$

Six-system

$$\begin{array}{r} 135 \\ \underline{42} \\ 221 \end{array} \qquad \begin{array}{r} 45312 \\ 5423 \\ \underline{355} \\ 55534 \end{array}$$

Seven-system

$$\begin{array}{r} 61 \\ \underline{355} \\ 446 \end{array} \qquad \begin{array}{r} 56543 \\ 3635 \\ \underline{216} \\ 64030 \end{array}$$

Eight-system

$$\begin{array}{r} 653 \\ \underline{471} \\ 1344 \end{array} \qquad \begin{array}{r} 64753 \\ 2567 \\ \underline{471} \\ 70233 \end{array}$$

Nine-system

$$\begin{array}{r} 387 \\ \underline{241} \\ 638 \end{array} \qquad \begin{array}{r} 784521 \\ 63677 \\ \underline{2467} \\ 861776 \end{array}$$

Problem. Check the above sums by the decimal system.

3.5 Subtraction. In subtraction, the method of "borrowing" will be used. Analysis of the decimal system technique may be helpful. Suppose 17 is to be subtracted from 42.

$$\begin{array}{r} 31 \\ \cancel{4}2 \\ \underline{17} \\ 25 \end{array}$$

Since 7 cannot be subtracted from 2, 1 ten is borrowed from 4 tens leaving 3 tens; 7 from 12, then, gives 5 and, finally, 1 from 3 is 2, and the answer is 25.

Again in the ten system suppose 46 is to be subtracted from 302.

$$\begin{array}{r} 291 \\ \cancel{3}\cancel{0}2 \\ \underline{46} \\ 256 \end{array}$$

Here since 6 cannot be subtracted from 2, and there are no tens to be borrowed, borrow 10 tens or 1 one hundred from the hundred place leaving 2 hundreds; from these 10 tens borrow 1 ten for the units place leaving 9 tens in the ten place; then, 6 from 12 gives 6, 4 tens from 9 tens gives 5 tens, and finally no hundreds from 2 hundreds is 2 hundreds and the answer is 256.

It should be remembered that, in each system of numeration, the number corresponding to the base of that system, or a power of that number, is the one borrowed.

Subtraction in other systems of numeration may be described in a similar manner. Subtraction, where borrowing is not required, is not difficult if one will check himself on the addition facts for that system of numeration. Thus, in the two system

$$\begin{array}{r} 11011 \\ -\ \underline{1001} \\ 10010 \end{array}$$

and in the five system

$$\begin{array}{r} 4332 \\ -\ \underline{2321} \\ 2011. \end{array}$$

The answers may be checked by adding the difference and the subtrahend or by translation to the decimal system.

When borrowing is necessary the procedure is similar to the method described in the ten system. Thus, in the binary system.

$$\begin{array}{rrr} 0 & 10 & \\ \cancel{1} & \cancel{0} & 1 \\ & 1 & 1 \\ \hline & 1 & 0 \end{array}$$

Starting at the right, 1 from 1 is 0. In the 2's column 1 cannot be subtracted from 0. Hence 1 four is borrowed from the 4's column leaving no fours. This four is really two 2's or 10 in the 2's column. Thus, in the 2's column 10 - 1 is 1. In the 4's column 0 - 0 is 0. The difference, therefore, is $10_{(2)}$.

Consider another example of borrowing in the binary system.

$$\begin{array}{rrrr} & 1 & & \\ 0 & \cancel{10} & 10 & \\ \cancel{1} & \cancel{0} & \cancel{0} & 1 \\ & 1 & 1 & 1 \\ \hline & & 1 & 0 \end{array}$$

In the units column 1 from 1 is 0. In the 2's column 1 cannot be subtracted from 0 and there are no fours to be borrowed. Hence, borrow 1 eight from the 8's column leaving no eights. Think of the eight as 10 in the 4's column, meaning two 4's. Then borrow 1 four leaving 1 four in the 4's column and write 10 in the 2's column meaning two 2's. Thus 10 - 1 is 1 in the 2's column; 1 - 1 is 0 in the 4's column; and 0 - 0 is 0 in the 8's column. The answer is $10_{(2)}$.

Following is one more example using the eight system of numeration.

$$\begin{array}{ccc} 4 & 12 & \\ \cancel{5} & \cancel{2} & 3 \\ \underline{2} & \underline{4} & \underline{1} \\ 2 & 6 & 2 \end{array}$$

In the units column 1 from 3 is 2. Borrow one 64 from the 64's column leaving four 64's. Think of the borrowed 64 as 8 eights, or 10 eights and add the 2 eights already in that column giving 12 eights. Hence 12 - 4 is 6 and 4 - 2 is 2. The difference, then, is $262_{(8)}$.

Remember, it is absolutely essential that the base being used is kept constantly in mind.

3.6 <u>Problems</u>. Following are examples of subtraction in various numeration systems. Dots are placed over columns to indicate that borrowing took place in those columns.

Two-system
```
...
101011
  1101
 11110
```

Three-system
```
...
22110
 1202
20201
```

Four-system
```
....
312023
 33231
212132
```

Five-system
```
....
43420
 3442
34423
```

Six-system
```
.. .
452050
 43443
404203
```

Seven-system
```
....
611012
256261
321421
```

Eight-system
```
....
12472
 6777
 3472
```

Nine-system
```
....
73421
 8678
63632
```

The results may be checked by addition or by translation into the decimal system.

3.7 Multiplication. Multiplication is performed the same way in all systems of numeration. In systems other than the decimal the chief difficulty is a tendency to think in terms of the decimal system. To avoid this, remember that the number denoting the system is always written as 10. The multiplication tables given in this chapter also will be useful.

Multiplication in the two-system is so easy that it bears out the statement that this is the simplest of all systems. There are of course only two digits, 0 and 1, and $0 \cdot 1 = 1 \cdot 0 = 0$; $1 \cdot 1 = 1$. Thus:

$$\begin{array}{r} 10011 \\ \underline{101} \\ 10011 \\ \underline{10011} \\ 1011111. \end{array}$$

The product of two numbers in the three-system is obtained as follows:

$$\begin{array}{r} 112 \\ \underline{221} \\ 112 \\ 1001 \\ \underline{1001} \\ 110222. \end{array}$$

The multiplication by 1 results in the original numeral 112. Multiplying 112 by 2 is done this way:

```
2•2 = 11.                        Write 1 and carry 1.
2•1 = 2, and 2 + 1 = 10.         Write 0 and carry 1.
2•1 = 2, and 2 + 1 = 10.         Write 10.
```

The product, therefore, is 1,001. The results may be checked by using the decimal system. Thus:

```
    112 =  14
    221 =  25
    ---    --
    112    70
   1001    28
  1001     --
-------
110,222 = 350.
```

3.8 Problems. Below are representative examples of multiplication. Problems such as these may be done in class and similar problems given to students as exercises.

```
Two-system              Three-system

    10110                    221
      101                     22
    -----                   ----
    10110                   1212
   101100                  1212
---------                 ------
1,101,110                 21,102


Five-system             Eight-system

                             347
      342                    526
       34                   ----
     ----                   2552
     3023                   716
     2131                  2203
   ------                -------
   24,333                232,232
```

Examples of multiplication using other bases are left to the teacher. A check may be made by translating to the decimal system.

3.9 Division. Division, too, follows the same pattern in all systems of numeration. The student, however, should be well versed in subtraction and multiplication before proceeding with division.

Below are examples of division:

Two-system

Divide 100011 by 111

```
       101
111)100011
     111
       11
       00
        111
        111
```

Three-system

```
     112
12)2122
    12
     22
     12
      102
      101
        1
```

Thus, 2, 122 ÷ 12 = 112 + $\frac{1}{12}$.

Four-system

```
      121
213)33210
    213
    1131
    1032
      330
      213
      111
```

Note that in the last example, when 1,131 was divided by 213, 2 was chosen as a quotient; 3 was not chosen because from inspection $3 \cdot 2 = 12$, and the first two digits of 1,131 are 11. Thus, 3 would have been too large.

Five-system

```
      302
123)43214
    424
     314
     301
      13
```

Thus, the answer is; $302 + \frac{13}{123}$, or $302 + \frac{4}{34}$.

3.10 <u>Problems</u>.

I. Add or subtract as indicated.

Two-system

1. 111 + 101
2. 1110 + 1010
3. 1011 - 1010
4. 1110 - 111
5. 10101 + 1010
6. 111010 - 10011

Five-system

1. 23 + 24
2. 413 + 224
3. 432 - 122
4. 414 + 223 + 143
5. 341 - 233
6. 301 - 213

II. Multiply or divide, as indicated.

Two-system

1. $11 \cdot 11$
2. $1110 \cdot 101$
3. $1000 \div 10$
4. $11001 \div 101$
5. $101101 \div 1001$
6. $100111 \div 1101$

Five-system

1. $24 \cdot 32$
2. $243 \cdot 23$
3. $100 \div 10$
4. $14 \div 3$
5. $242 \div 13$
6. $333 \div 104$

Check the preceeding problems by the decimal system. Any desired number of problems similar to the above may be presented using other systems of numeration.

CHAPTER IV

RELATED NUMERATION SYSTEMS

4.1 Introduction. The writing of numbers in the two-system of numeration becomes very cumbersome. However, this writing may be simplified. The simplification which follows is introduced into the construction of the binary electronic computing devices so that the recording and the typing of the final numerical results become comparatively simple.[1]

4.2 Bases two and eight. Note the following combinations of the digits "0" and "1" in the two-system of numeration.

000 represents 0	100 represents 4
001 represents 1	101 represents 5
010 represents 2	110 represents 6
011 represents 3	111 represents 7

These triplets may be employed in translating numbers written in the two-system of numeration. By translating the triplets according to the system above,

11,001,010,000,101,110,100,011,111

becomes 312,056,437.

The numeral 312,056,437 is, however, not written in the decimal system of numeration, but rather the eight-system

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[1]Aaron Bakst, Mathematical Puzzles and Pastimes (New York: D. Van Nostrand Company, Inc., 1954), p. 40.

(which is also known as the octal system). Note that the digits 8 or 9 do not occur in the translation, nor could they possibly occur since the translation is into the eight-system.

The advantage of the octal system over the two-system of numeration lies in the fact that in the octal system fewer digits are required for writing numbers. The translation of a numeral written in the octal system of numeration into the two-system of numeration can be performed with almost no effort. Electronic computing devices make provisions for such translations automatically.[2] Thus, for example, the numeral $15,675,217,346_{(8)}$ in the octal system is translated into

1,101,110,111,101,010,001,111,011,100,110

in the binary system. Note that each digit in the octal representation of the number is represented by a group of three digits in the binary system.

4.3 Problems. The following problems are merely representative of a kind that could be presented at this point.

1. Translate $11,110,101,011,111,010,000,100,010_{(2)}$ into the octal system of numeration.

---

[2] *Ibid.*, p. 41.

2. Translate the octal numeral, 157,634,527,100 into the two-system of numeration.

4.4 Bases two and four. The "counting" in different numeral systems may be related to some "primary" number systems.[3] Such a relationship between the two-system and the octal system has just been discussed. A similar relation exists between the two-system and the four-system of enumeration. By writing the first four numbers in the two-system of numeration we have:

00 represents 0	10 represents 2
01 represents 1	11 represents 3

Then, a number written in the two-system of numeration, say,

11011001010110

may be rewritten, pairing off the digits from right to left, and obtain

11,01,10,01,01,01,10.

Now, employing the stated relationship, rewrite this number as

3121112

in the four-system of numeration. If these two numerals were translated into the decimal system, the results would be the same:

$$2^{13} + 2^{12} + 2^{10} + 2^{9} + 2^{6} + 2^{4} + 2^{2} + 2 = 13{,}910_{(10)}.$$
$$(3\cdot4^{6}) + (4^{5}) + (2\cdot4^{4}) + (4^{3}) + (4^{2}) + (4) + (2) = 13{,}910_{(10)}.$$

[3]Ibid., p. 51.

If a number is written in the four-system of numeration, then it can be quickly translated into the two-system of numeration. For example, suppose that the number written in the four-system is

230110312.

The four relations discussed above make possible the translation of this numeral into the two-system of numeration, resulting in

10,11,00,01,01,00,11,01,10.

4.5 Problems.

1. Translate the two-system numeral

110100010110111

into the four-system.

2. Translate the four-system numeral

30221011033210

into the two-system.

4.6 Summary. In the discussion of related systems of numeration it has been shown that the two-system can very conveniently be translated into the four-system or the eight-system and conversely. The two-system would therefore be defined as the "primary" system in relation to the four-system and the eight-system respectively. To translate from the two-system to the four-system of numeration, group the digits in the binary number by two's from right to left, then translate each group individually into the symbols 0, 1, 2, or 3 as they may occur. To translate from the two system to the eight-system of numeration, group the digits in the binary number by three's from right to left, then translate each group individually into the symbols 0, 1, 2, 3, 4, 5, 6, or 7 as they may occur.

To translate the four-system, or the eight-system of numeration into the two-system, translate each digit in the higher base system number to the equivalent number symbol in the two-system. This process is more conveniently performed by working from left to right. More examples:

1. $101101011101_{(2)} = 10,11,01,01,11,01_{(2)} = 231131_{(4)}$.
2. $101101011101_{(2)} = 101,101,011,101_{(2)} = 5535_{(8)}$.
3. $23123_{(4)} = 10,11,01,10,11_{(2)}$.
4. $7234165_{(8)} = 111,101,011,100,001,110,101.$

4.7 Problems.

1. What relationship do you suppose exists between the two-system and the four-system; and the two-system and the eight system of numeration that makes such a convenient translation possible? [Ans. 4 and 8 are perfect powers of 2.]

2. Do you suppose a "primary" relation exists between the three-system and the nine-system of numeration.

3. Write a number in the three-system of numeration. Try to translate this numeral into the nine-system directly. Check your results by translating each numeral into the decimal system. Do they check, that is, are the resulting decimal numerals the same?

CHAPTER V

FRACTIONS AND "DECIMALS"

5.1 <u>Introduction</u>. Fractions and decimals and their relation to other systems of numeration, and conversely, will be given brief consideration here. Some students may develop considerable interest in this area because of the unfamiliar and often strange appearing results.

The decimal numeral or decimal fraction as known in the ten system of numeration has its counterpart in other systems of numeration. The numerals and fractions in other systems should not, strictly speaking, be referred to as decimal numerals or fractions since the base of the system being used is not ten. Rather than invent special names for these numerals expressed in different numeration systems the author will use the word "decimal" but use a subscript in parentheses to indicate the base number of the system being used. Thus, $.01_{(2)}$ is a decimal$_{(2)}$ written in the two system of numeration. "Decimal" in quotation marks shall mean a numeral other than base ten but no specific base implied.

5.2 <u>"Decimals"</u>. "Decimal" numerals in other systems of numeration have place value just as in the decimal$_{(10)}$ system. Just as

$$.235_{(10)} = 2(\frac{1}{10}) + 3(\frac{1}{10^2}) + 5(\frac{1}{10^3})$$

so $$.101_{(2)} = 1(\frac{1}{2}) + 0(\frac{1}{2^2}) + 1(\frac{1}{2^3})$$

and $$.234_{(5)} = 2(\frac{1}{5}) + 3(\frac{1}{5^2}) + 4(\frac{1}{5^3})$$

In other words

$$.235_{(10)} = \frac{2}{10} + \frac{3}{100} + \frac{5}{1000} = \frac{235}{1000}_{(10)}$$

$$.101_{(2)} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8}_{(10)}$$

and $$.234_{(5)} = \frac{2}{5} + \frac{3}{25} + \frac{4}{125} = \frac{69}{125}_{(10)}$$

To convert a "decimal" in another system to a decimal$_{(10)}$ simply convert the "decimal" into a fraction$_{(10)}$ by the above method and then divide.

5.3 <u>Fractions</u>. A decimal$_{(10)}$ fraction may be changed into a fraction in any other numeration system simply by translating its numerator and denominator into the new system. Thus

$$\frac{3}{4}_{(10)} = \frac{11}{100}_{(2)}$$

A fraction written in a system other than the ten system may be converted to a "decimal" form of the same base by the division process described in unit 2.15. Thus in the three system

$$\frac{1}{11}_{(3)} = .0202\dots_{(3)}$$

since

$$\begin{array}{r} .0202\cdots \\ 11\overline{)1.0000} \\ \underline{22} \\ 100 \\ \underline{22} \\ 1 \end{array}$$

Notice that

$$\frac{1}{4}_{(10)} = .2500_{(10)}$$

$$\frac{1}{11}_{(3)} = .0202\cdots_{(3)}$$

and

$$\frac{1}{4}_{(10)} = \frac{1}{11}_{(3)}$$

therefore

$$.2500_{(10)} = .0202\cdots_{(3)}$$

5.4 <u>"Decimal" conversion</u>. The process of changing a decimal$_{(10)}$ to the equivalent "decimal" in another system is rather involved. One method of course (if the decimal$_{(10)}$ is terminal) is to convert the decimal$_{(10)}$ to an equivalent fraction$_{(10)}$, then translate the numerator and denominator of the fraction$_{(10)}$ into the system being dealt with before dividing. Thus

$$.2500\cdots_{(10)} = \frac{1}{4}_{(10)}$$

but

$$\frac{1}{4}_{(10)} = \frac{1}{100}_{(2)}$$

and by division

$$\frac{1}{100}_{(2)} = .0100\cdots_{(2)}$$

therefore

$$.2500\cdots_{(10)} = .0100\cdots_{(2)}$$

Another method of conversion could be as follows.

Example. Change $.7_{(10)}$ to the binary scale.

Solution. $.7_{(10)} = 1.4$ halves or $1(\frac{1}{2}) + .4$ halves

$.4$ halves $= .8$ fourths or $0(\frac{1}{4}) + .8$ fourths

$.8$ fourths $= 1.6$ eighths or $1(\frac{1}{8}) + .6$ eighths

$.6$ eighths $= 1.2$ sixteenths or $1(\frac{1}{16}) + .2$ sixteenths

and so on.

Thus $.7_{(10)} = .1011\cdots_{(2)}$

That is $.7_{(10)} = 1(\frac{1}{2}) + 0(\frac{1}{4}) + 1(\frac{1}{8}) + 1(\frac{1}{16})\cdots$

To convert a $\text{numeral}_{(10)}$ such as $25.25_{(10)}$ to a numeral in another system of numeration requires two distinct operations.

Example. Change $25.25_{(10)}$ to the binary system.

```
     1  R 1
   2)3  R 0
   2)6  R 0
  2)12  R 1
  2)25.
```

Thus $25._{(10)} = 11001_{(2)}$

The $.25_{(10)}$ may be translated by either of two ways.

Thus $.25_{(10)} = \frac{1}{4}_{(10)} = 0(\frac{1}{2}) + 1(\frac{1}{4})$.

and $.25_{(10)} = .01_{(2)}$

or $.25_{(10)} = .50$ halves or $0(\frac{1}{2})$

$.50$ halves $= 1.00$ fourths or $1(\frac{1}{4})$

and $.25_{(10)} = .01_{(2)}$.

Therefore $25.25_{(10)} = 11001.01_{(2)}$

5.5 Problems.

1. Verify some of the "decimal" equivalents given in Table IV by procedures described in this unit.

2. Change the $\text{decimal}_{(10)}$ fraction $\frac{5}{6}_{(10)}$ to a binary fraction.

3. Change $\frac{11}{1000}_{(2)}$ to a $\text{decimal}_{(2)}$ equivalent.

4. Change $.3333_{(10)}$ to a $\text{decimal}_{(2)}$.

5. Change $.101000\cdots_{(2)}$ to a $\text{fraction}_{(10)}$.

TABLE IV

COMMON FRACTIONS EXPRESSED AS "DECIMALS" IN VARIOUS NUMERATION SYSTEMS

Common Fractions	Ten System 10ths 100ths 1000ths 10000ths	Two System halves 4ths 8ths 16ths	Three System thirds 9ths 27ths 81sts	Four System fourths 16ths 64ths 256ths	Eight System eighths 64ths 512ths 4096ths
1/2	.5000	.1000	.1111	.2000	.4000
1/3	.3333	.0101	.1000	.1111	.2525
1/4	.2500	.0100	.0202	.1000	.2000
1/5	.2000	.0011	.0121	.0303	.1463
1/6	.1666	.0010	.0111	.0222	.1252
1/7	.1428	.0010	.0102	.0210	.1111
1/8	.1250	.0010	.0101	.0200	.1000
1/10	.1000	.0001	.0022	.0121	.0631
1/12	.0833	.0001	.0020	.0111	.0525
5/6	.8333	.1101	.2111	.3111	.6525
3/8	.3750	.0110	.1010	.1200	.3000

CHAPTER VI

PROBABILITY

6.1 Introduction. Some of the simpler ideas of permutations, combinations, and probability may be interjected into a ninth grade mathematics course usually with wide acceptance by the students. The element of chance tends to lend intrigue to almost any endeavor in any walk of life. A child has experience in the playing of games and in other endeavors where the element of chance will bear on the outcome of his experience. He recognizes the element of chance in his choice of an answer to a true or false question on a test, especially if he is ill prepared.

Thus, chance is a facet of mathematics familiar to the child but usually not in the mathematical sense.

The two problems selected for this chapter and the presentation of tree diagrams as a method of analyzing certain situations involving the element of chance were selected to demonstrate to the student the role that mathematics can play in predicting the outcome of varied situations involving chance. Further, it is expected that these problems may well suggest a form of mathematics heretofore unknown to him.

The problems are not intended to introduce any formal approach to probability theory, nor is there any direct reference to permutations or combinations. Any formal approach to this topic is left to the discretion of the teacher.

6.2 Buffon's Needle Problem. An interesting number used as a multiplier in computation of the circumference of a circle is the number known as π. It is approximately equal to 3.14159. We know that it is impossible to determine π exactly, but we shall see now that it is possible to approximate its value during the process of a very simple and unusual experiment.

On a large piece of paper or drawing board construct a series of parallel lines such that the distance between them is twice the length of an ordinary needle. Place the paper or drawing board on a horizontal surface and drop the needle onto the paper. Continue dropping the needle a large number of times, a hundred or a thousand times; the greater the number of times the greater the probability will be that a closer result will be obtained. Each time the needle is dropped note whether it crosses some line, considering it a crossing when even the end of the needle touches a line. Now, if the total number of times the needle was dropped is divided by the number of times it crossed a line, the result will be the approximate value of π. That is, if B is the number of times the needle is dropped and A is the number of times the needle touches or crosses a line, then $\frac{B}{A} \doteq \pi$.

<u>Explanation</u>. Suppose that the number of crossings is A, and assume that any part of the needle has the same chance of falling across any of the lines. If the needle is two inches long, then, since every part of the needle has the same chance of falling across a line, the number of crossings for one inch of the needle would be just one-half that for two inches, or $\frac{A}{2}$. If the needle is divided into n equal parts, the number of crossings for each part is $\frac{A}{n}$. The number of crossings for two such parts is $\frac{2A}{n}$, and for ten such parts $\frac{10A}{n}$. From the preceeding discussion we arrive at the conclusion that the number of the crossings is proportional to the length of the needle. Thus, if the length of the needle is r, then $A = Kr$ where K is a constant.

Now suppose we have a needle that is bent into a circle and the radius of the circle is equal in length to the original needle. When such a circle is dropped onto the paper (the distance between the lines is thus equal to the diameter of the circle), it will either cross one line twice or will touch two lines. Let us suppose that the number of times the circular needle is dropped is B, then the number of crossings is 2B, because every time this circular needle is dropped it must either come in contact with one line twice or touch two lines. The length of the circular needle (if its radius is r, which is the length of the original needle) is $2\pi r$. Thus the circular needle is 2π times the length of the original straight needle. We also have established that the number of crossings is proportional to the length of the needle. Thus, the number of possible crossings of the circular needle is 2π times the number of possible crossings of the original needle.

In other words,

$$2B = A \cdot 2\pi$$

and from this

$$\pi \doteq \frac{B}{A} \text{ approximately.}$$

That is

$$\pi \doteq \frac{\text{Number of times the needle was dropped}}{\text{Number of times the needle crossed a line}}.$$

In terms of probability, or the chance that the needle will cross a line it may be stated that

$$\frac{A}{B} = \frac{\text{Number of times the needle crossed a line}}{\text{Number of times the needle was dropped}},$$

which is the probability of the needle crossing a line. Since $\pi \doteq \frac{B}{A}$, then $\frac{1}{\pi} \doteq \frac{A}{B}$, and the stated probability is given by $\frac{1}{\pi} \doteq 0.31831$.

In other words the chance of the needle crossing a line is a little less than one time out of three and a little better than three times out of ten.

Summary of the explanation.

Let A = the number of crossings of the straight needle of length r.

B = the number of times *either* needle is dropped (circular or straight).

C = the length of the circular needle of radius r.

1. $A = Kr$ or $r = \frac{A}{K}$
2. $2B$ = the number of times the circular needle crosses.
3. $\therefore 2B = KC$, and since $C = 2\pi r$

   $2B = K(2\pi r)$
4. $\therefore 2B \doteq K \cdot 2\pi \cdot \frac{A}{K}$ by substitution.
5. and $B \doteq \pi A$
6. $\therefore \quad \pi \doteq \frac{B}{A}$.

Note. In step 4 above the approximation symbol was introduced because of the experimental relation between B and A. Also in step 4, K is assumed to be the same proportionality constant for both the straight and the circular needle.

A further consideration. The needle need not be straight; suppose that it is bent as shown in Figure 6.2. Suppose that BC contains m parts of the needle (after it is

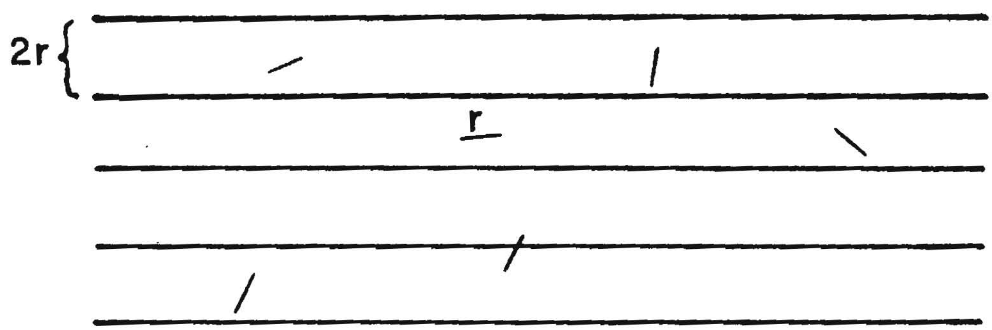

FIGURE 6.1

BUFFON'S NEEDLE PROBLEM

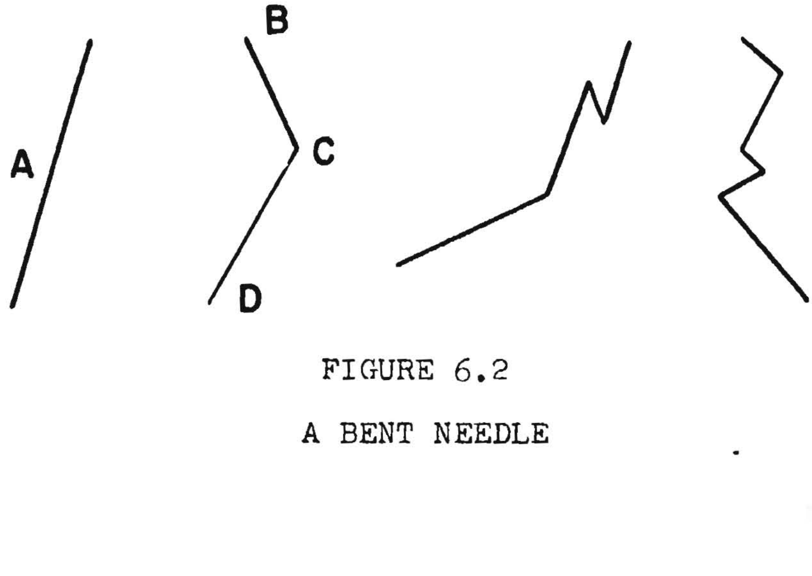

FIGURE 6.2

A BENT NEEDLE

divided into n equal parts). Then the remaining portion of the needle (CD) will contain (n - m)parts. Their respective number of crossings will be $\frac{mA}{n}$ and $\frac{(n - m)A}{n}$ and the total sum of the crossings is still equal to A. However it should be noted that a bent needle may fall so that it will cross the same line several times. If this happens, all the crossings must be counted.[1]

6.3 Three card problem. Here is a famous problem in probability that is said to have trapped even skilled mathematicians into error from time to time. Yet it is easily stated, and appears innocent enough.

Three cards are identical in appearance except for their coloring, which is as follows: one card is red on both sides, one is white on both sides, and one is red on one side and white on the other. I shuffle them in a closed bag, and then reach in and draw one out and lay it on the table, without looking at or letting you see the side that is down. Suppose the side that is up is red. I then say, "Obviously this is not the white--white card. Therefore it is either the red--white or the red--red. I'll bet you even money that it is the red--red." If you take this bet and we repeat the game often enough, you will go home a

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[1]Aaron Bakst, Mathematics--Its Magic and Mastery (New York: D. Van Nostrand Company, Inc., 1941), pp. 350-52.

lot poorer than when you came. The chances that it is the red--red are not even, but two to one in my favor.

This problem was stated and the solution given in an article in the October 1950 Scientific American by Warren Weaver, the director of the natural sciences division of the Rockefeller Foundation. A spirited exchange of letters between Dr. Weaver and a professional gambler who challenged the correctness of the solution appeared in the December 1950 correspondence column of the same magazine.[2]

Solution. Suppose a card is drawn from the bag and turned face up on the table. If red is the color turned up this means that the white--white card is still in the bag and the card on the table must be either red--red or red--white. At this point the uninitiated will believe that there is an equal chance of the card being red--red or red--white. However, of the two cards in question, three sides are red and only one side is white. This means there are two ways red could be up such that the down side of the card would be red, while there is only one way that red could be up such that the down side of the card would be white. Hence the odds are two to one in favor of the red--red card. Of course if white turns up on the draw of the card the odds would be two to one in favor of white--white by similar

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[2]C. Stanley Ogilvy, Through the Mathescope (New York: Oxford University Press, 1956), pp. 32-33.

reasoning. Hence by calling red--red or white--white depending on the up color of the drawn card it is obvious that two of the three cards in the bag are winners while only one is a loser.

6.4 Tree diagrams. Problems are many and varied which deal with the processes of making an exhaustive search for all the possible outcomes in a situation where more than one outcome exists. In pure mathematics such problems are often found in situations involved with mathematical probability, permutations, and combinations. Much of the language of this subject is necessarily beyond the level of presentation intended here; therefore this work will be treated principally by example with the use of technical terms avoided as much as possible.

The term logical possibilities shall be used to mean the several outcomes or possible results that may be determined from a given physical situation. A very useful tool for analyzing logical possibilities is the drawing of a "tree" diagram. This device will be illustrated by several examples.

Problem 1. Consider the following problem which is of a type often studied in probability theory. "There are two urns; the first contains two black beads and one white bead, while the second contains one black bead and two white beads. Select an urn at random and draw two beads in succession from it. Now there is certainly more than one way in which this operation could be performed and we are interested in how many different ways it could be done."

Solution. Draw a "tree" diagram as shown in Figure 6.3. Start at a single point and draw two "branches" leading to each of the first two logical possibilities, that is, the choice of either the first urn or the second urn. From the first urn three branches are drawn to represent the logical possibilities of selecting any one of the three beads in it. The two black beads are made distinguishable by identifying one as B_1 and the other as B_2. From each of these branches, B_1, B_2 and W, draw two branches to represent the remaining possibilities after the first bead is selected. The diagram now represents all the logical possibilities or different ways in which two beads may be drawn in succession if the first urn is chosen. The branches from the second urn are drawn in a similar manner with the two white beads being distinguished by W_1 and W_2.

The student should note that there are twelve logical possibilities or possible outcomes, all different, and these are the only outcomes possible. He should trace each of the twelve paths from start to end in order to enforce this concept.

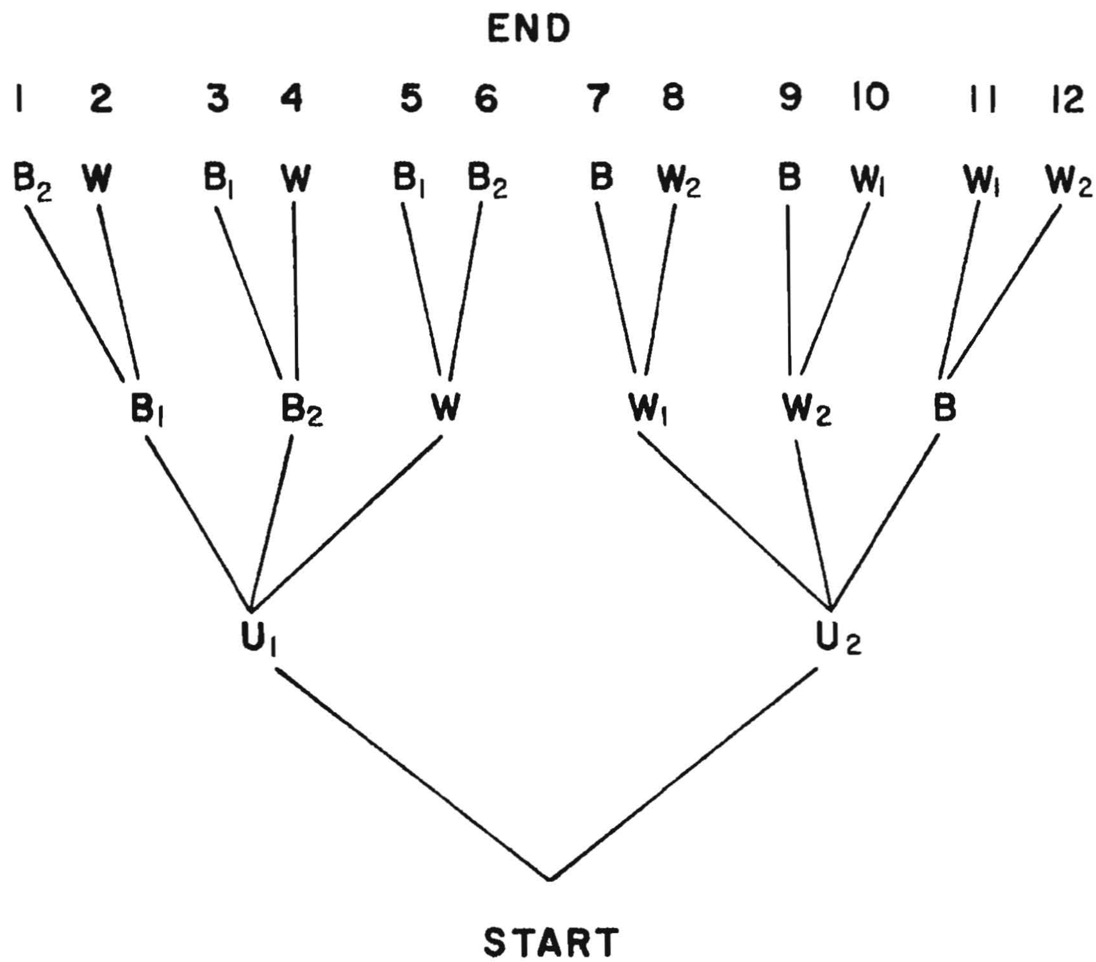

FIGURE 6.3

TREE OF LOGICAL POSSIBILITIES

<u>Questions</u>.

1. In how many ways may a black bead be drawn followed by a white bead? (Answer: Four ways. Paths 2, 4, 11, and 12.)

2. In how many ways may two black beads be drawn? (Answer: Two ways. Paths 1 and 3.)

3. If by the chance or probability of a particular outcome occuring we mean the number of ways it can happen divided by the total number of all outcomes, what is the chance of two black beads being drawn in succession? (Answer: The chance is $\frac{2}{12}$ or $\frac{1}{6}$. That is, one chance out of six.)

4. What is the probability of a black bead being selected followed by a white? (Answer: The probability is $\frac{4}{12}$ or $\frac{1}{3}$. That is,one chance out of three.)

5. What is the probability that the second bead drawn will be white? (Answer: The probability is $\frac{6}{12}$ or $\frac{1}{2}$. That is, one chance out of two.)

6. What is the chance that the first bead chosen will be black? (Answer: Refer to the section of the tree that represents the logical possibilities on the first draw. The first bead can be chosen in six different ways while three of these ways result in the selection of a black bead. Therefore the chance of drawing a black bead

on the first draw is $\frac{3}{6}$ or $\frac{1}{2}$. That is, one chance out of two.)

7. What is the probability that the first urn will be selected? (Answer: $\frac{1}{2}$, or one chance out of two.)

8. If the first urn is selected, what is the chance that the second bead drawn will be black? (Answer: $\frac{4}{6}$ or $\frac{2}{3}$; two out of three.)

9. If the first urn is chosen what is the probability that two beads of different colors will be chosen? (Answer: $\frac{4}{6}$ or $\frac{2}{3}$; two out of three.)

10. If the first urn is chosen, what is the probability that the first bead chosen will be black and the second one white? (Answer: $\frac{2}{6}$ or $\frac{1}{3}$; one out of three.)

Problem 2. This problem is similar to Problem 1 except that one urn has two black beads and two white beads in it, while the second urn contains one white bead and four black beads. Select an urn and draw two beads from it. Construct the tree diagram of logical possibilities. How many possibilities are there? (Answer: 32.) After the tree diagram is properly drawn the teacher may ask questions similar to those following Problem 1.

Problem 3. As another example, construct the tree of logical possibilities for the outcomes of a World Series played between the Dodgers and the Yankees. In Figure 6.4

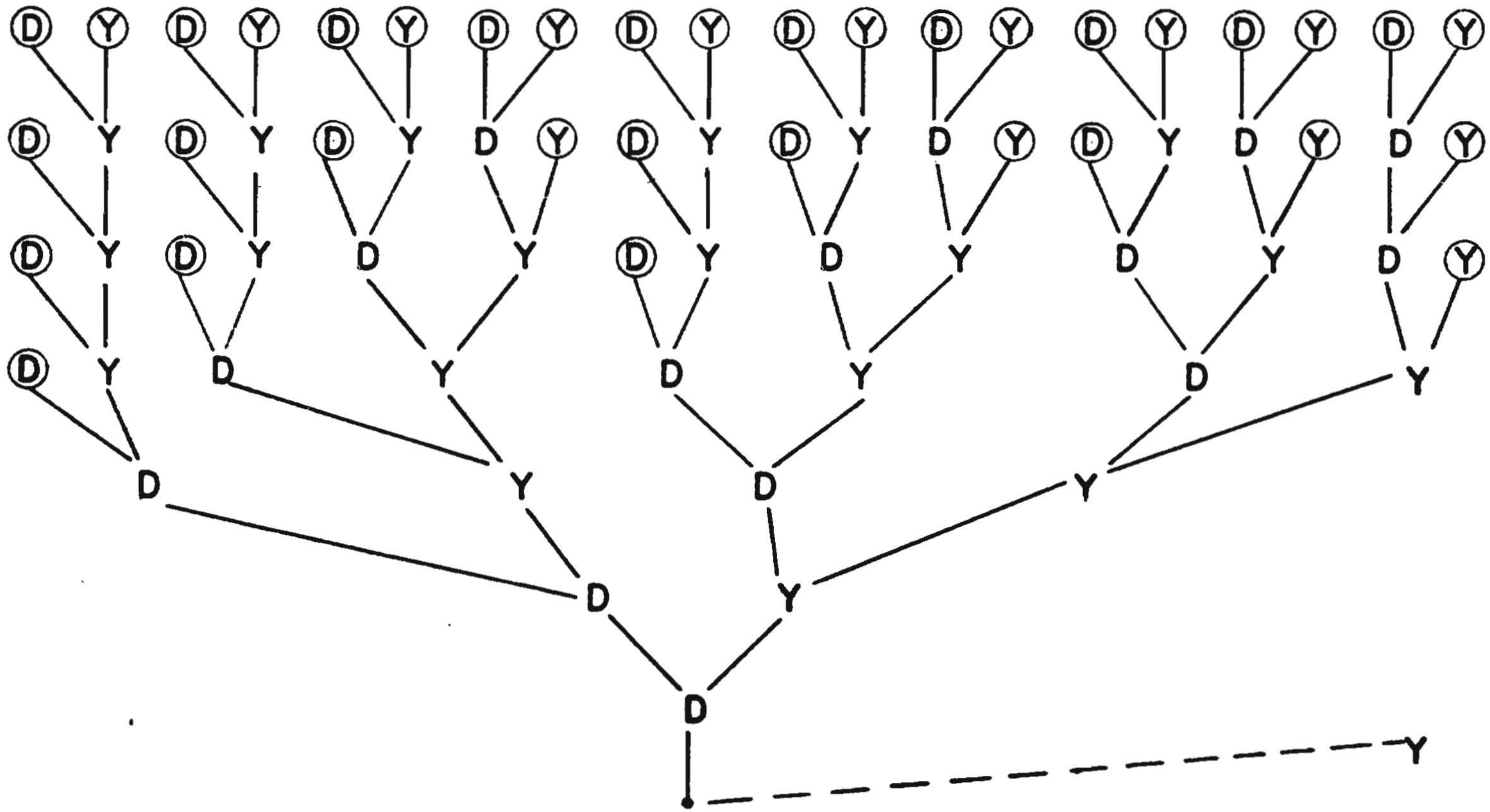

FIGURE 6.4

TREE DIAGRAM FOR THE WORLD SERIES

is shown half of the tree corresponding to the case when the Dodgers win the first game (the dotted line at the bottom leads to the other half of the tree when the Yankees win the first game). In the figure a "D" stands for a Dodger win and a "Y" for a Yankee win. A circled letter indicates that the series has ended at that stage with the circled letter indicating the winner of the series. There are 35 possible outcomes (corresponding to the circled letters) in the half-tree shown, so that the World Series can end in 70 different ways. This example is different from the previous one in that the paths of the tree end at different levels corresponding to the fact that the World Series ends whenever one of the teams has won four games.[3]

Problem 4. Construct the other half of the tree corresponding to the fact that the Yankees win the first game.

Questions.

1. In 1955 the Dodgers lost the first two games of the World Series but won the series in the end. In how many ways can the series go so that the winning team loses the first two games? (Answer: 10.)

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[3]John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson, Introduction to Finite Mathematics (Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1957), pp. 25-30.

2. What is the chance that a team will win the series after losing the first two games? (Answer: $\frac{10}{70}$ or $\frac{1}{7}$, one out of seven.)

3. In how many ways can the World Series be played (see Figure 6.4) if the Dodgers win the first game and

(a) No team wins two games in a row. (Answer: 1.)

(b) The Dodgers win at least the odd-numbered games. (Answer: 5.)

(c) The winning team wins four games in a row. (Answer: 4.)

(d) The losing team wins four games. (Answer: 0.)

4. What is the probability of each occurrence happening in the preceeding question?

(a) $\frac{1}{70}$ or 1 out of 70.
(b) $\frac{5}{70}$ or $\frac{1}{14}$; 1 out of 14.
(c) $\frac{4}{70}$ or $\frac{2}{35}$; 2 out of 35.
(d) $\frac{0}{70}$; or no chance at all.

Problem 5. If a family is to have four children in how many ways may the births occur by sex classification? (Answer: 16.) Draw a tree to show all the possibilities.

Questions.

1. In how many ways may the family consist of exactly two boys and two girls? (Answer: 6.)

2. In how many ways may the family consist of at least two boys? (Answer: 11.)

3. What is the probability that the family will consist of exactly two boys and two girls? (Answer: $\frac{6}{16}$ or $\frac{3}{8}$; three out of eight.)

4. What is the probability that the family will contain no boys? (Answer: $\frac{1}{16}$ or one out of sixteen.)

CHAPTER VII

FACTS FALLACIES AND DIVERSIONS

7.1 Introduction. The remainder of this thesis is devoted to problem situations of a varied nature, selected with the primary intent of arousing interest and encouraging mathematical thinking. There are problems which involve the use of one or more simple mathematical principles and concepts but deal with phenomena or experiences with which the student may not have had much previous contact. There are problems which may require a certain amount of experimentation and assembling of pertinent data before convincing the student that a solution is possible. Some problems may lead to the need of acquiring new techniques and operations which have not been studied previously. Finally, there are some problems which lead to the conjecturing and eventual proof of specific statements.

The teacher should find that some of the problems are appropriate only for Algebra students while others may be used in both the general mathematics and the Algebra class. The ordering of problems from first to last in this chapter has been determined by the depth of mathematical knowledge required of the student to cope with the problem successfully.

The author suggests that problems such as these should be freely intermingled with the other more routine problems. The inclusion of such problems may make rote learning slower and somewhat less efficient than drill procedures, but may also produce less mechanical behavior and more productive thinking. It is hoped that students will find some interest and challenge in facing and coping with new and changing problem situations.

A reference to the bibliography will afford the teacher an opportunity to select and refer to numerous books which offer a wealth of problems similar to the ones presented here, as well as many types not included here.

7.2 Wine and water problem. Let us suppose that we have in one glass a certain quantity of water and in another glass an equal quantity of wine. We take a teaspoonful of wine from the second glass, put it in the glass of water, and stir. We then take a teaspoonful of the mixture and put it back in the wine glass. Is the quantity of water now in the first glass greater or less than the quantity of wine now in the second glass?

A rousing good argument can be started with this one--but only because nearly everyone tries to do it the hard way. It is an elementary illustration of something that often happens in more serious mathematics: the right attack "breaks" the problem in a minimum time.

Explanation. Suppose for simplicity that we start with 4 teaspoonsful each of water and wine. If we put one teaspoonful of wine in the water, the resulting five teaspoonsful of mixture is $\frac{1}{5}$ wine and $\frac{4}{5}$ water. When we transfer one teaspoon of the mixture to the glass of wine, we are returning $\frac{1}{5}$ of a teaspoonful of wine--thus leaving $\frac{4}{5}$ of a teaspoon of wine in the water--and are adding $\frac{4}{5}$ of a teaspoon of water to the wine. Thus there are equal quantities--$\frac{4}{5}$ of a teaspoonful--of wine in water and water in wine. Incidentally, it makes no difference whether or not the mixture is stirred! Finally, the operation with the spoon may be repeated as many times as desired--the answer to the original question will be the same.[1]

7.3 A salary problem. A large business firm was once planning to open a new branch in a certain city, and advertised positions for three clerks. Out of a number of applicants the personnel manager selected three promising young men and addressed them in the following way: "Your salaries are to begin at the rate of $3000 per year, to be paid every half-year. If your work is satisfactory, and we

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[1]Eugene P. Northrop, Riddles in Mathematics (New York: D. Van Nostrand Company, Inc., 1944), pp. 14-16.

keep you, your salaries will be raised. Which would you prefer, a raise of $300 per year or a raise of $100 every half-year?"

One of the three applicants, after a moment's reflection took the second of the two alternatives and was promptly put in charge of the other two. His alertness of mind had resulted in not only a higher position than his companions, but in a higher annual income as well.

The two possibilities may be treated in the following manner:

	$300 raise yearly	$100 raise half-yearly
1st year:	$1500 + $1500 = $3000	$1500 + $1600 = $3100
2nd year:	$1650 + $1650 = $3300	$1700 + $1800 = $3500
3rd year:	$1800 + $1800 = $3600	$1900 + $2000 = $3900
4th year:	$1950 + $1950 = $3900	$2100 + $2200 = $4300

How much more than his companions would the bright young man have earned at the end of ten years?[2]

7.4 Ring and circle problem. In Figure 7.1 one would not immediately suspect that the two shaded portions of the figure have equal areas.

Assume that the radius of the inner circle is of unit length 1 and that each successive concentric circle is

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[2]Eugene P. Northrop, Riddles in Mathematics (New York: D. Van Nostrand Company, Inc., 1944), pp. 10-11.

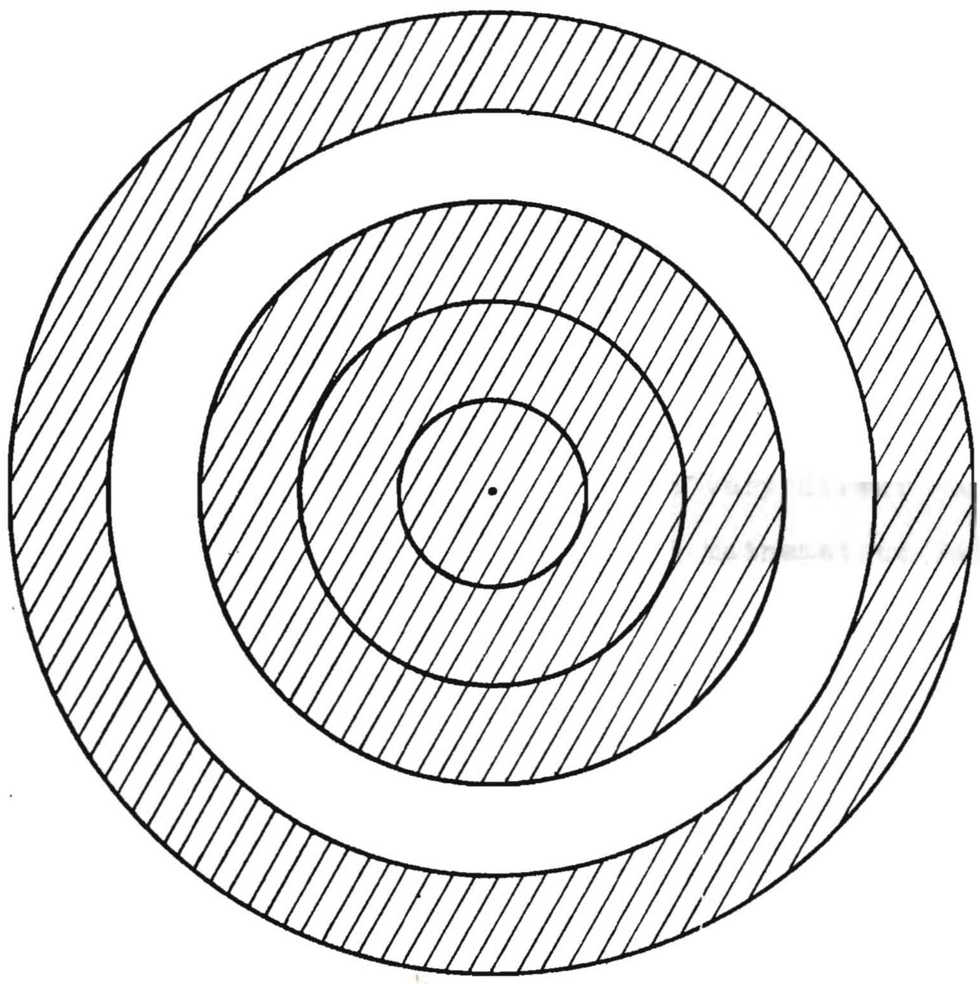

FIGURE 7.1

CONCENTRIC CIRCLES

increased in radius by unit length 1. Can you prove that the two shaded portions are equal in area?

Proof. If the radius of the largest circle is taken as 5, then the inner radius of the shaded ring is 4, and the radius of the shaded circle is 3. Hence the area of the shaded circle is

$$\pi r^2 = \pi \cdot 3^2 \text{ or } 9\pi \text{ square units.}$$

The area of the shaded ring is

$$\pi \cdot 5^2 - \pi \cdot 4^2 = 25\pi - 16\pi = 9\pi \text{ square units.}$$

This proof, as presented here, is very direct and brief. A student, especially in general mathematics may well use the fraction $\frac{22}{7}$ or the decimal 3.14 as an approximation of π and become involved in considerable more arithmetic.[3]

7.5 Rope around the equator! Suppose there were a rope fitting tightly around the equator of the earth. Also imagine that the surface of the earth at the equator is perfectly uniform, that is no mountains or valleys, bumps or irregularities to interfere with our thinking that the equator is a perfect circle. Now suppose that this rope is cut at one place and we splice in an additional piece 10

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[3]Eugene P. Northrop, Riddles in Mathematics (New York: D. Van Nostrand Company, Inc., 1944), p. 48

feet longer than the original one. Finally we have a means of fixing this rope so that (because of its extra length) it will be the same distance from the equator all the way around the earth.

Now how large a space would there be between the rope and the earth?

Would it be large enough for

(a) a man, 6 feet tall, to walk through,

(b) an average dog to walk through,

(c) a piece of tissue paper to just slip through?

Remember the distance around the equator and therefore the length of the original rope is approximately 25,000 miles.

Explanation. (1) $C = 2\pi r$ for any circle no matter how large or how small.

If we increase the radius of a circle with radius r by an amount x (see Figure 7.2) and make a new, larger, circle whose radius is now $(r + x)$ the new circumference would now be $C' = 2\pi(r + x)$. This may now be written

$$(2) \quad C' = 2\pi r + 2\pi x$$

using the distributive law for multiplication over addition. If we now compare this with the value of C given above, namely

$$(1) \quad C = 2\pi r$$

we see that C' is more than $2\pi r$ by an amount $2\pi x$. In other

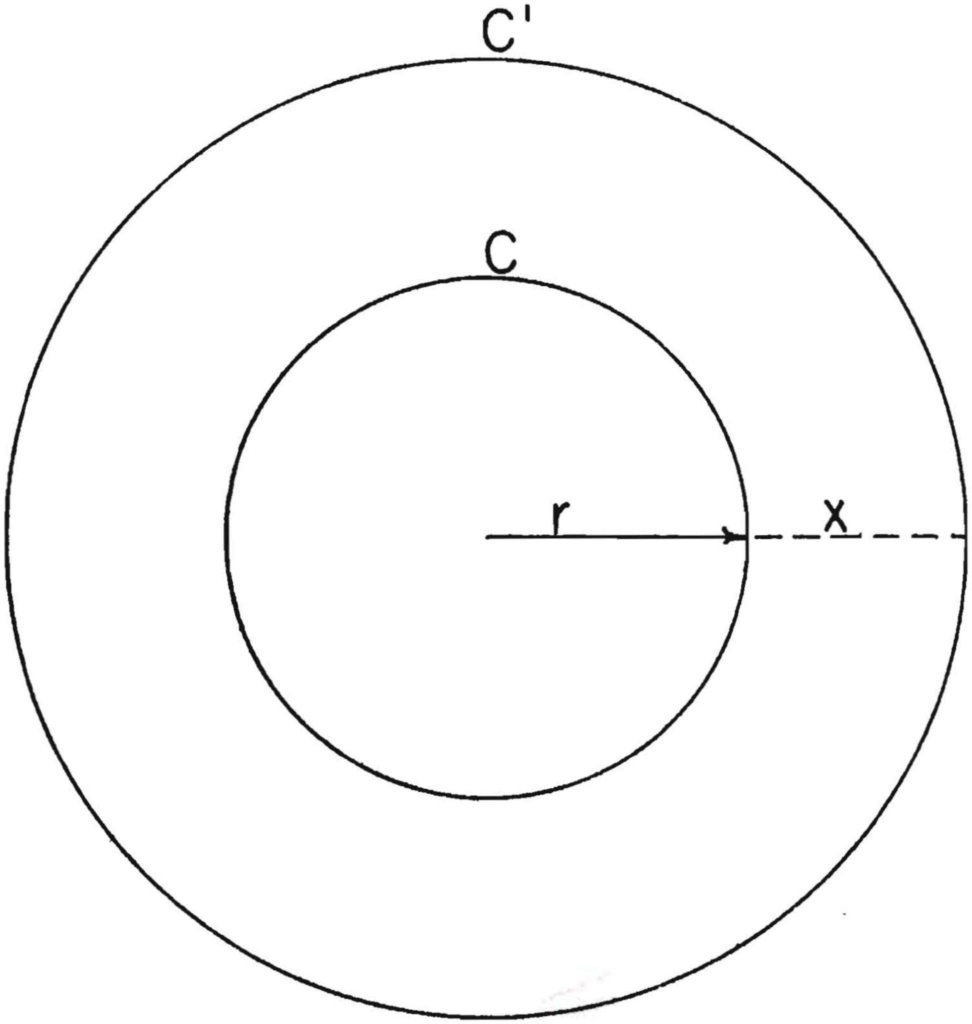

FIGURE 7.2

RELATION BETWEEN RADIUS

AND CIRCUMFERENCE

words subtracting (1) from (2)

$$(2) \quad C' = 2\pi r + 2\pi x$$

$$(1) \quad \underline{C \;\; = 2\pi r \qquad\quad}$$

$$(3) \quad C'-C = 2\pi x.$$

Since in this problem (C' - C) is 10 feet then

$$(4) \quad 10 = 2\pi x$$

and $$x = \frac{5}{\pi} \text{ or about 1.6 feet.}$$

So by inserting an extra 10 feet of rope into our 25,000 miles of rope we have increased the radius by over $1\frac{1}{2}$ feet and our average dog should have little trouble in walking under the rope.

Problem. If a 6 foot man could walk around the earth at the equator, how much farther than his feet would his head travel?[4]

7.6 Cigarette paper problem. Paper to be fed to cigarette machines comes in long bands wound in a tight roll around a wooden spool. The diameter of such a paper roll is 16 inches, that of the spool itself, 4 inches. If the paper is $\frac{1}{250}$ of an inch thick and we assume there is no measurable space between the layers, how long is the paper?[5]

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[4]Lillian R. Lieber, The Education of T. C. Mits (New York: W. W. Norton and Company, Inc., 1944), pp. 32-39

[5]Joseph De Grazia, Math is Fun (New York: Emerson Books, Inc., 1954), p. 94.

<u>Solution</u>. Imagine cutting through the paper lengthwise to the spool such that the mass of paper could be removed from the spool and laid out on a flat surface as in Figure 7.3. Now the cross section of this mass of paper is an isosceles trapezoid with one base having a length equal to the circumference of the spool and the other base equal in length to the circumference of the roll of paper. Thus one base is 4π inches and the other 16π inches in length. The height of the trapezoid is

$$\frac{1}{2}(16 - 4) \text{ or } 6 \text{ inches.}$$

The average length of a single sheet of paper in this mass is the length of the median or

$$\frac{1}{2}(16\pi + 4\pi) \text{ or } 10\pi \text{ inches.}$$

There are

$$6(250) \text{ or } 1500 \text{ sheets of paper,}$$

hence, $1500(10\pi)$ is the total length of paper contained on the spool.

$$1500(10\pi) = 15000\pi \text{ inches}$$

or $$47{,}100 \text{ inches approximately.}$$

This is approximately 3925 feet!

<u>Problems</u>.

1. If it requires $1\frac{1}{4}$ inches of paper to manufacture one cigarette, how many cigarettes could be made from one spool of paper? (Answer: 33,680.)

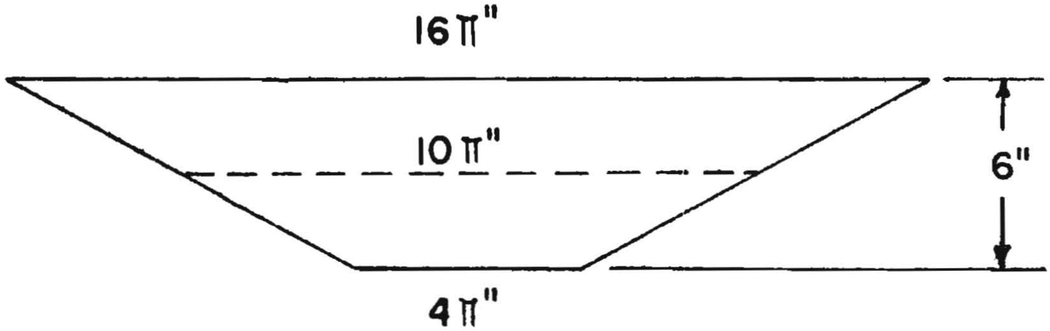

FIGURE 7.3

PAPER REMOVED FROM SPOOL

2. How many packs, if there are 20 cigarettes in a pack? (Answer: 1684.)

7.7 A circle problem. Construct a circle O having a one-inch radius and construct two diameters AB and CD perpendicular to each other. Now select any point E on the circle and construct EF parallel to CD meeting AB at F and EG ⊥ CD meeting CD at G.

How long is line GF? (See Figure 7.4.)

Solution. Many students have worked a considerable length of time on this problem without success because they became fixed in their attack on the problem. The focus of attention is the right triangle. The pythagorean theorem is a powerful tool but when that fails the able problem solver is one who is able to shift his attention.

The teacher is in a good position to help the student realize the dangers of rigidity and help him to broaden his approach to the problem. A statement such as the following could be helpful; "When you are having trouble with a problem look at it in a different manner than you have been doing. You have been thinking about triangle GOF or triangle GEF, have you tried thinking about the quadrilateral EGOF? Try it. Get in the habit of asking

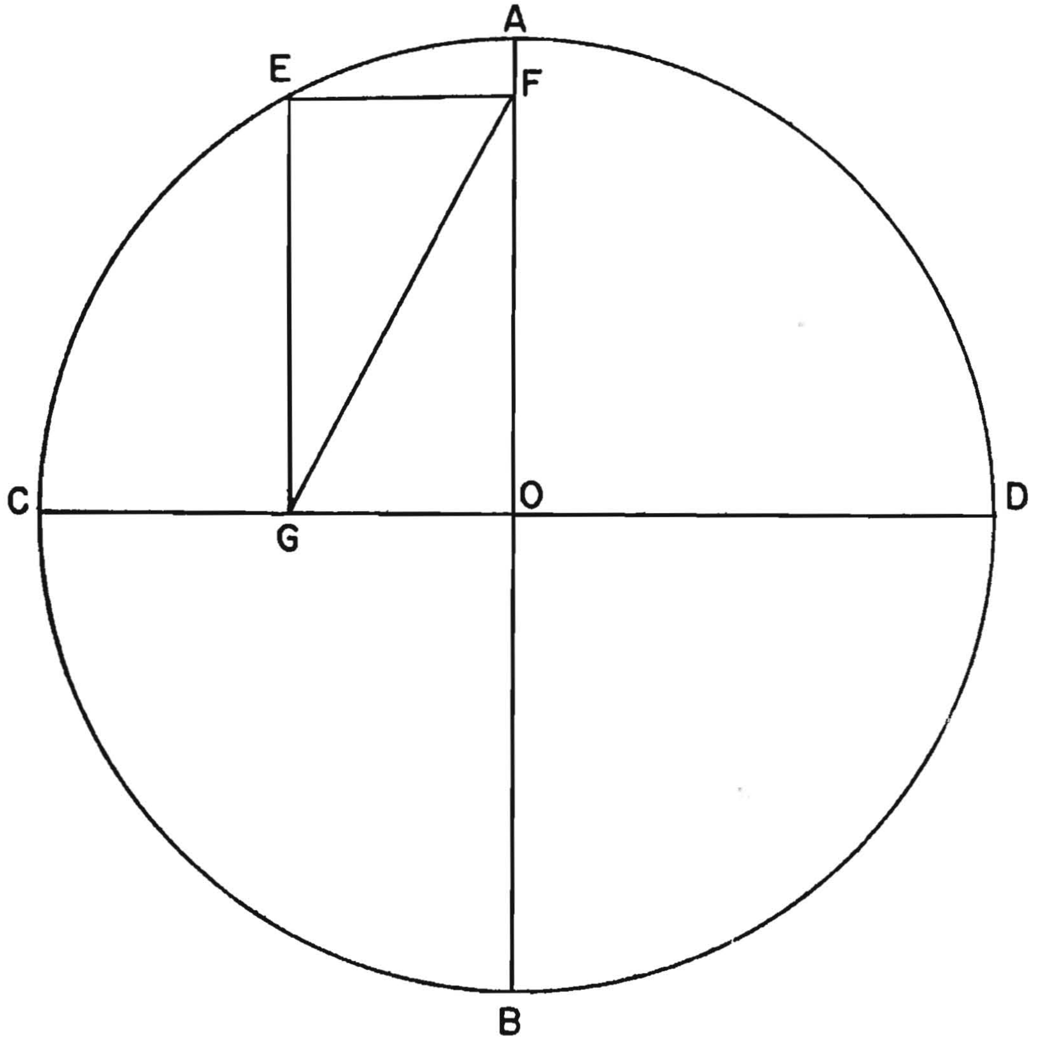

FIGURE 7.4

FIND THE LENGTH OF GF

yourself questions about the problem as you work on it."
Of course GF is one inch long since GF = OE.[6]

7.8 The grindstone problem. A couple of shopworn carpenters who had an axe to grind decided to go halvers and buy a very large grindstone. They were able to get one at a very low price, so they each invested an equal amount in it. The stone measured 5 feet and 6 inches in diameter but because of the spindle the carpenters decided that the stone would be quite useless when its diameter was reduced to 18 inches by constant wear. Having nothing better to do they further decided that one would use the stone until his share of it was worn away at which time the second carpenter would inherit his share of the stone. Now what the carpenters wanted to know was how much of the stone the first one could grind away from it and still leave an equal amount for the second one to use.[7]

Solution. In Figure 7.5 ring A represents the portion of the stone to be used by the first carpenter, ring B by the second, and circle C the remainder of the stone not

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[6] Kenneth B. Henderson and Robert E. Pingry, "Problem-Solving in Mathematics," The Learning of Mathematics--Its Theory and Practice, Twenty-first Yearbook of the National Council of Teachers of Mathematics, (Washington, D. C.: NCTM, 1953), pp. 256-57.

[7] Frederick A. Collins, Fun with Figures (New York: D. Appleton and Company, 1928), p. 22.

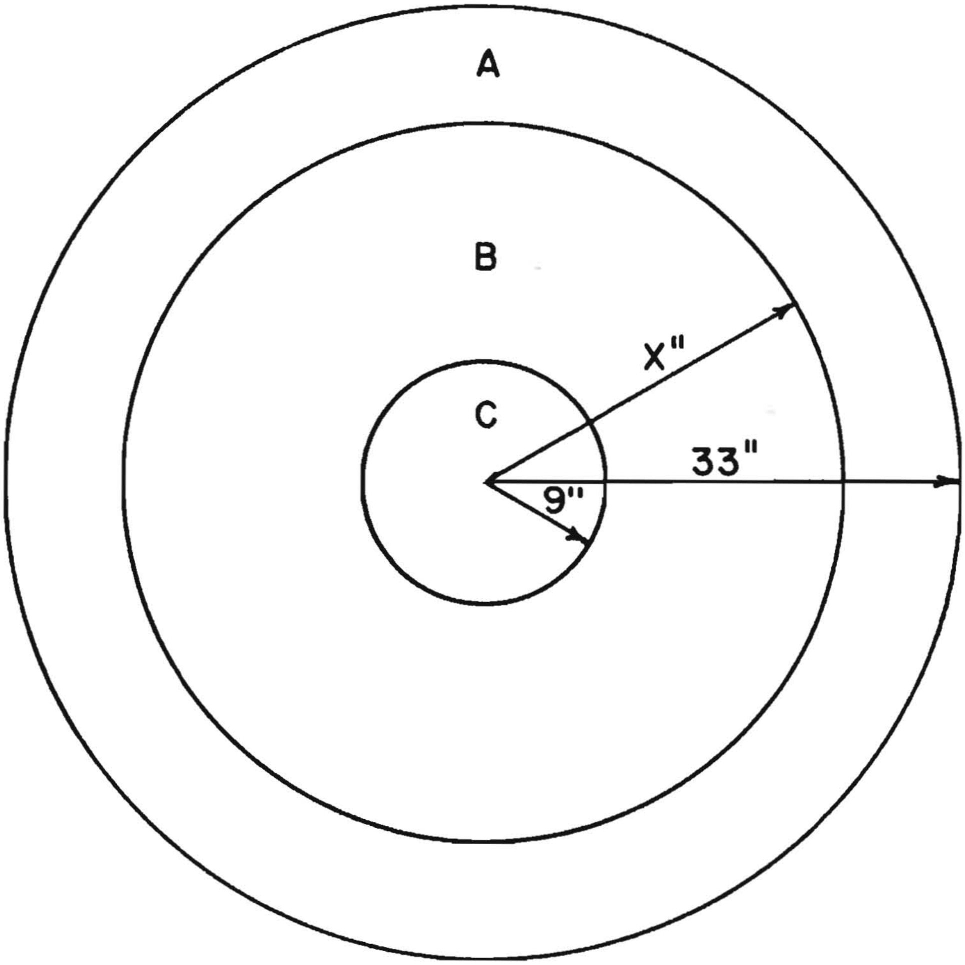

FIGURE 7.5

THE CARPENTERS GRINDSTONE

usable because of the spindle. For the carpenters to share equally in the use of the stone the area of ring A must equal the area of ring B. The area of ring A is given by $\pi(33)^2 - \pi x^2$. The area of ring B is $\pi x^2 - \pi(9)^2$. Therefore

$$\pi x^2 - \pi(9)^2 = \pi(33)^2 - \pi x^2$$

or

$$x^2 - 81 = 1089 - x^2$$

$$2x^2 = 1170$$

$$x^2 = 585$$

$$x = 24.186''$$

Since the radius of the original stone is 33 inches the first carpenter may wear away 33 inches minus 24.186 inches or 8.814 inches. Thus the first carpenter may reduce the radius by 8.8 inches resulting in a reduction of diameter by 17.6 inches or 1 foot, 5.6 inches.

7.9 A "Circular" Paradox. Problem: Consider the two equal circular disks, A and B, of Figure 7.6. If B is kept fixed and A is rolled around B without slipping, how many revolutions will A have made about its own center when it is back in its original position?

Explanation. The answer, if obtained without the aid of actual disks, is almost invariably incorrect. It is generally argued that since the circumferences are equal, and since the circumference of A is laid out once along that of B, A must make 1 revolution about its own center. But if

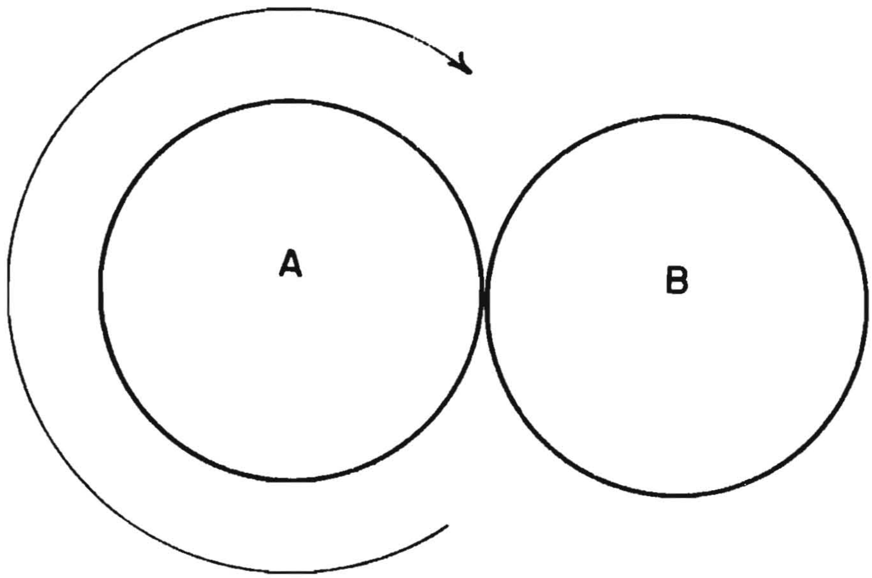

FIGURE 7.6

ROLLING A DISK ABOUT AN

EQUAL DISK

the experiment is tried with, say, two coins of the same size, it will be found that A makes 2 revolutions. This fact can be shown diagrammatically as follows:

In Figure 7.7, let P be the extreme left-hand point of A when A is in its original position. A moment's thought will make it clear that when A has completed half its circuit about B, the arc of the shaded portion of A will have been laid out along that of the shaded portion of B, and P will again be the extreme left-hand point of A. Hence A must have made 1 revolution about its own center. The same argument holds for the arcs of the unshaded portions of A and B when A has completed the second half of its circuit about B.[8]

7.10 <u>Curves of constant breadth</u>. In moving heavy objects by means of a slab and rollers, would it be possible to use rollers whose cross sections are not circles, but some other kind of curve? In other words, are circles the only curves of constant breadth? The intuitive answer is yes; the correct answer is no.

By a curve of constant breadth we shall mean exactly what the slab-and-roller idea implies. That is to say, if such a curve is placed between and in contact with two fixed

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[8]Northrop, <u>op</u>. <u>cit</u>., pp. 55-56.

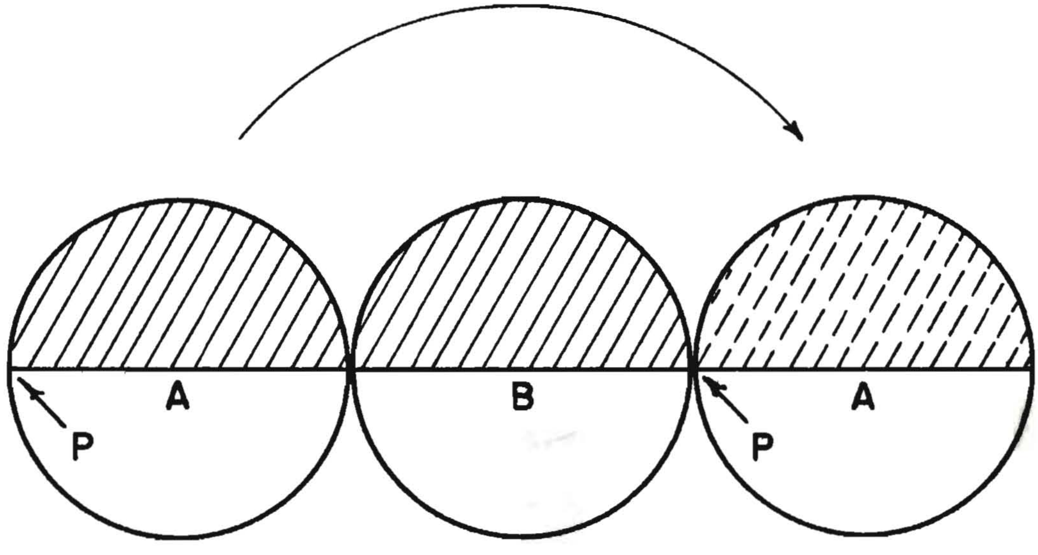

FIGURE 7.7

THE ROLLING DISK AT THE

HALF-WAY POINT

parallel lines, then it will remain in contact with the two fixed lines regardless of how it is turned.

The simplest curve of constant breadth--aside from the circle--is shown in Figure 7.8(a). To construct it, first construct the equilateral triangle ABC and denote the length of each of its sides by r. With A as center, and with radius r, draw the arc BC. With B as center, and with radius r, draw the arc CA. Finally, with C as center, and with radius r, draw the arc AB. This curve can be made smooth by prolonging the sides of the triangle any distance, say S, as in Figure 7.8(b). Here the arcs DE, FG, and HI, with centers at A, B, and C respectively, are all drawn with radius S; and the arcs EF, GH, and ID, with centers at C, A, and B respectively, are all drawn with radius r + S.

In Figure 7.9, the second of these curves is shown placed between two fixed parallel lines. It is evident from the figure that the curve will remain in contact with the two lines regardless of how it is turned, for the distance PQ between the highest and lowest points of the curve is always the sum of the two constant radii, S and r + S, and so is always the same.

Other curves of constant breadth may be constructed using regular polygons as base figures. The student may well be inclined toward experimenting with such base figures as the pentagon and heptagon.

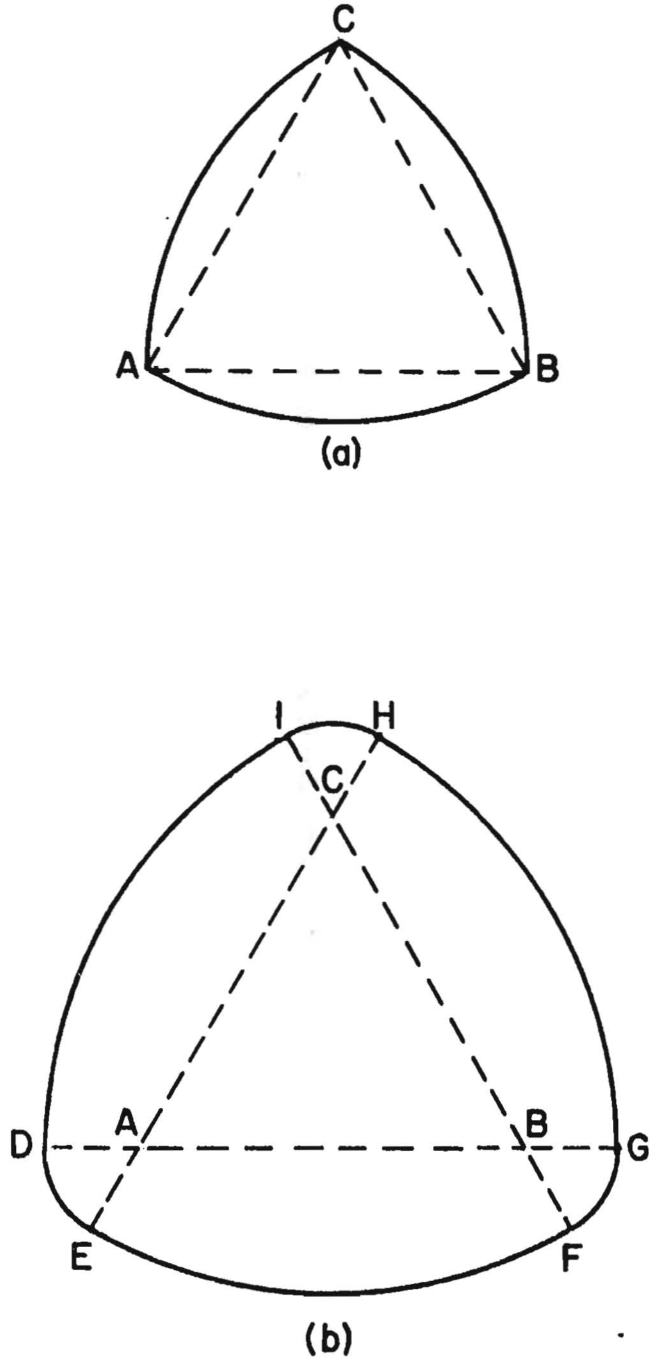

FIGURE 7.8

CURVES OF CONSTANT BREADTH

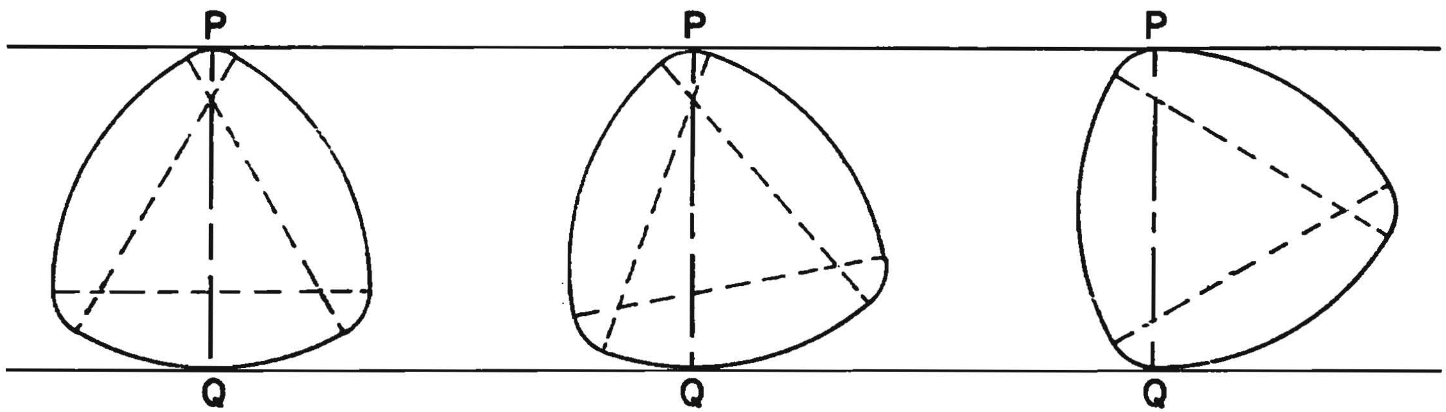

FIGURE 7.9

PARALLELISM OF A CURVE OF

CONSTANT BREADTH

It is well to note that although any roller whose cross section is a curve of constant breadth can be used in place of a circular roller for the moving of objects on a slab, a wheel in the shape of either of the curves in Figure 7.8 could never be used in place of a circular cart wheel or a circular gear. For these curves have no real center--no point, that is, which is equidistant from all points on the curve. The circle is the only curve which has this particular property.

Curves of constant breadth need not be regular in shape, as were the two just examined. The irregular curve of Figure 7.10 is constructed as follows: With A as center, and with any radius AB, swing arc BC. With C as center, and with the same radius (the radius remains constant throughout), swing AD. With D as center, swing CE. With E as center, swing DF. With F as center, swing EG. With B as center, swing AG. (G is the point of intersection of the last two arcs.) Finally, with G as center, swing FB. This curve has corner points which can be rounded off by extending the lines AB, AC, and the like, as was done in the transition from diagrams (a) to (b) in Figure 7.8.

There should be wooden models similar to the geometric shapes in Figure 7.8 and thick enough to roll along on edge. Pupils should be asked whether any other

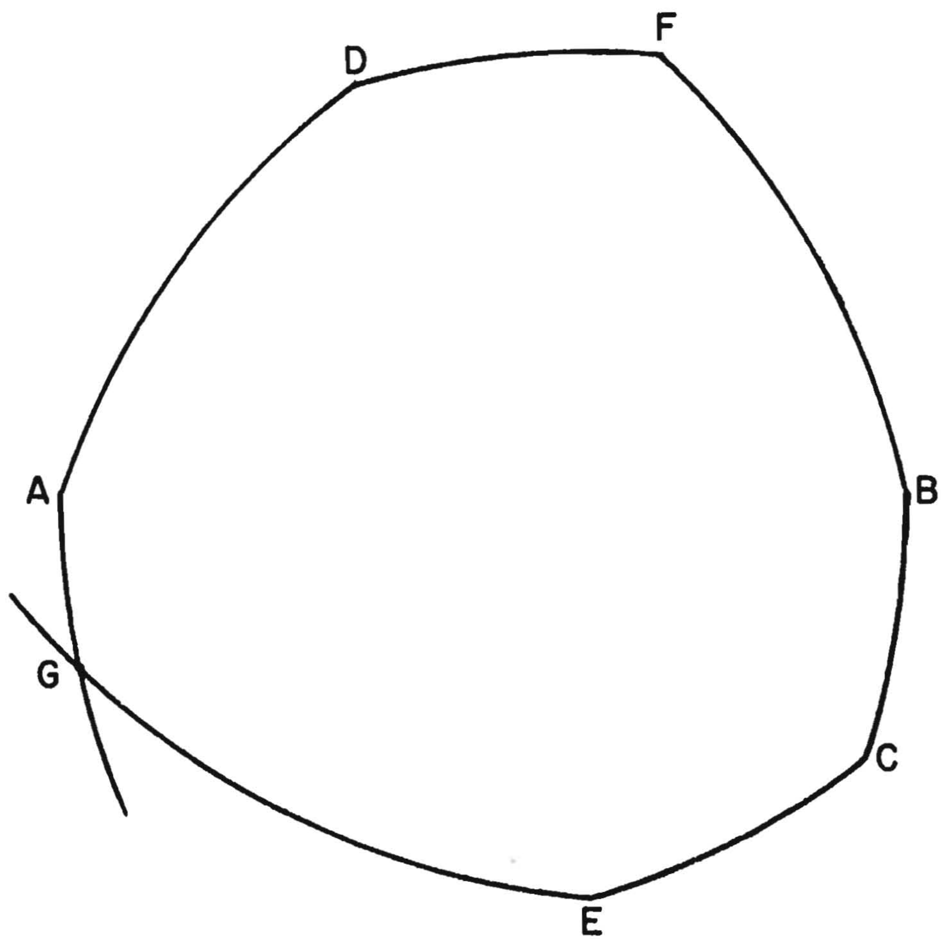

FIGURE 7.10

AN IRREGULAR CURVE OF

CONSTANT BREADTH

shape except a circle will roll along a line so that the highest point is always the same distance from the line. Their mental imagery usually says no. Even when the shapes are shown to them they doubt it. Actual experiment is usually necessary to convince them. The theory by which these figures are constructed might be given to them, or they could be asked to work it out for themselves.[9]

7.11 Slab and roller problem. Difficulties are generally encountered in the problem of a slab supported by rollers--a device frequently used in moving safes, houses, and other heavy objects.

If the circumference of each roller in Figure 7.11 is 1 foot, how far forward will the slab have moved when the rollers have made 1 revolution?

Explanation. The usual argument is to the effect that the distance the slab moves must be equal to the circumference of the rollers, or 1 foot. The correct answer is not 1 foot, but 2 feet.

Suppose we resolve the motion into two parts. First think of the rollers lifted off the ground and supported at their centers. Then if the centers remain stationary, 1 revolution of the rollers will move the slab forward 1 foot.

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[9] Northrop, *op. cit.*, pp. 57-59.

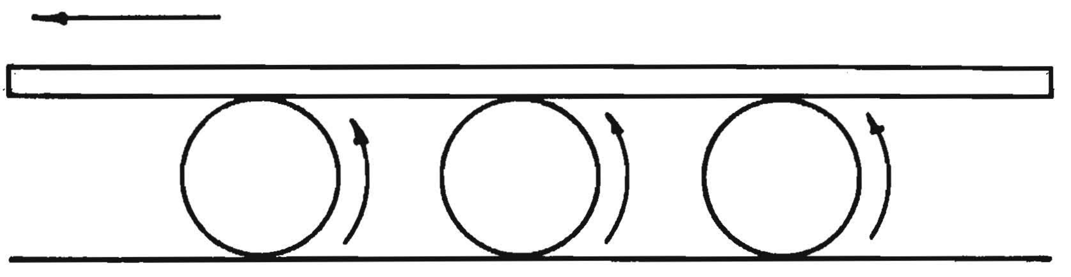

FIGURE 7.11

A SLAB ON ROLLERS

Next think of the rollers on the ground and without the slab. Then 1 revolution will carry the centers of the rollers forward 1 foot. If we now combine these two motions, it should be evident that 1 revolution of the rollers will carry the slab forward a distance of 2 feet.[10]

7.12 A simple addition problem. An interesting property of a finite arithmetic progression is the fact that we can obtain the sum of its numbers without recourse to addition of all of them. Consider the following indicated sum where the terms establish a sequence known as an arithmetic progression. Suppose we wish to add the first fifteen numbers.

$$\begin{array}{ccccccccccccccc} 1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +10 & +11 & +12 & +13 & +14 & +15 \\ \underline{15} & +\underline{14} & +\underline{13} & +\underline{12} & +\underline{11} & +\underline{10} & +\underline{9} & +\underline{8} & +\underline{7} & +\underline{6} & +\underline{5} & +\underline{4} & +\underline{3} & +\underline{2} & +\underline{1} \\ 16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 & +16 \end{array}$$

Below the sequence of the first fifteen numbers write the same sequence, term by term, but in reversed order, then add the terms vertically. Note that in each case the sum is 16. Now there are fifteen such sums, hence the sum of the two sequences is $15 \cdot 16 = 240$. But we added two sequences of the first fifteen numbers. Therefore the sum of one such sequence is one-half of 240, or 120.

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[10]Northrop, *op. cit.*, p. 47.

This procedure may be applied to the addition of any arithmetic progression. Note that the addition of the first fifteen numbers was reduced to adding the first and the last term, and the sum thus obtained was multiplied by the number of terms in the progression. Finally the product was divided by 2.

Problem 1. Find the sum of all the whole numbers from 40 to 100 inclusive by the above method.

Problem 2. Find the sum of the even numbers from 2 to 100 inclusive.[11]

7.13 An experimental paradox. A well known paradox involves the dissection and rearrangement of a figure. It is a good example of the pitfalls of "experimental geometry," a topic generally discussed in the early stages of any course in plane geometry. However with a little preparation the presentation of this experiment to a general mathematics or algebra class would certainly whet the curiosity of some.

The student is shown how to deduce experimentally the fact that the sum of the angles of any triangle is a straight angle, or 180°. To do so, he makes a triangle of paper or cardboard, cuts off the three angles, and

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[11] Aaron Bakst, Mathematics--Its Magic and Mastery (New York: D. Van Nostrand Company, Inc., 1941), p. 234.

rearranges them as shown in Figure 7.12. Let us see to what sort of contradiction this method of proof, not backed up by sound logical argument, can lead.

Suppose we take a square piece of paper and divide it into 64 small squares, as in a chessboard, or use rectangular coordinate graph paper and outline a square 8 units by 8 units. We then cut it into two triangles and two trapezoids in the manner indicated in Figure 7.13(a) and rearrange the parts as in Figure 7.13(b).

Now the resulting rectangle has sides which are respectively 5 units and 13 units long, so that its area is $5 \cdot 13 = 65$ square units, whereas the area of the original figure was $8 \cdot 8 = 64$ square units. Where did that additional square unit come from?

Explanation. The truth is that the edges of the parts 1, 2, 3, and 4 do not actually coincide along the diagonal PQ, but form a parallelogram PSQR which is shown in exaggerated proportions in Figure 7.14. The area of this parallelogram is the elusive square unit. The angle SPR is so small that the parallelogram is never noticed unless the cutting and rearrangement is done with great care. Probably the simplest explanation would be to show that the slope of PS is $\frac{2}{5}$ and the slope of SQ is $\frac{3}{8}$. Certainly $\frac{15}{40}$ and $\frac{16}{40}$ are not the same. The same relation can of course be shown

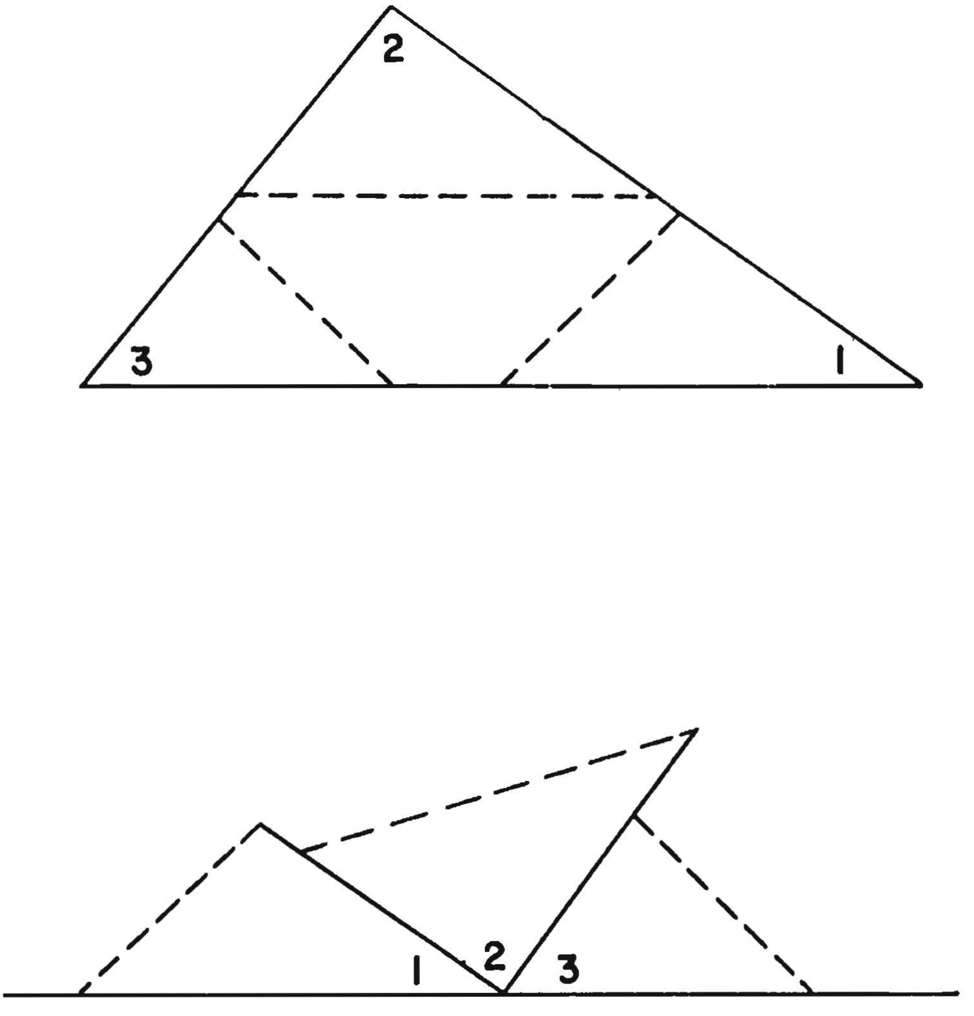

FIGURE 7.12

THE SUM OF THE ANGLES OF

A TRIANGLE IS 180°

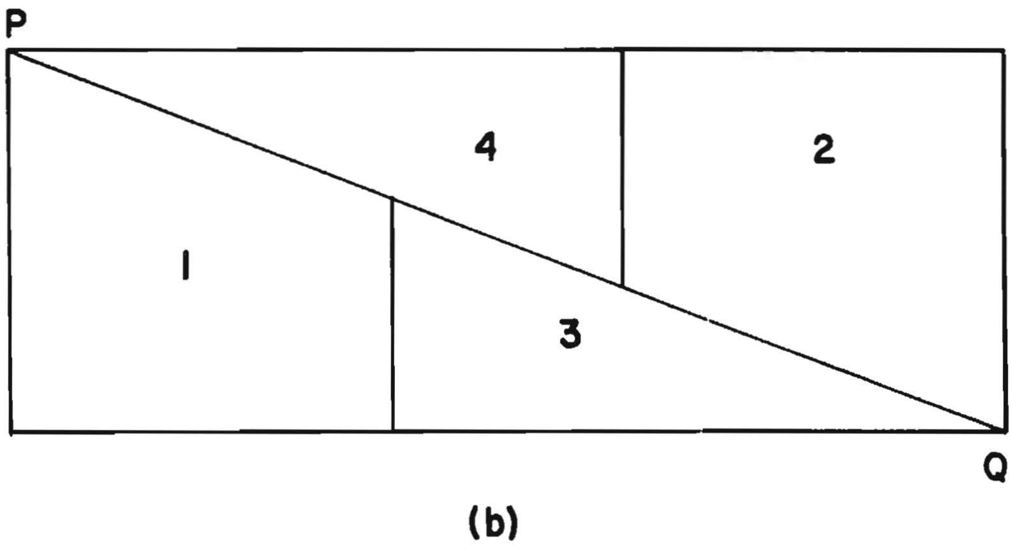

FIGURE 7.13

THE DISSECTION AND REARRANGEMENT

OF A SQUARE

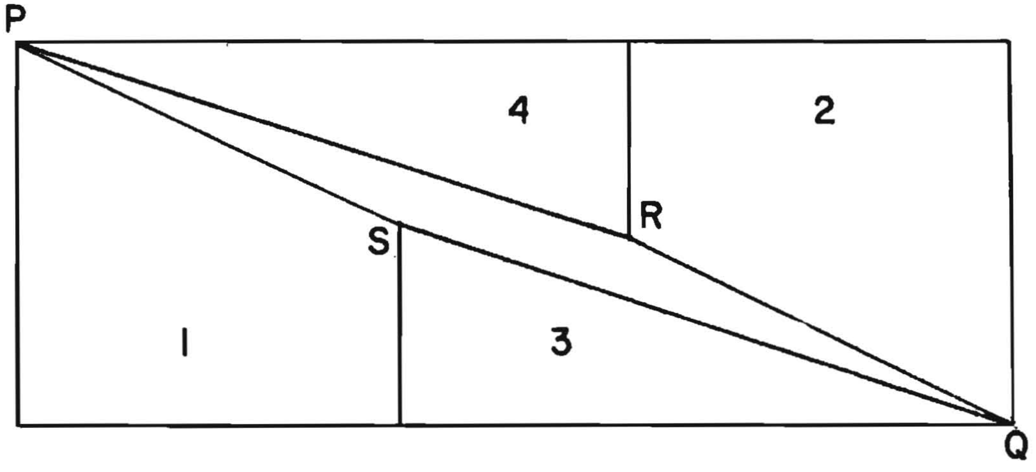

FIGURE 7.14

THE FALLACY

for QR and RP. This problem should illustrate very well that because physical materials have been drawn, cut, and rearranged one should not feel that this is convincing proof.[12]

7.14 <u>The persistent 9</u>. When any two-digit number is written in reverse order, the new number is also a two-digit number. We shall consider here that the reverse of a two-digit number ending in zero is also a two-digit number. For example, the reverse of 20, which is 02 shall be considered a two-digit number. If the digits in a two-digit number are different, the difference between the number and its "reverse" possesses an unusual property; it is always divisible by 9. For example, the reverse of 74 is 47, and 74 - 47 = 27. The reverse of 83 is 38, and 83 - 38 = 45. The reverse of 20 is 02, and 20 - 02 = 18. All of these differences, 27, 45, and 18, can be divided by 9 with no remainder. Would it be possible to show that this remarkable property is always true without trying <u>all</u> two-digit numbers.

<u>Proof</u>. This fact may be verified by writing a general expression for a two-digit number such as $10a + b$ where $a \neq b$. The reverse of this number is then written as

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[12]Northrop, <u>op</u>. <u>cit</u>., pp. 49-50.

$10b + a$. The difference of these two numbers is $(10a + b) - (10b + a) = 10a + b - 10b - a$ and is equal to

$$9a - 9b = 9(a - b).$$

Since 9 is a factor, this result shows that the difference between a two-digit number and its reverse is always divisible with no remainder by 9.

Question. Is it possible that the difference between a two-digit number and its reverse be greater than 81.

Answer. No! Since each difference must be divisible by 9 the quotient so obtained would have to be $(a - b)$. But $(a - b)$ must be equal to or less than 9 since 9 is the largest possible value given $a = 9$ and $b = 0$. Hence the maximum value of the difference $9(a - b)$ is $9 \cdot 9$ or 81. The only possible differences may be

$1 \cdot 9 = 9$	$4 \cdot 9 = 36$	$7 \cdot 9 = 63$
$2 \cdot 9 = 18$	$5 \cdot 9 = 45$	$8 \cdot 9 = 72$
$3 \cdot 9 = 27$	$6 \cdot 9 = 54$	$9 \cdot 9 = 81.$

The sum of any such difference and its "reverse" is always equal to 99. Thus, $27 + 72 = 99$, $45 + 54 = 99$, and $72 + 27 = 99$. Can you prove this statement to be true?

Proof. Let $10a + b$ be the original two digit number, where a is greater than b, and subtract from it $10b + a$. Thus

	$10a + b$
Subtract	$\underline{10b + a}$
Difference	$(10\ a - 10 - 10b) + (10 + b - a)$
or	$10(a - b - 1) + (10 + b - a)$
reverse	$\underline{10(10 + b - a) + (a - b - 1)}$

add $10a - 10b - 10 + 100 + 10b - 10a + 10 + b - a + a - b - 1$ which when simplified is 99.

<u>Problem</u>. When any three-digit number is written in reverse order, the new number is also a three-digit number. If at least two of the digits are different, the difference between the number and its "reverse" is always evenly divisible by 99. For example, the reverse of 635 is 536, and $635 - 536 = 99$. The reverse of 841 is 148 and $841 - 148 = 693$. The reverse of 512 is 215, and $512 - 215 = 297$. Each of these differences is divisible by 99 with no remainder.

<u>Proof</u>. This fact may be verified if we write a general expression for a three-digit number as $100a + 10b + c$. (Assume $a \neq c$ and for convenience a greater than c.) The reverse of this number is then written as $100c + 10b + a$. The difference of these two numbers is

$$(100a + 10b + c) - (100c + 10b + a)$$

$$= 100a + 10b + c - 100c - 10b - a$$

and is equal to

$$99a - 99c = 99(a - c).$$

This result indicates that the difference between any three-digit number and its reverse is always divisible by 99 with no remainder.

Question 1. What other numbers would this difference be evenly divisible by other than 99. (Answer: 3, 9, 11, 33.)

Question 2. What is the largest number value for $99(a - c)$. (Answer: 891.)

Why: $a - c$ is greatest in value when $a = 9$ and $c = 0$. Therefore $99(9 - 0) = 99 \cdot 9$ or 891.

The difference between a three-digit number and its reverse cannot ever be greater than 891. In other words, the differences may be

$1 \cdot 99 = 099$	$4 \cdot 99 = 396$	$7 \cdot 99 = 693$
$2 \cdot 99 = 198$	$5 \cdot 99 = 495$	$8 \cdot 99 = 792$
$3 \cdot 99 = 297$	$6 \cdot 99 = 594$	$9 \cdot 99 = 891.$

The sum of any such difference and its "reverse" is always equal to 1,089. Thus, $297 + 792 = 1089$, $495 + 594 = 1089$. Notice that $297 + 792 = 3 \cdot 99 + 8 \cdot 99 = 11 \cdot 99$

and $495 + 594 = 5 \cdot 99 + 6 \cdot 99 = 11 \cdot 99$. In other words, the sum of a difference and its "reverse" is always equal to $11 \cdot 99 = 1,089$.

Can this last statement be proven algebraically?

The foregoing problems might well be presented to a class originally as a trick. By having several students select a two-digit or three-digit number and perform the several operations on them and then announce to their surprise that the result is 99 or 1089 as the case might be.[13]

7.15 Why 1089. Here is an interesting exercise. Think of any number comprised of three digits; in order to avoid negatives it is preferable to make the hundreds digit larger than the ones digit. Thus, selecting three numbers at random, such as 584, 753 and 872, we carry out successively the following operations--reverse and subtract, again reverse and add.

	584	753	872
Reverse	485	357	278
Remainder or Difference	099	396	594
Reverse	990	639	495
Sum	1089	1089	1089

No matter what digits are selected the result is always the same. Indeed it is not difficult to prove algebraically that such must be the case.

Proof. Let the three digits be a, b, and c, respectively; then, carrying out the above operations,

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[13] Aaron Bakst, Mathematical Puzzles and Pastimes (New York: D. Van Nostrand Company, Inc., 1954), pp. 171-172.

	a	b	c	
Reverse	c	b	a	
Difference	$a - c - 1$	9	$10 + c - a$	(Why?)
Reverse	$10 + c - a$	9	$a - c - 1$	
Sum	9	(18)	9	
that is	10	8	9	

Obviously this must be true whatever digits are assigned to a, b, and c, provided only that a is not equal to c.[14] Why?

7.16 A division fallacy.

Let	$a = b;\ a \neq 0$
Multiply both sides by a:	$a^2 = ab$
Subtract b^2 from both sides:	$a^2 - b^2 = ab - b^2$
Factor:	$(a + b)(a - b) = b(a - b)$
Divide both sides by $(a - b)$:	$a + b = b$
But $a = b$; therefore	$2b = b$
Divide both sides by b:	$2 = 1$

Explanation. Of course the trouble is between the fourth and fifth lines. Since $a = b$, the quantity $(a - b)$ must equal zero. The fourth line is correct, the fifth line is not. We have broken what R. P. Agnew[15] calls the fundamental commandment of mathematics: Thou Shalt Not Divide By

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[14] J. Newton Friend, Numbers: Fun and Facts (New York: Charles Scribner's Sons, 1954), pp. 65-66.

[15] R. P. Agnew, Differential Equations (New York: McGraw-Hill, 1942), p. 35.

zero. Division by zero is against the rules of the game, and whenever you try it you will get something meaningless like 2 = 1.[16]

7.17 A square root fallacy.

$$(x + 1)^2 = x^2 + 2x + 1$$

$$(x + 1)^2 - (2x + 1) = x^2$$

$$(x+1)^2 - (2x+1) - x(2x+1) = x^2 - x(2x + 1)$$

$$(x+1)^2-(x+1)(2x+1) + \tfrac{1}{4}(2x+1)^2 = x^2 - x(2x+1) + \tfrac{1}{4}(2x+1)^2$$

$$[(x+1) - \tfrac{1}{2}(2x+1)]^2 = [x - \tfrac{1}{2}(2x+1)]^2$$

$$(x+1) - \tfrac{1}{2}(2x+1) = x - \tfrac{1}{2}(2x+1)$$

$$x + 1 = x$$

$$1 = 0.$$ [17]

Explanation. The line with the square brackets says $[\frac{1}{2}]^2 = [-\frac{1}{2}]^2$. This is correct. However, taking the square root of both sides leaves $\frac{1}{2} = -\frac{1}{2}$, which is most certainly not correct. You cannot take the square root of both sides of an equation without first inspecting for the possibility of sign trouble.[18]

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[16] C. Stanley Ogilvy, Through the Mathescope (New York: Oxford University Press, 1956), pp. 38-39.

[17] C. Stanley Ogilvy, Through the Mathescope (New York: Oxford University Press, 1956), p. 39.

[18] Ibid., p. 139.

7.18 <u>A law of multiplication</u>. Why does negative two times negative five give positive ten? Attempts at justifying this rule include illustrations such as the following. It costs the state $5 per day to feed, house, and clothe each prisoner at a penitentiary. Two convicts escape. Hence the prison counts -5 dollars, times -2 prisoners, and shows a profit of +10 dollars on its books for every day the fugitives are at large.

Such an illustration proves nothing. The true state of affairs is not in the least mysterious. If a certain law of multiplication of positive numbers (the Distributive Law) is to hold for negative numbers, too, then the rest follows logically.

$$\begin{aligned}(-2)(-5) &= (-2)(-5) + (0)(+5)\\ &= (-2)(-5) + (-2+2)(+5)\\ &= (-2)(-5) + (-2)(+5) + (+2)(+5)\\ &= (-2)(-5+5) + (+2)(+5)\\ &= (-2)(0) + (+2)(+5)\\ &= (+2)(+5)\\ &= +10\end{aligned}$$

Any numbers or letters can be used in place of 2's and 5's, to show that

$$(-a)(-b) = (+a)(+b).^{19}$$

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[19] C. Stanley Ogilvy, <u>Through the Mathescope</u> (New York: Oxford University Press, 1956), p. 41.

7.19 Square roots by successive approximation. Square roots of numbers are obtained either from tables or there are special methods for their extraction by calculation.

The usual method of extraction as found in algebra textbooks is lengthy, little understood, (even by the teacher) and requires much numerical work. Now we may learn a method which is not cumbersome, yet yields satisfactory results.

This method utilizes one important idea employed in mathematics: If we have a fraction (or a number very small in comparison with some other number), the square of the fraction is so small that it may be discarded in computation. For example, suppose we have a decimal fraction 0.01 which is part of a number, say 4.21. This number may be written as 4.2 + 0.01. Now if we square this number, we have

$$4.21^2 = (4.2 + 0.01)^2 = 4.2^2 + 2(4.2)\,(0.01) + 0.01^2$$

or
$$4.21^2 = 17.64 + 0.084 + 0.0001.$$

The square of 0.01, which is 0.0001, may be disregarded and discarded; if 4.21 is correct to three significant digits, its square will also be correct to three significant digits without the 0.0001, hence 0.0001 or 0.01^2 is of no value to us.

With this in mind we may proceed with the extraction of square roots. Suppose we wish to calculate $\sqrt{14}$. (To check on our method obtain its value as given in tables of square roots and have that $\sqrt{14} = 3.742$.) We know that $\sqrt{14}$ is greater than 3 since $3^2 = 9$, and is less than 4 since $4^2 = 16$. Therefore let

$$\sqrt{14} = 3 + x$$

where x is some fraction. Square both sides of the equation. We then have

$$14 = 3^2 + 2\cdot 3x + x^2$$

or

$$14 = 9 + 6x + x^2.$$

Now since x is a fraction, x^2 is also a fraction, but much smaller than x. We therefore discard x^2 and have

$$14 \doteq 9 + 6x.$$

Solving this equation for x we have that

$$14 - 9 \doteq 6x, \text{ or } 6x \doteq 5 \text{ and}$$

$$x \doteq \frac{5}{6}.$$

From this we have that

$$\sqrt{14} \doteq 3\tfrac{5}{6}$$

which is only an approximate value. In decimals then

$$\sqrt{14} \doteq 3.83\cdots$$

Since we have one approximation to the value of $\sqrt{14}$, we may use this as a basis and apply the same method once more. We may say then that

$$\sqrt{14} = 3\tfrac{5}{6} + y$$

where y may be either positive or negative. Again squaring both sides of the equation we have

$$14 = (\tfrac{23}{6})^2 + 2(\tfrac{23}{6})y + y^2.$$

Since y is a fraction and we discard its square and have the equation

$$14 \doteq \frac{529}{36} + \frac{23}{3}y$$

and from this, by solving for y, we have that

$$y \doteq \frac{3}{23}(\frac{504-529}{36})$$

or $y \doteq \frac{-25}{276}$, or $y \doteq -\frac{25}{275}$ or $y \doteq -\frac{1}{11}$. Then

$$\sqrt{14} \doteq 3\tfrac{5}{6} - \frac{1}{11} \doteq 3.83 - 0.09 \doteq 3.74.$$

This is a second approximation, and we may proceed with another calculation to obtain a third approximation. We write

$$\sqrt{14} = 3.74 + z$$

where z may be either positive or negative. Squaring both sides of the equation we have

$$14 = 3.74^2 + 7.58z + z^2.$$

We again discard z^2 and have

$$14 \doteq 13.9876 + 7.58z$$

and

$$7.58z \doteq 0.0124.$$

From this we obtain that

$$z \doteq 0.0016 \text{ or,}$$

$$z \doteq 0.002.$$

Then

$$\sqrt{14} \doteq 3.74 + 0.002$$

or

$$\sqrt{14} \doteq 3.742$$

and this checks with the value of the square root as obtained from a table.[20] Thus, starting with the one significant digit, 3, four significant digits have been obtained since the number of correct digits is virtually doubled at each stage of the approximation process.[21]

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[20]Aaron Bakst, Mathematics--Its Magic and Mastery (New York: D. Van Nostrand Company, Inc., 1941), pp. 218-20.

[21]Yudell L. Luke, "Numerical Analysis and High School Mathematics," The Mathematics Teacher, November, 1957. pp. 507-12.

CHAPTER VIII

SUMMARY AND CONCLUSION

8.1 Summary. The content of this thesis was organized for the teacher of ninth grade mathematics with the purpose of supplementing text material in the area of both general mathematics and algebra. The criterion for the selection of materials was three-fold; (1) the problems and demonstrations should be clearly mathematical in principle and not merely a collection of puzzles, games, or situations involving a play on words, (2) the nature of the problems should be that of the unusual or out of the ordinary so as to enhance the opportunity to create an atmosphere of curiosity and interest, and (3) the work should be commensurate to the maturity and intellectual level of the ninth grade mathematics student.

In Chapter I, section 1.7, suggestions have been presented to the teacher for the use of this material. Experience may indicate numerous variations of the methods suggested and flexibility in presentation may well be desirable and necessary in many teaching situations.

Chapters II through V, on systems of numeration, are included in this work to demonstrate the basic structure of the decimal numeration system. The use of any or all of these chapters would require considerable class time and would likely be most effective as an integrated unit of study. The basic concepts of the fundamental operations of arithmetic are, however, dominant in this work and the time required for a thorough treatment of this unit, whether in general mathematics or algebra, may well be justified. The material presented in this unit is more comprehensive than that found in any one reference used by the author.

The problems in Chapters VI and VII should introduce to the student a variety of mathematical methods and principles and tend to stimulate both curiosity and interest in mathematics. This material is primarily a representative sample of problems judged by the author to satisfy the criterion established in the early part of this chapter.

8.2 Conclusion. The teacher of mathematics in the secondary school should be aware of the potential for stimulating intellectual curiosity and interest in mathematics through the use of supplemental material. The problem situations presented in this thesis need not be restricted to use at only the ninth grade level but may be used to a similar advantage at higher grade levels in the secondary school.

The author suggests that secondary school libraries should contain at least several books in the area of recreational mathematics and that mathematics teachers should encourage the use of these books. Several references listed in the bibliography, identified by an asterisk, were judged by the author as very acceptable and are recommended for the high school library.

Finally, it is strongly recommended that mathematics teachers use to advantage The National Council of Teachers of Mathematics publication The Mathematics Teacher as a means of stimulating their own curiosity and interest and as a source of mathematics often applicable in the modern mathematics classroom.

8.3 Suggestion for further study. The material in this thesis has not as yet been used in the classroom by the author. A suggestion is made, therefore, that further study could consist of accumulating data by experimental means in an attempt to evaluate the effectiveness of such material as a motivating factor in ninth grade mathematics.

BIBLIOGRAPHY

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A. BOOKS

*Bakst, Aaron. Mathematical Puzzles and Pastimes. New York: D. Van Nostrand Company, Inc., 1954.

*Bakst, Aaron. Mathematics--Its Magic and Mastery. New York: D. Van Nostrand Company, Inc., 1941.

Ball, Walter W. Mathematical Recreations and Essays. New York: The Macmillan Company, 1956.

Ball, W. W. Rouse. Mathematical Recreations and Essays. New York: The Macmillan Company, 1905.

Butler, Charles H. and F. Lynwood Wren. The Teaching of Secondary Mathematics. Second edition. New York: McGraw-Hill Book Company, Inc., 1951.

Collins, A. Frederick. Fun with Figures. New York: D. Appleton and Company, 1928.

DeGrazia, Joseph. Math is Fun. New York: Emerson Books, Inc., 1954.

Friend, John N. Numbers Fun and Facts. New York: Charles Scribner's Sons, 1954.

*Gamow, George. One Two Three···Infinity. New York: The Viking Press, 1957.

Harkin, Duncan. Fundamental Mathematics. New York: Prentice-Hall, Inc., 1941.

Kemeny, John G., J. Laurie Snell, and Gerald L. Thompson. Introduction to Finite Mathematics. Englewood Cliffs, N. J.: Prentice-Hall, Inc., 1957.

*Lieber, Lillian R. The Education of T. C. Mits. New York: W. W. Norton and Company, Inc., 1944.

Meyer, Jerome S. Fun With Mathematics. New York: The World Publishing Company, 1952.

Morton, Robert, _et. al_. _Making Sure of Arithmetic_. Teachers Edition. Morristown, New Jersey: Silver Burdett Company, 1958.

☆Newman, James R. (ed.). _The World of Mathematics_. 4 vols. New York: Simon and Schuster, 1956.

☆Northrop, Eugene P. _Riddles in Mathematics_. New York: D. Van Nostrand Company, Inc., 1944.

☆Ogilvy, Stanley C. _Through the Mathescope_. New York: Oxford University Press, 1956.

Polya, G. _How to Solve It_. Garden City, New York: Doubleday and Company, Inc., 1957.

Reid, Constance. _From Zero to Infinity_. New York: Thomas Y. Crowell Company, 1955.

Richardson, Moses. _Fundamentals of Mathematics_. Revised edition. New York: The Macmillan Company, 1958.

Steinhaus, H. _Mathematical Snapshots_. New York: G. E. Stechert and Company, ____.

Welchons, A. M., W. R. Krickenberger, and Helen R. Pearson. _Algebra--Book Two_. Revised edition. New York: Ginn and Company, 1957.

B. PUBLICATIONS OF LEARNED ORGANIZATIONS

National Council of Teachers of Mathematics. _The Learning of Mathematics--Its Theory and Practice_. Twenty-first Yearbook of the National Council of Teachers of Mathematics. Washington, D. C., 1953.

______. _The Growth of Mathematical Ideas--Grades K Through 12_. Twenty-fourth Yearbook of the National Council of Teachers of Mathematics. Washington, D. C., 1959.

C. PERIODICALS

Bakst, Aaron. "Mathematical Recreations," *The Mathematics Teacher*, March, 1953.

Luke, Yudell L. "Numerical Analysis and High School Mathematics," *The Mathematics Teacher*, November, 1957.