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# A PROBLEM IN CELESTIAL MECHANICS

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by

Dean Arvin Marsh

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# CHAPTER I

# INTRODUCTION

It is the purpose of this paper to give the reader a general knowledge of celestial mechanics and to acquaint the reader with a specific problem in this field. The reader must have a strong undergraduate background in mathematics and a workable knowledge of vectors.

## I. THE PROBLEM

This specific problem dealt with an artificial satellite and the action, or relative direction of motion, of an object released from the satellite. The object released can be thought of as an astronaut. This brought the problem closer to the recent problems of the Russian and American astronauts' walk in space.

With the examination of the equations of satellite motion it was possible to predict the probable path of the released object. Also, with slight alteration of a radius vector the predictions of changes in related orbital elements could be obtained.

Celestial mechanics is a far-reaching and very complex field of astronomy. Only the basic concepts of celestial mechanics were considered. Many things such as orbit perdurbations, orbit improvements, and orbit determinations, to name a few, were not considered because of their complexity and irrelevance. Therefore, some formulas and ideas must be accepted by the reader.

If completely solved, the problem should be verified by practical applications. The results of the recent experiments in space will give this verification.

A reference frame is very important in locating objects in space. Until the vast space has been examined, awareness of the relevance of everyday words such as "velocity," "distance," and "size" is inconceivable. To expound on this, an observer on a railroad platform was watching a passing train. Assume, in this hypothetical case, that there were two boys on the train playing catch with a ball. The train was moving in a direction perpendicular to the observer. Boy number one threw the ball in the same direction as the motion of the train. The ball's velocity, as noted by the observer on the platform, was larger than the velocity of the ball as seen by the boys on the train. The question is then, which velocity was the correct velocity? Was it the velocity as seen by the boys or by the observer? Maybe the true velocity was with respect to the North Pole of the earth, the moon, the sun, or even the center of the galaxy. To examine all of the possible velocities of an artificial satellite would be very impractical. Therefore, a few of the useful coordinate systems or reference frames used in celestial mechanics have been explained.

After examining the coordinate systems, the laws which govern the path of the satellite were derived. The sun, the moon, the earth, the satellite all have a gravitational attraction for one another. However, only the attracting forces of the satellite and the earth were considered.

The problem had many compromises or simplifications, such as neglecting the gravitational attraction of the object released on the satellite. This assumption is justified if the mass of the object is small relative to that of the satellite. A velocity has been given to the released object in the nature of v'. The relative position of the object with respect to the satellite in the orbit was found in the solution. Once into the problem the reference frame will be neglected and only the position in the orbit will be considered.

## II. HISTORY

Astronomy being the oldest of the sciences dates back more than 2000 years. Then, the main concern was to define our universe in terms of some physical idea. The Ptolemaic theory was the first and most lasting theory of that day. The Ptolemaic theory stated that the earth was the center of the universe with all other heavenly bodies travelling around the earth in very complex paths called epicycles. The Ptolemaic earth-centered universe was in complete agreement with the doctrine of the church and therefore, widely accepted. The theory was very lasting because of this church agreement and because there was nothing to disprove it. The observations of the day were not accurate enough to confirm or disprove the theory.

In the middle of the 16th century, Copernicus unlocked the key to the physical system of the universe. Still, his system was in no better agreement with the observations than the Ptolemaic system. At the time of Copernicus, and before, it was believed that the true and only curved path was a circle. In this new system the path of the heavenly bodies were perfect circles.

Using a lifetime of accurate observations made by Tycho Brahe, Kepler achieved success in explaining the true character of the motion of planets.[1] Finally, with the refinement of Kepler's laws of planetary

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[1] A. D. Dubyago, The Determination of Orbits (New York: Macmillan Company, 1961), pp. 4-6.

motion and the development by Newton of the law of universal gravitation, the period of practical astronomy ended and theoretical astronomy began.[2]

The discovery of new planets increased the desire to solve the general problem of the determination of orbits. Working with this problem were men such as Euler, Lambert, Lagrange, Laplace, and Gauss, who were prominent in almost every field of science. With Gauss, the theory of orbit determination reached a certain degree of completeness.

In recent years the whole technique of calculations has changed. Previously, astronomical calculations were carried out with the aid of logarithms. In our day, widespread use of fast and reliable computing machines has eliminated the use of logarithms in orbit computations.

Considerable success has been achieved in working out methods of calculation of perturbations. In 1909, Cowell and Crommelim presented a method in which, by means of numerical integration, they obtained the perturbed coordinates themselves. This method is based on an almost direct utilization of the differential equations of motion written in their simplest form--in rectangular coordinates.[3]

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[2]Forest Ray Moulton, _An Introduction to Celestial Mechanics_ (New York: Macmillan Company, 1964), pp. 29-35.

[3]A. D. Dubyago, _op. cit._, pp. 9-21.

## CHAPTER II

## COORDINATES AND ORBIT DESCRIPTION

In the following paragraphs, terms used in this discourse will be defined.

### I. DEFINITIONS

A satellite is a body revolving around a planet. It is normally of negligible mass compared with its parent planet. The word "revolve" will be used in this paper to imply motion around a point. "Rotate" will imply motion about an axis. Thus, the earth revolves around the sun, but rotates on its axis.

The earth is nearly spherical, with a radius of about 4000 miles or 6400 km. It is flattened at the poles, the deviation from a sphere being slight, but important for precise work. In most modern systems which depend on practical celestial mechanics, the distortions caused by a pear-shaped earth are not significant. Any deviations in the true revolution of a body, namely a satellite, are known as perturbations.[1] In this problem, a small satellite around the earth will be dealt with. Therefore, the definitions will be earth-satellite oriented.

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[1]J. M. A. Danby, Fundamentals of Celestial Mechanics (New York: The Macmillan Company, 1962), pp. 1-4.

The path of a satellite around the earth is known as the orbit of the satellite. In this orbit the point of closest approach is called the perigee; the point when the satellite is farthest away is called the apogee. A set of quantities which characterize an orbit is called the elements of the orbit.

The prediction of the motions of the heavenly bodies was one of the earliest problems in astronomy. At first these predictions were based on the Ptolemaic system of epicycles, but as more and more accurate observations accumulated, this system was found to be unsatisfactory.

Later, the motions of the planets were based on the laws which Kepler discovered through his analysis of Tycho Brahe's observations of Mars. Most of the work today in orbit mechanics is based upon assumptions that satellites obey Newton's laws of motion and gravitation. In a later chapter it will be shown that from these assumptions the Keplerian laws of planetary motion can be derived. Therefore, it will be assumed that the following laws govern satellite motion.

Kepler's Laws: (revised for purpose of this paper)

1. The orbit of each satellite is an ellipse, with the center of the earth at one of its foci.

2. Each satellite revolves so that the line joining it to the center of the earth sweeps out equal areas in equal intervals of time.

3. The ratio of the square of the time of one revolution to the cube of the mean distance from the center of the earth is the same for all satellites.[2]

Newton's Laws:

1. A particle will continue in a state of rest or of uniform motion in a straight line unless acted upon by some force.

2. The action of a force upon a particle produces an acceleration which is proportional to the force and in the direction of the force, and inversely proportional to the mass of the particle. (F = ma)

3. For every acting force there is an oppositely directed force of equal magnitude.[3]

4. Every particle of matter attracts every other particle with a force that is directly proportional to the product of their masses and is inversely proportional to the square of the distance between them. ($F = k^2 \frac{m_1 m_2}{r^2}$ Universal Gravitation)[4]

## II. COORDINATE SYSTEMS

The celestial sphere is the imaginary sphere which surrounds the earth at a unit radius from the center of the earth. The heavenly objects are considered as being on this celestial sphere. The celestial sphere is used as a method of reasoning.

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[2]Peter Van De Kamp, Elements of Astromechanics (San Francisco: W. H. Freeman and Company, 1964), pp. 19-23.

[3]Paul Herget, The Computation of Orbits (Paul Herget, 1948), p. 1.

[4]Forest Ray Moulton, An Introduction to Celestial Mechanics (New York: The Macmillan Company, 1964), pp. 2-8.

The principal motions of the earth are its axial rotation and its revolution around the sun. As seen from the observer, all objects on the celestial sphere rise in the east, travel across the celestial sphere, and set in the west.

Since the observations are from the earth, the earth's orbital motion becomes apparent as a motion of the sun among the stars. Once in approximately 365.25 days the sun appears to have made a complete circuit with respect to the stars. The apparent great-circle path of the sun on the celestial sphere during the course of the year is called the ecliptic.

The great circle obtained by projecting the plane of the earth's equator until it intersects the celestial sphere is the celestial equator. Figure (2.1) shows the earth at the center of the celestial sphere. The coordinate systems used in this paper are right-handed systems.

In Figure (2.1) the point P is the north celestial pole. The point where the sun crosses the equator from south to north about March 21 each year is called the vernal equinox (H). The vernal equinox is not fixed on the sphere, but moves westward along the equator at a rate of one revolution every 26,000 years. The point H is used in the inertial coordinate system to determine the X-axis, the center of the earth as the origin and

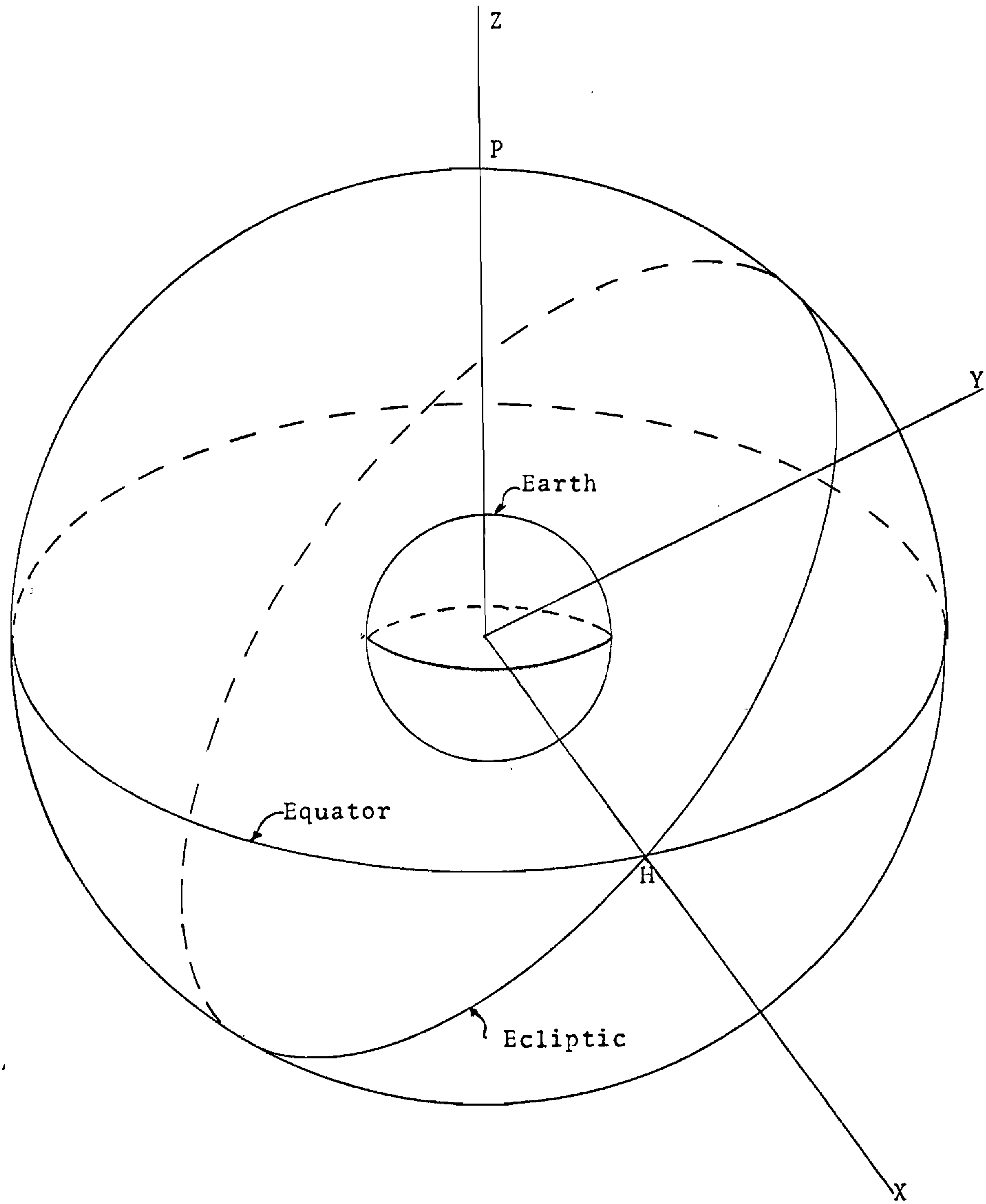


Figure 2.1

the north celestial pole, P, as the Z coordinate. (Figure 2.1) The X Y-plane is in the plane of the celestial equator.[5]

The other principal coordinate system, reference frame, is also earth centered. This system is an earth-fixed reference frame, where the X and Y axes rotate at the same rate as the earth. An arbitrary point has been chosen on the equator and an imaginary great circle, known as the prime meridian, has been drawn through this point and the north pole. This great circle which passes through Greenwich, England, is defined as the 0° line of longitude and also serves as the reference line for time zones on the earth. Using this 0° longitude point on the equator as the X coordinate of the earth-fixed system to pass through this point and the earth's center, Figure (2.2) shows the earth-fixed coordinate system.

When converting from one coordinate system to the other, the X Y-planes coincide and thus, the Z coordinates are the same for any position in space. The problem in conversion is to find the X and Y coordinates (earth-fixed) with respect to the X and Y inertial coordinates. This conversion depends on the rotation of the earth which is a measure of time.

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[5]V. M. Blanco and S. W. McCuskey, Basic Physics of the Solar System (Massachusetts: Addison-Wekley Publishing Company, Inc., 1961), pp. 1-6.

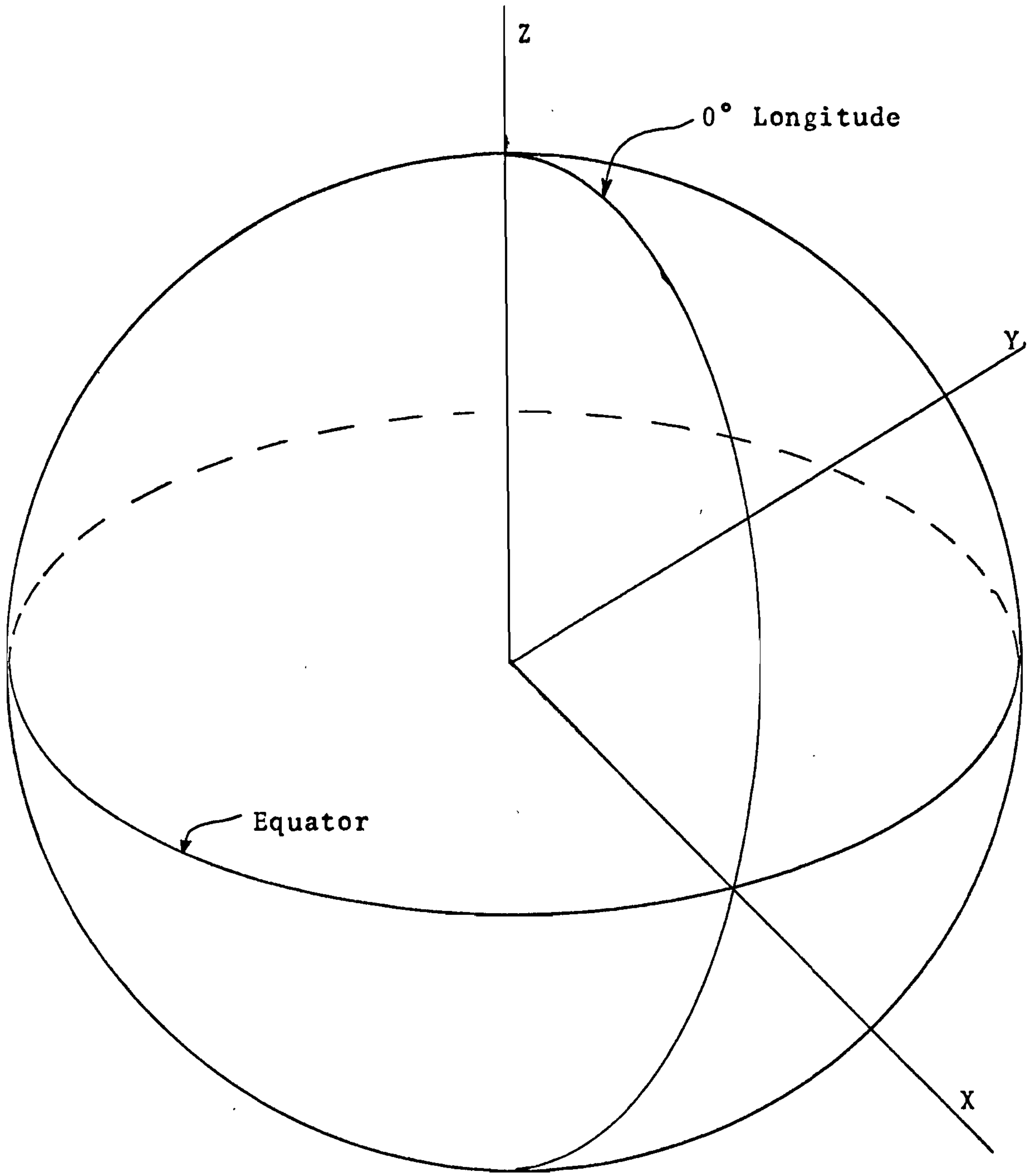


Figure 2.2

The equation of motion of the equinox and the equinox position at given times are given in the American Epheneris. With these facts it is possible to convert to inertial coordinates from earth fixed and conversely.

When an observation is made on the surface of the earth, the inertial coordinates of the observation are not known and neither are the earth-fixed coordinates. If the observer's position on the earth's surface is known, it is possible to convert to earth-fixed coordinates by a series of angle rotations of the observations.

## III. ORBIT DESCRIPTION

In order to find the position of the plane of the orbit of the satellite in space, it is necessary to use the inertial coordinates X, Y, Z. In Figure (2.3) where E E' is in the celestial equatorial plane and A A' is in the orbital plane, the X axis is directed to the point of the vernal equinox H, and the X Y plane is made to coincide with the plane of the celestial equator.[6]

The straight line O N is called the line of nodes, and its intersections with the celestial sphere are the nodes of the orbit. The node in which the satellite passes from the southern hemisphere of the celestial

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[6]A. D. Dubyago, op. cit., pp. 32-33.

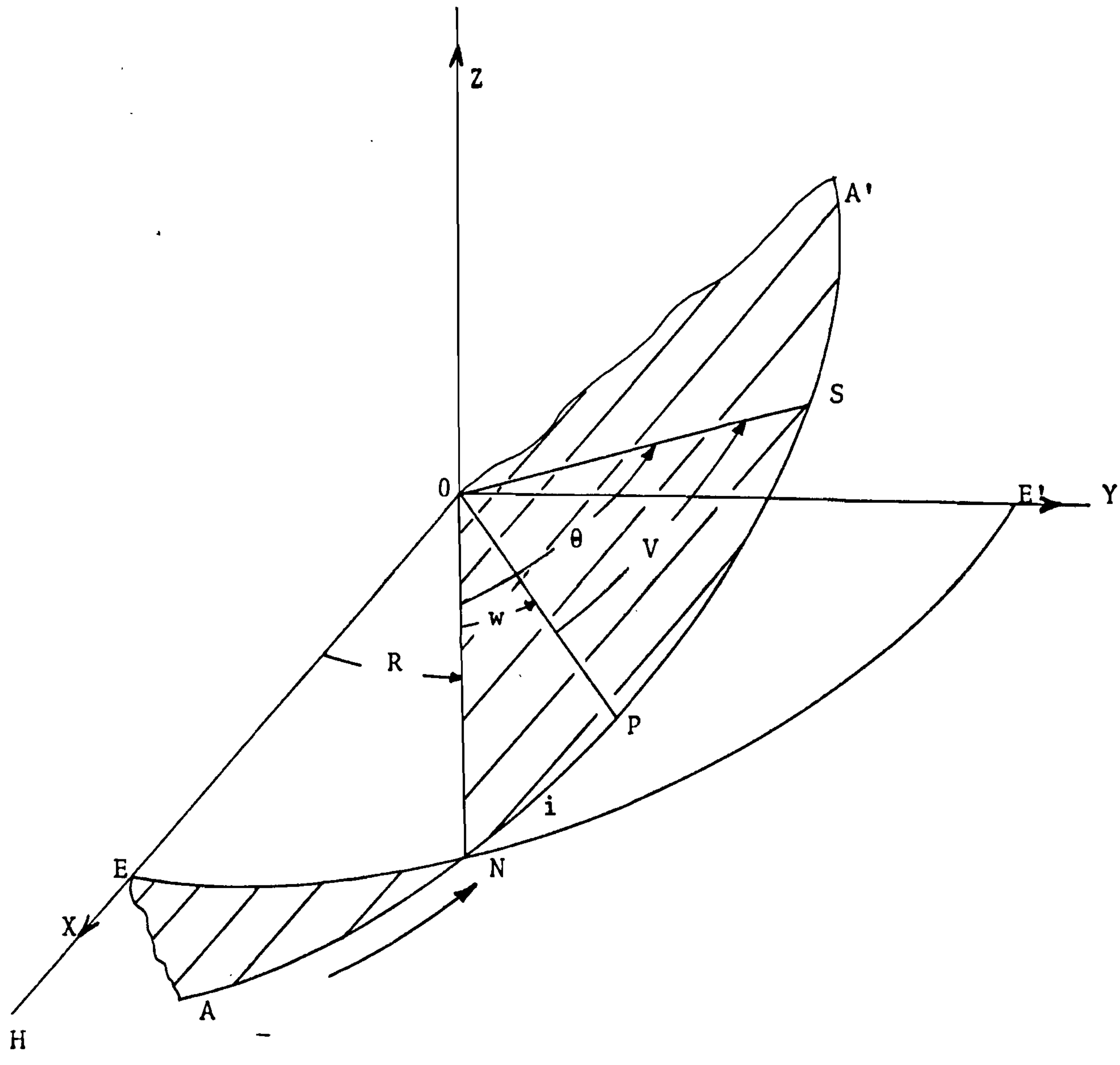


Figure 2.3

sphere into the northern, bears the name ascending node and has the symbol N. The angle XON designated by the symbol R, is called the right ascension of the ascending node and is one of the elements of the orbit. The second element determining the plane of the orbit is the inclination i of the plane of the celestial equator, equal to E'NA'. If $i < 90°$, then the body moves in a so-called direct motion. If $i > 90°$, then the motion is called retrograde. Most of the United States' satellites have an inclination $i < 90°$. The rotation of the earth will aid in a launching of a direct motion orbit, whereas in a retrograde orbit this rotation velocity of the earth would have to be overcome.[7]

The remaining elements describe the satellite position in the plane of its orbit. In Figure (2.3) the point P is the point of closest approach in the orbit, called the perigee, and s is the position of the satellite. Then θ denotes the angle NOS, customarily called the argument of latitude of the satellite. The true anomaly V, is equal to the difference between argument of latitude θ and the argument of perigee w, written

$$V = \theta - w \tag{2.1}$$

[7]J. M. A. Danby, *op. cit.*, pp. 155-157.

When the two-body problem is solved in Chapter III, six constants of integration are found; these are then expressed in terms of six other constants, the elements of the orbit. These new constants are the following:

i - the inclination of the orbit.
R - the longitude of the ascending node.
w - the argument of perigee.
a - the semi-major axis
e - the eccentricity
T - the moment of passage of the perigee.

The first three elements give the position of the plane of the orbit in space and the direction of the major axis of the orbit. These depend upon the selection of the coordinate system.

The size and the form of the eliptical orbit are determined with the aid of the next two elements.[8]

The last element fixes the position of the body in the orbit at a definite moment.

Chapter IV will expound on the geometric interpretation of the orbit size and position in space.

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[8]V. M. Blanco and S. W. McCluskey, op. cit., p. 31.

## CHAPTER III

## TWO-BODY PROBLEM

The motion of a satellite around the earth is governed by central forces. A central force is one which acts along the line joining the two bodies. Newton's law of gravitation states that any material body in the universe attracts any other body with a force which varies directly as the product of their masses and inversely as the square of the distance between them, and this force acts along the line joining the bodies.

$$F = - \frac{k^2 m_1 m_2}{r^2} \tag{3.1}$$

where $k^2$ is the universal gravitational constant.
$F$ is the attracting force between the two bodies.
$m_1$ and $m_2$ are their respective masses.
$r$ is the distance between $m_1$ and $m_2$.

The negative sign in (3.1) denotes an attractive force.

Newton's second law of motion states that

$$F = ma \tag{3.2}$$

where $a$ is the acceleration of a body of mass $m$ subject to a force F.[1]

Consider two masses $m_1$ and $m_2$ situated at points $r_1$ and $r_2$ in a rectangular coordinate system in which

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[1]A. D. Dubyago, _op. cit._, pp. 22-23.

Newton's laws of motion hold.[2] (Figure 3.1) Let $m_1$ have the coordinates $(B_1,B_2,B_3)$ and $m_2$ have the coordinates $(C_1,C_2,C_3)$. Then the projection of the force expressed in (3.1) on the $A_i$ axis is

$$F_i = -\frac{k^2\ m_1\ m_2}{r^2}\cdot \mathrm{Cos}(r,A_i) \qquad i = 1,2,3 \qquad (3.3)$$

where $(r,A_i)$ is meant to be the angle between the direction along a line from $m_1$ to $m_2$ and the $A_i$-axis.[3] However,

$$\mathrm{Cos}(r,A_i) = \frac{(C_i-B_i)}{r} \qquad i = 1,2,3,$$

substituted into (3.3) gives

$$F_i = -\frac{k^2\ m_1\ m_2}{r^3}\,(C_i-B_i) \qquad i = 1,2,3. \qquad (3.4)$$

The projection of the force expressed in (3.2) on the $A_i$-axis is

$$F_i = ma_i = m\,\frac{d^2\ A_i}{dt^2} \qquad i = 1,2,3 \qquad (3.5)$$

where $A_i$ is a distance on the axis. Now equating formulas (3.4) and (3.5) for the body $m_1$ we find

$$m_1\,\frac{d^2\ B_i}{dt^2} = \frac{k^2\ m_1\ m_2}{r^3}\,(C_i-B_i)$$

or

$$\frac{d^2\ B_i}{dt^2} = \frac{k^2\ m_2}{r^3}\,(C_i-B_i) \qquad i = 1,2,3 \qquad (3.6)$$

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[2]Forest Ray Moulton, op. cit., pp. 140-142.

[3]A. D. Dubyago, op. cit., pp. 24-25.

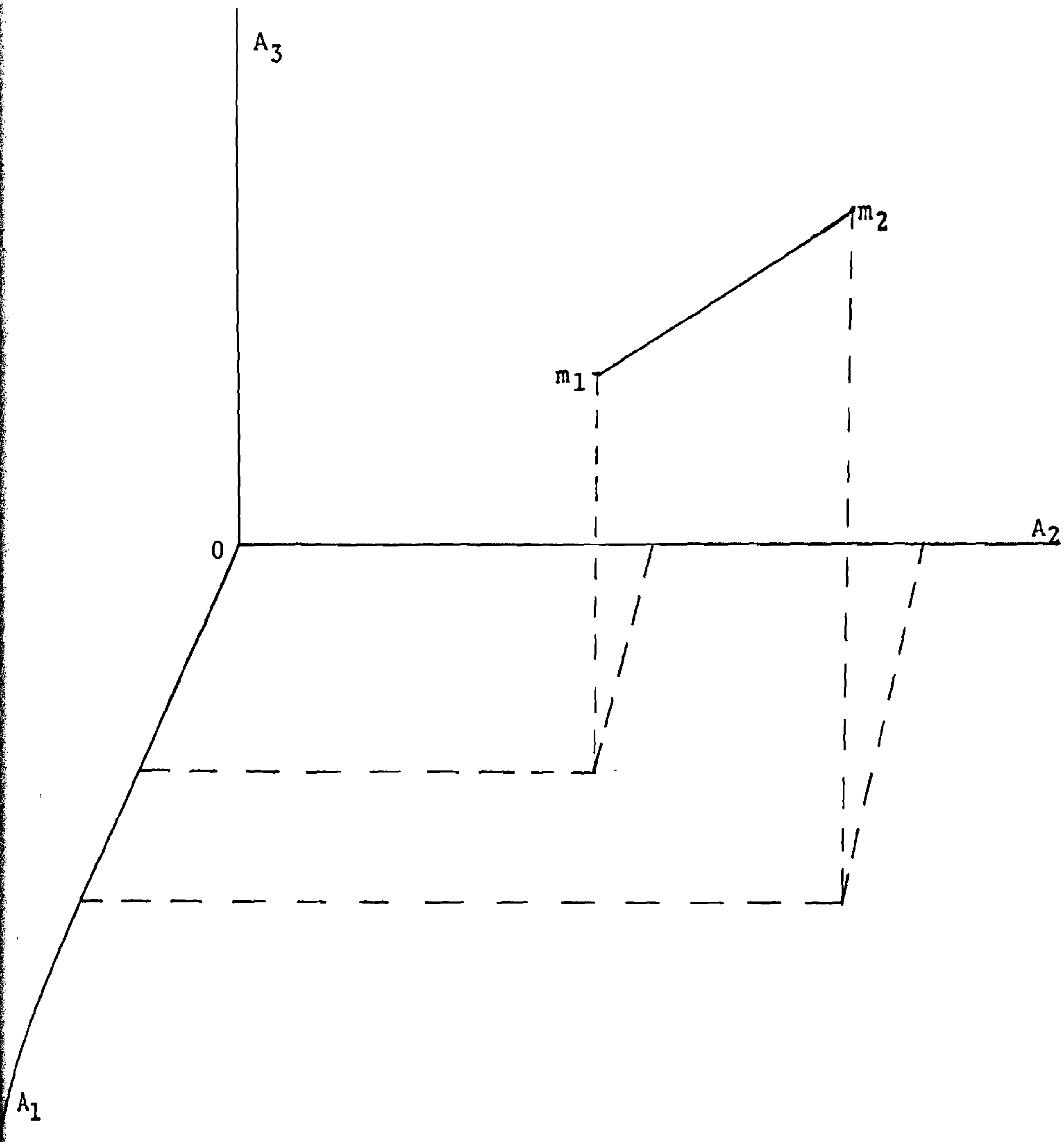


Figure 3.1

and similarly for $m_2$

$$\frac{d^2 C_i}{dt^2} = -\frac{k^2 m_1}{r^3}(C_i - B_i) \qquad i = 1,2,3. \quad (3.7)$$

These two equations may be subtracted to obtain

$$\frac{d^2}{dt^2}(C_i - B_i) = -\frac{k^2(m_1 + m_2)}{r^3}(C_i - B_i) \quad i = 1,2,3. \quad (3.8)$$

Now introducing a new coordinate system with origin at the center of $m_1$, its axes are paralleled to those of the old frame.[4] Then calling these new axes $X_1, X_2, X_3$, the coordinates of $m_2$ become

$$X_i = C_i - B_i \qquad i = 1,2,3. \quad (3.9)$$

With this substitution, equation (3.8) becomes

$$\frac{d^2 X_i}{dt^2} = -\frac{k^2(m_1 + m_2)}{r^3} X_i \qquad i = 1,2,3. \quad (3.10)$$

With the notation

$$k^2 (m_1 + m_2) = K^2,$$

equation (3.10) becomes

$$\frac{d^2 X_1}{dt^2} = -\frac{K^2 X_1}{r^3},$$

$$\frac{d^2 X_2}{dt^2} = -\frac{K^2 X_2}{r^3},$$

$$\frac{d^2 X_3}{dt^2} = -\frac{K^2 X_3}{r^3}. \qquad (3.11)$$

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[4]Forest Ray Moulton, op. cit., pp. 143-145.

These three are then the differential equations of motion which will give six constants of integration.[5] These constants are related to six orbital elements.

A change to the familiar coordinate notation X,Y,Z, can be made by setting $X_1 = X$, $X_2 = Y$, $X_3 = Z$. Then system (3.11) becomes

$$\frac{d^2 X}{dt^2} = - \frac{K^2 X}{r^3},$$

$$\frac{d^2 Y}{dt^2} = - \frac{K^2 Y}{r^3},$$

$$\frac{d^2 Z}{dt^2} = - \frac{K^2 Z}{r^3}. \qquad (3.11)$$

Multiplying the first of these equations by -Y, the second by X and then adding, yields[6]

$$\frac{X\, d^2 Y}{dt^2} - Y \frac{d^2 X}{dt^2} = 0,$$

$$\frac{Y\, d^2 Z}{dt^2} - Z \frac{d^2 Y}{dt^2} = 0,$$

$$\frac{Z\, d^2 X}{dt^2} - X \frac{d^2 Z}{dt^2} = 0. \qquad (3.12)$$

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[5]A. D. Dubyago, op. cit., p. 24.

[6]Forest Ray Moulton, op. cit., p. 144.

This can also be written as

$$\frac{d}{dt}\left(X\frac{dY}{dt} - Y\frac{dX}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(Y\frac{dZ}{dt} - Z\frac{dY}{dt}\right) = 0,$$

$$\frac{d}{dt}\left(Z\frac{dX}{dt} - X\frac{dY}{dt}\right) = 0. \qquad (3.13)$$

(3.13) can then immediately be integrated to give

$$X\frac{dY}{dt} - Y\frac{dX}{dt} = a_1,$$

$$Y\frac{dZ}{dt} - Z\frac{dY}{dt} = a_2,$$

$$Z\frac{dX}{dt} - X\frac{dZ}{dt} = a_3. \qquad (3.14)$$

Multiplying this system by Z, X, Y respectively, and adding, gives[7]

$$a_1 Z + a_2 X + a_3 Y = 0. \qquad (3.15)$$

This is the equation of a plane passing through the origin of the coordinates ($m_1$). Since this plane passes through a definite point of space, it is completely determined by two parameters.

Multiplying both sides of each equation in (3.14) yields

$$XdY - YdZ = a_1\, dt,$$

$$YdZ - ZdY = a_2\, dt,$$

$$ZdX - XdZ = a_3\, dt.$$

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[7]Paul Herget, op. cit., p. 26.

Let the moving body at the moment t be at point P with coordinates X,Y,Z, and at moment t + dt be at point P' with coordinates X + dX, Y + dY, Z + dZ, (Figure 3.2). Then, X d(Y) - Y d(X) expresses twice the area of triangle OQQ', which is the projection of triangle OPP' on plane XOY. Now, the area OQQ' equals the area OPP' multiplied by the cosine of the angle between the plane of motion of the body AOA' and the plane XOY.[8]

The problem has thus been simplified by reducing it to a problem in the plane. The next choice is a reference frame in the orbital plane with the earth as the center, calling the coordinates X and Y; these are not, of course, the same X and Y previously used. Thus, the following equations are left to be solved.

$$\frac{d^2 X}{dt^2} = - \frac{K^2 X}{r^3},$$

$$\frac{d^2 Y}{dt^2} = - \frac{K^2 Y}{r^3}, \qquad (3.16)$$

which will give four constants of integration.

As before, multiplying (3.16) by -Y and X respectively, and adding, gives

$$X \frac{d^2 Y}{dt^2} - Y \frac{d^2 X}{dt^2} = 0. \qquad (3.17)$$

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[8] A. D. Dubyago, *op. cit.*, p. 26.

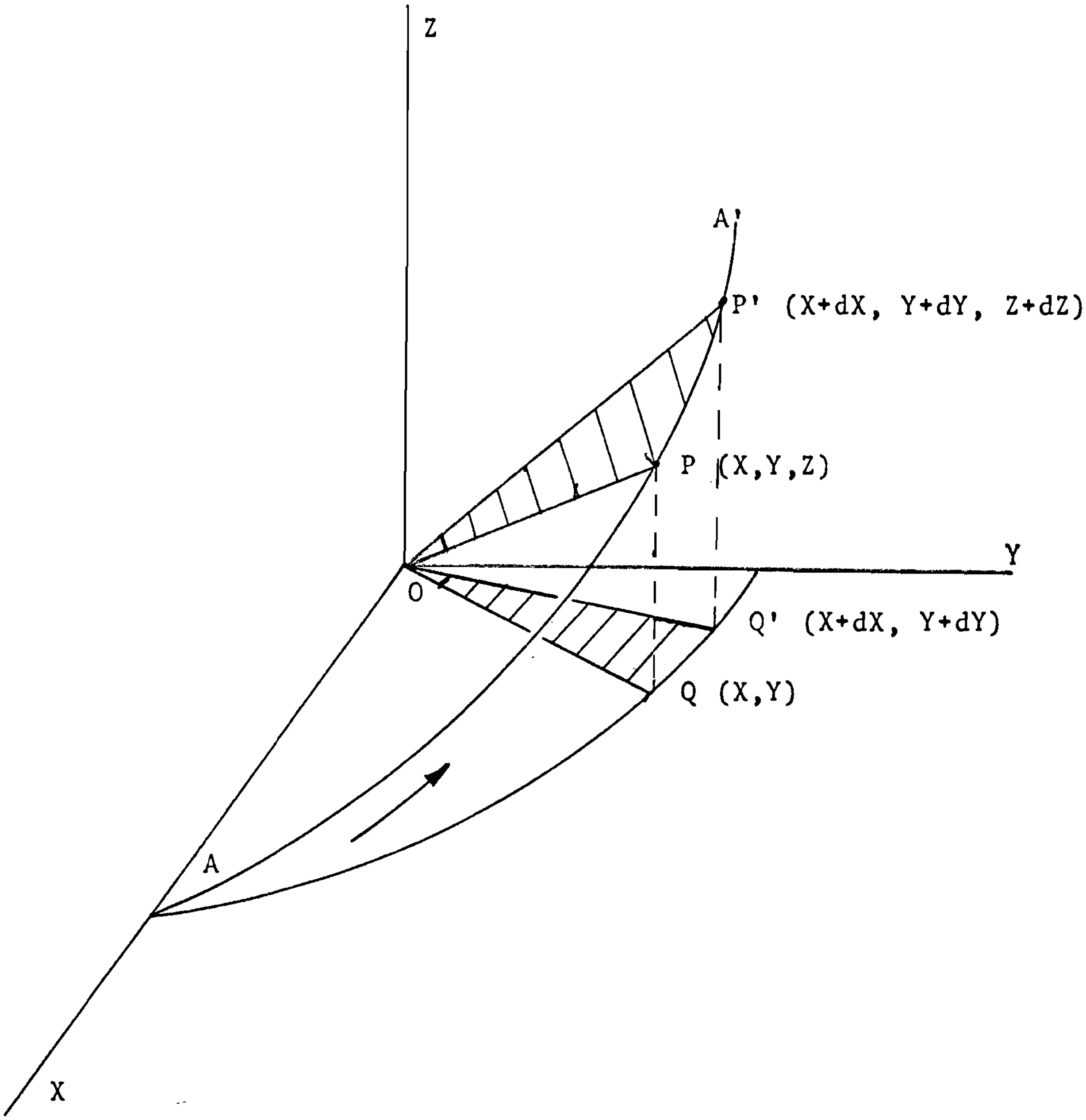


Figure 3.2

Integrating this, yields

$$X \frac{dY}{dt} - Y \frac{dX}{dt} = C_3. \tag{3.18}$$

It is now advantageous to introduce polar coordinates. Let

$$X = r \cos \theta,$$

$$Y = r \sin \theta,$$

then

$$\frac{dX}{dt} = \cos \theta \frac{dr}{dt} - r \sin \theta \frac{d\theta}{dt},$$

$$\frac{dY}{dt} = \sin \theta \frac{dr}{dt} + r \cos \theta \frac{d\theta}{dt}. \tag{3.19}$$

Substituting (3.19) into (3.18) gives

$$r^2 \frac{d\theta}{dt} = C_3 \tag{3.20}$$

From Figure (3.3)

$$d(A) = \frac{1}{2} r \, d\theta \, r$$

or

$$2d(A) = r^2 \, d\theta. \tag{3.21}$$

Comparing (3.20) and (3.21) gives

$$2 \frac{dA}{dt} = C_3,$$

which immediately integrates to[9]

$$2 A = C_3 t + C_4 \tag{3.22}$$

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[9] Paul Herget, op. cit., p. 26.

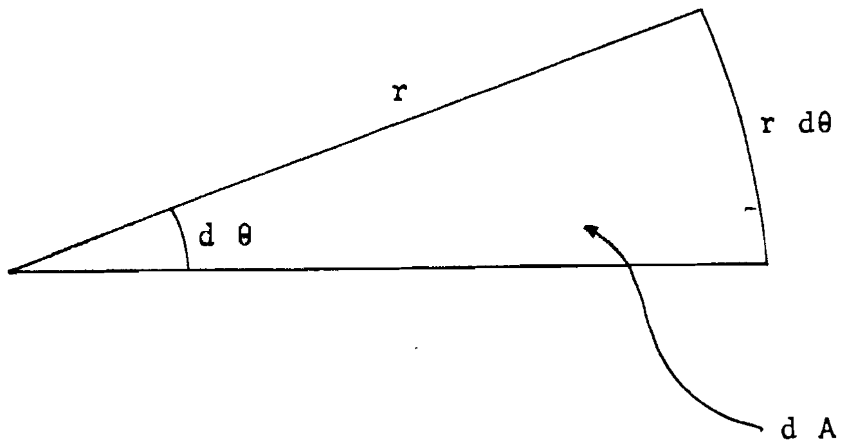


Figure 3.3

Formula (3.22) expresses Kepler's Second Law:
Each body revolving around the earth moves in a plane passing through the center of the earth; moreover, the area being described by the radius vector of the body changes proportionally with time.

Multiplying (3.16) by

$$2\,\frac{dX}{dt} \text{ and } 2\,\frac{dY}{dt},$$

respectively, and adding, yields

$$2\,\frac{d^2\,X}{dt}\,\frac{dX}{dt} + 2\,\frac{d^2\,Y}{dt}\,\frac{dY}{dt} = -2\,\frac{K^2}{r^3}(X\,\frac{dX}{dt} + Y\,\frac{dY}{dt}). \tag{3.23}$$

Differentiating

$$r^2 = X^2 + Y^2,$$

with respect to time gives

$$r\,\frac{dr}{dt} = X\,\frac{dX}{dt} + Y\,\frac{dY}{dt}\,. \tag{3.24}$$

Substituting this into (3.23) yields

$$2\,\frac{d^2\,X}{dt^2}\,\frac{dX}{dt} + 2\,\frac{d^2\,Y}{dt^2}\,\frac{dY}{dt} = -2\,\frac{K^2}{r^2}\,\frac{dr}{dt}\,. \tag{3.25}$$

But the left-hand side of (3.25) can be written as

$$2\,\frac{d^2\,X}{dt^2}\,\frac{dX}{dt} + 2\,\frac{d^2\,Y}{dt^2}\,\frac{dY}{dt} = \frac{d}{dt}((\frac{dX}{dt})^2 + (\frac{dY}{dt})^2) \tag{3.26}$$

with (3.26) the (3.25) becomes[10]

$$\frac{d}{dt}((\frac{dX}{dt})^2 + (\frac{dY}{dt})^2) = -2\,\frac{K^2}{r^2}\,\frac{dr}{dt}\,. \tag{3.27}$$

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[10]A. D. Dubyago, op. cit., p. 28.

Integrating this equation gives[11]

$$(\frac{dX}{dt})^2 + (\frac{dY}{dt})^2 = 2\,\frac{K^2}{r} + C_5 \qquad (3.28)$$

which in many different forms is known as the energy integral.[12] So the change in kinetic and potential energy are equal. Transforming to polar coordinates, using (3.19), (3.28) becomes

$$(\frac{dr}{dt})^2 + r^2\,(\frac{d\theta}{dt})^2 = 2\,\frac{K^2}{r} + C_5 \qquad (3.29)$$

But since

$$\frac{dr}{dt} = \frac{dr}{d\theta}\;\frac{d\theta}{dt}$$

(3.29) becomes

$$(\frac{d\theta}{dt})^2\,((\frac{dr}{d\theta})^2 + r^2) = 2\,\frac{K^2}{r} + C_5 \qquad (3.30)$$

Using (3.20), replacing $(\frac{d\theta}{dt})^2$ by $\frac{C_3^2}{r^4}$, (3.30) then reads

$$\frac{C_3^2}{r^4}\,((\frac{dr}{d\theta})^2 + r^2) = 2\,\frac{K^2}{r} + C_5\;.$$

Thus solving for $d\theta$ gives

$$d\theta = \frac{C_3\;dr}{r^2\sqrt{\frac{2K^2}{r} + C_5 - \frac{C_3^2}{r^2}}} \qquad (3.31)$$

which, since

$$-d\left(\frac{C_3}{r}\right) = \frac{C_3}{r^2}\,dr,$$

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[11]Forest Ray Moulton, op. cit., p. 28.

[12]J. M. A. Danby, op. cit., p. 64.

(3.31) may be written in the form[13]

$$d\theta = \frac{-d(\frac{C_3}{r})}{\sqrt{C_5 + \frac{K^4}{C_3^2} - (\frac{K^2}{C_3} - \frac{C_3}{r})^2}} \quad . \tag{3.32}$$

Now let

$$\frac{K^2}{C_3} - \frac{C_3}{r} = -J.$$

Since then

$$d(J) = d(\frac{C_3}{r}) \tag{3.33}$$

(3.32 becomes, using (3.33),

$$d\theta = \frac{-d(J)}{\sqrt{C_5 + \frac{K^4}{C_3^2} - J^2}} \tag{3.34}$$

which, when integrated, gives

$$\theta = \cos^{-1} \frac{J}{\sqrt{C_5 + \frac{K^4}{C_3^2}}} + C_6 \tag{3.35}$$

or using (3.33) again,

$$\theta = \cos^{-1} \frac{\frac{C_3}{r} - \frac{K^2}{C_3}}{\sqrt{C_5 + \frac{K^4}{C_3^2}}} + C_6$$

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[13]Forest Ray Moulton, _op_. _cit_., p. 148.

or

$$\cos(\theta - C_6) = \frac{\frac{C_3}{r} - \frac{K^2}{C_3}}{\sqrt{C_5 + \frac{K^4}{C_3^2}}}.$$

Solving the equation for r gives[14]

$$r = \frac{C_3}{\frac{K^2}{C_3} + \sqrt{C_5 + \frac{K^4}{C_3^2}} \cos(\theta - C_6)} \qquad (3.36)$$

which is the polar equation of a conic section with the origin at one of its foci. This equation demonstrates Kepler's first law, namely that the satellites move in ellipses, the earth being one of the foci.

In order to write the orbital elements (shown in an earlier chapter) into terms of the above constants of integration, where

$$r = \frac{P}{1 + e\cos(\theta - w)} \qquad (3.37)$$

is the ordinary equation of a conic with origin at the "right-hand" focus. P is the semi-latus rectum and w the angle, between the major axis and the polar axis.

Multiplying numerator and denominator of (3.36) by $\frac{C_3}{K^2}$ yields

$$r = \frac{\frac{C_3^2}{K^2}}{1 + \sqrt{1 + \frac{C_3^2 C_5}{K^4}} \cos(\theta - C_6)} \qquad (3.38)$$

[14]A. D. Dubyago, _op. cit._, p. 29.

A comparison of (3.37) and (3.38) yields

$$p = \frac{C_3^2}{K^2}$$

$$e = \sqrt{1 + \frac{C_3^2 \; C_5}{K^4}}$$

$$w = C_6 \qquad (3.39)$$

The elements i and R, describing the orientation of the orbital plane in space, are related to our constants of integration by the following equations:[15]

$$a_1 = C_3 \cos i$$

$$\left.\begin{aligned} a_2 &= \pm\, C_3 \sin i \sin R \\ a_3 &= \mp\, C_3 \sin i \cos R \end{aligned}\right\} \begin{aligned} &\text{upper sign if } a_1 \text{ positive} \\ &\text{lower sign if } a_1 \text{ negative} \end{aligned}$$

(3.40)

Note that

$$C_3 = \sqrt{a_1^2 + a_2^2 + a_3^2} \qquad (3.41)$$

is thus not a new constant of integration.

The elements in Chapter II can now be written in terms of the constants of integration. Where

$$i = \cos^{-1} \frac{a_1}{C_3} \,,$$

$$R = \sin^{-1} \frac{a_2}{C_3 \sin i} \,,$$

$$w = C_6$$

$$a = \frac{C_1^2}{K(1 - e^2)} \,,$$

---

[15]Forest Ray Moulton, op. cit., p. 146.

$$e = \sqrt{1 + \frac{C_3^2 \; C_5}{K^4}} \, ,$$

$$T = \frac{2A - C_2}{C_1} \, .$$

# CHAPTER IV

## DEVELOPMENT OF THE ORBITAL ELEMENTS FOR THE ASTRONAUT

This chapter yields orbital elements for the released astronaut in terms of the elements of the capsule.

### I. EQUATIONS OF AN ELLIPSE

A look at the simple equation of an ellipse, as shown in Figure (4.1), will aid in the development of the astronaut's orbital elements. The conic section is the locus in a plane of all points having constant ratio e between the distance r from a fixed point F (focus) and the distance d from a fixed line (directrix):[1]

$$r = e\, d \qquad (4.1)$$

The distance from focus to directrix is

$$DF = d + r \cos V = r/e\ (1 + e \cos V) \qquad (4.2)$$

Here the true anomaly V is measured in the direction of orbital motion from the radius of perigee FP. Since

$$p = e\, DF \qquad (4.3)$$

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[1]Peter Van de Kamp, Elements of Astromechanics, (San Francisco: W. H. Freeman and Company, 1964), p. 9.

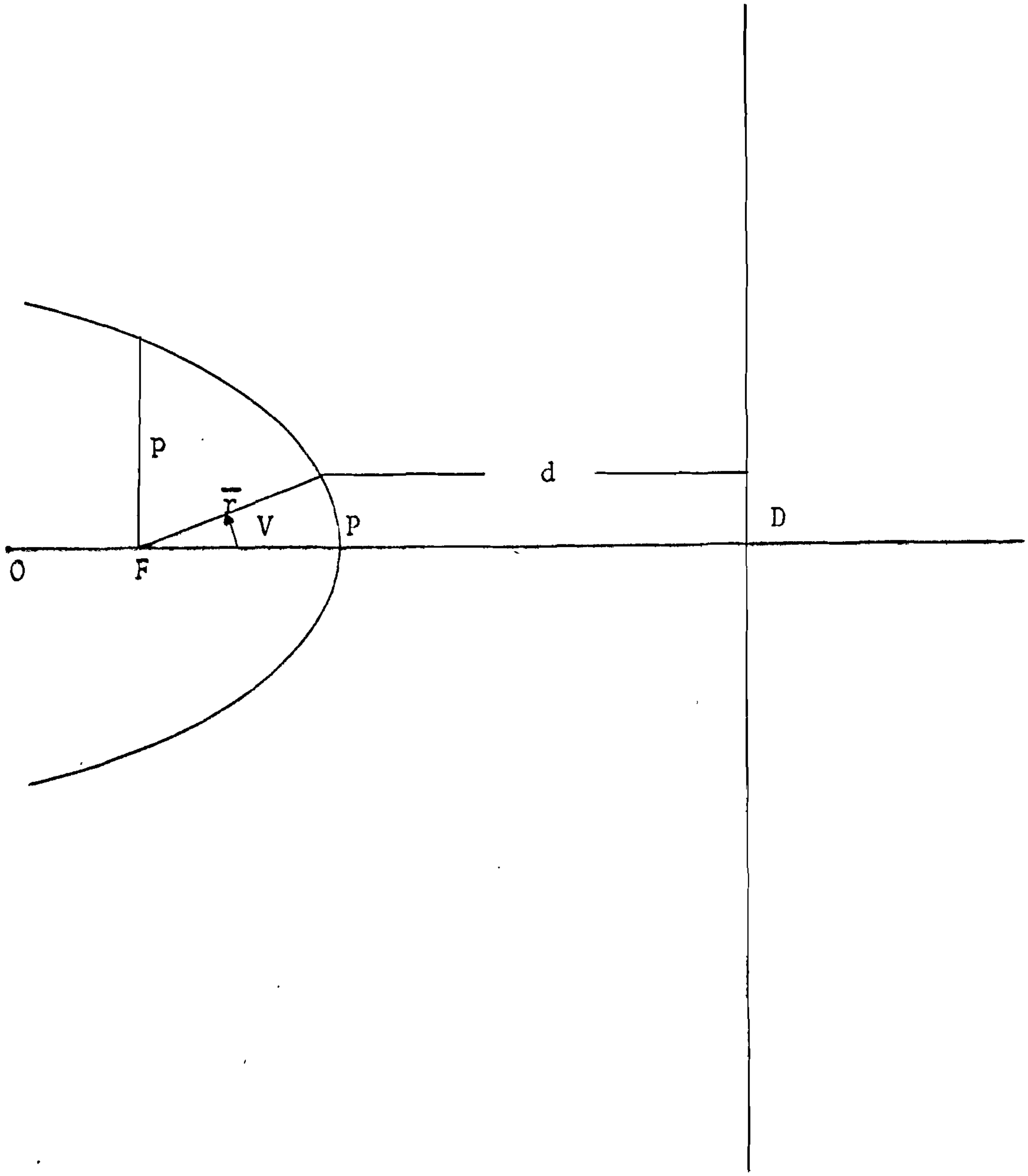


Figure 4.1

eliminating DF from (4.2) and (4.3) yields

$$r = \frac{p}{1 + e \cos V} \quad (4.4)$$

the well-known equation of the conic section in polar coordinates, with the origin at the focus; e represents the eccentricity. If $0 < e < 1$ the orbit will be elliptical in shape. The radius of perigee will occur when V is zero. The radius of apogee will exist when V is $\pi$.

The elliptic orbit is of primary importance in celestial mechanics. The radius of perigee and apogee are

$$r_p = \frac{p}{1 + e}, \quad (4.5)$$

$$r_a = \frac{p}{1 - e}, \quad (4.6)$$

The average of these,

$$\frac{r_a + r_p}{2} = \frac{p}{1 - e^2}, \quad (4.7)$$

is called the mean distance, which equals the semi-major axis a, hence

$$p = a\,(1 - e^2) \quad (4.8)$$

and

$$r_a = a\,(1 + e), \quad r_p = a\,(1 - e). \quad (4.9)$$

Thus for an elliptic orbit, equation (4.4) written in polar coordinates and referred to the focus, may also be written as

$$r = \frac{a\,(1 - e^2)}{1 + e \cos V} \tag{4.10}$$

The distance OF, called the linear eccentricity, is given by

$$c = ea. \tag{4.11}$$

## II. KEPLER'S EQUATION

There is a single but important derivation of Kepler's equation (4.15) which explains the geometric significance of E, known as the eccentric anomaly.[2]

Construct the ellipse in which the body moves, and also its auxiliary circle AS'P (Figure 4.2). The angle PFS is the true anomaly v; the angle POS' will be defined as the eccentric anomaly E.[3]

The true anomaly v and the eccentric anomaly E, are related as follows:

$$r \cos v = a\,(\cos E - e),$$

$$r \sin v = a \sin E \sqrt{1 - e^2}\ . \tag{4.12}$$

Squaring and adding equation (4.12) gives

$$r = a\,(1 - e \cos E). \tag{4.13}$$

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[2] J. M. A. Danby, _op_. _cit_., pp. 126-127.

[3] Forest Ray Moulton, _op_. _cit_., pp. 159-160.

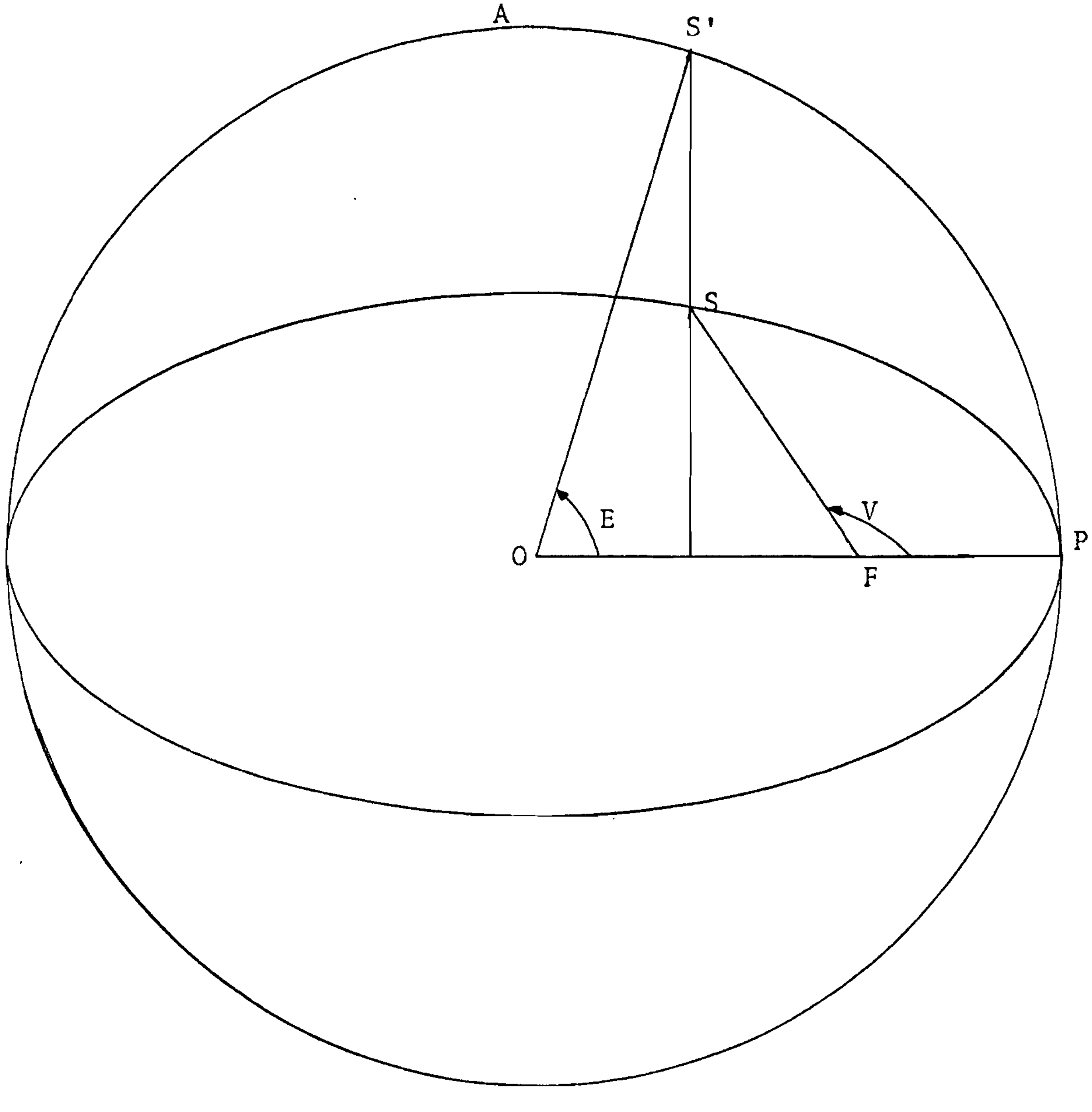


Figure 4.2

In the equation

$$n(t-T) = E-e \sin E \tag{4.14}$$

n represents the mean angular motion of the body in its orbit; (t-T) represents a change in time. The quantity n(t-T) is the angle which would have been described by the radius vector if it had moved uniformly with the average rate. This imaginary angle denoted by M is called the mean anomaly. The following equation is known as Kepler's equation,[4]

$$n(t-T) = M = E-e \sin E. \tag{4.15}$$

A convenient relation between v and E is found by eliminating r from (4.12) and (4.13).[5]

$$\cos v = \frac{\cos E-e}{1 - e \cos E} \tag{4.16}$$

The linear distance of the focus F to the center is ae. The other "empty" focus, equidistant on the other side of the center, does not appear explicitly in the equation in polar coordinates and has no physical significance in astromechanics. It is, however, significant in the construction of the ellipse, which is usually carried out in rectangular coordinates. The relation

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[4]*Ibid*., p. 159.

[5]Peter Van de Kamp, *op. cit*., p. 15.

between the distances r and r' from a point on the ellipse to the two foci is of particular interest. We have

$$r = a\,(1 - e\cos E) \qquad (4.17)$$

It is easily seen that for the empty focus

$$r' = a\,(1 + e\cos E). \qquad (4.18)$$

Hence,

$$r + r' = 2a \qquad (4.19)$$

which is the expression showing that the sum of the radius vectors connecting any point on the ellipse with the two foci is equal to the length of the major axis.[6] This relationship is shown in Figure (4.3).

## III. ASTRONAUT'S ORBITAL ELEMENTS

The four elements for the capsule, as listed, will be perturbed to give new elements for the astronaut's orbit.

- r = radius vector
- a = semi-major axis
- e = eccentricity
- P = period of revolution

Neglecting the argument of perigee, right ascension of ascending node and the inclination of the orbit will not affect the solution of the problem a great deal.

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[6] Ibid., p. 16.

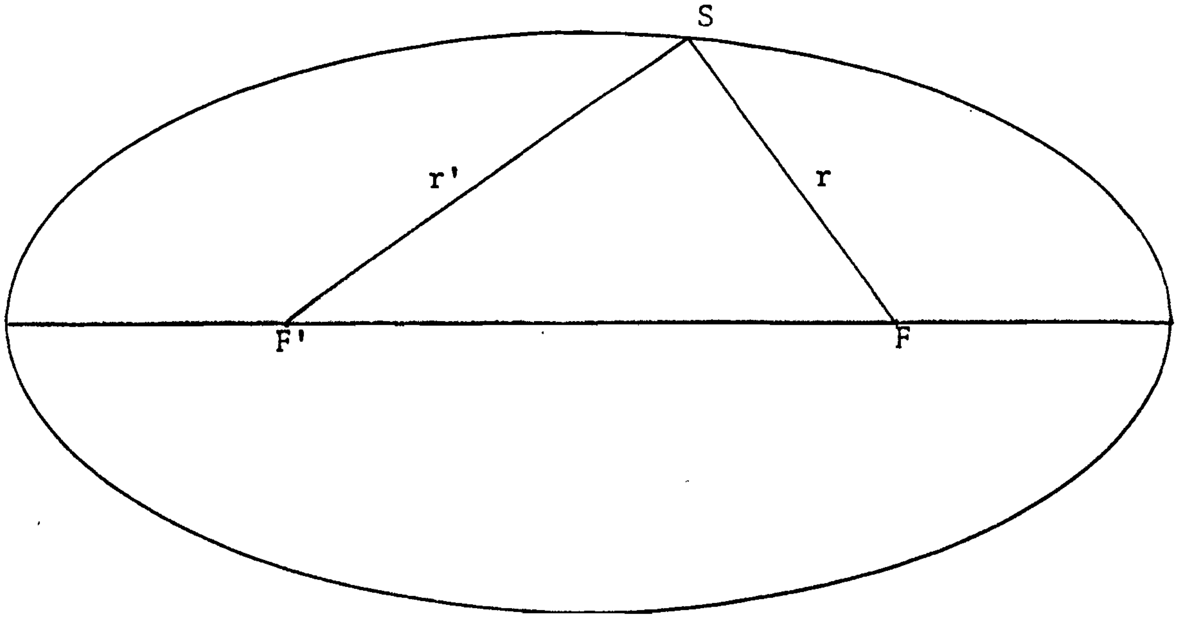


Figure 4.3

Note that the elements used, namely a, e, and P, are elements which describe the position of the satellite in the orbit.

The following are equations already derived in the paper or attained from another source.[7]

$$V_o^2 = k^2(m_1 + m_2)\ (\frac{2}{r} - \frac{1}{a}) \qquad (4.20)$$

where $V_o$ is the velocity of the capsule,

k is the universal gravitational constant,

$m_1$, $m_2$ are masses of two objects,

r is the distance from the capsule to the center of the earth, and

a is the semi-major axis.

$$n = \frac{k\sqrt{m_1 + m_2}}{a^{3/2}} \qquad (4.21)$$

where n is the mean angular motion.

$$M = n(t-T) \qquad (4.22)$$

where M is the mean anomaly,

(t-T) is the change in time,

$$n = \frac{2\pi}{P} \qquad (4.23)$$

where P is the period of revolution.

$$M = E - e \sin E \qquad (4.24)$$

where e is the eccentricity, and

E is the eccentric anomaly.

$$r = a(1 - e \cos E). \qquad (4.25)$$

---

[7]Forest Ray Moulton, _op_. _cit_., pp. 164-165.

The method for solution will be to assign the astronaut a velocity differing from the capsule's velocity by $dV_o$. Once the variation of the other elements due to the $dV_o$ are found, the elements of the astronaut can be derived.

From equation (4.19) differentiating gives

$$dr + dr' = 2da. \qquad (4.26)$$

Now making the assumption that $dr = dr'$, yields

$$2dr = 2da. \qquad (4.27)$$

The assumption is justified as an average over the complete orbit.

Next, differentiating equation (4.20) gives

$$2V_o dV_o = k^2(m_1 + m_2)\ [- \frac{2dr}{r^2} + \frac{da}{a^2}]. \qquad (4.28)$$

Substituting into (4.28) from (4.27) yields

$$2V_o dV_o = k^2(m_1 + m_2)\ [- \frac{2dr}{r^2} + \frac{dr}{a^2}].$$

or

$$\frac{dr}{dV_o} = \frac{2V_o}{k^2(m_1 + m_2)\ [\frac{1}{a^2} - \frac{2}{r^2}]}\ , \qquad (4.29)$$

which is the change in r with respect to a change in $V_o$.

Dividing equation (4.27) by $dV_o$ gives

$$\frac{2dr}{dV_o} = \frac{2da}{dV_o}\ .$$

So

$$\frac{da}{dV_o} = \frac{2V_o}{k^2(m_1 + m_2) \quad [\frac{1}{a^2} - \frac{2}{r^2}]} \tag{4.30}$$

Differentiating (4.25) and solving for dE yields

$$dE = \frac{dr - da + ae \cos E\, da + a \cos E\, de}{ae \sin E} . \tag{4.31}$$

Then

$$M = E - e \sin E,$$

gives

$$dE = dM + \sin E\, de. \tag{4.32}$$

Equating (4.32) and (4.31) yields

$$(dM + \sin E\, de)\, ae \sin E = dr - da + ae \cos E\, da$$
$$+ a \cos E\, de$$

or

$$(ae \sin^2 E - a \cos E)de = dr + (ae \cos E - 1)da$$
$$- ae \sin E\, dM.$$

Solving for de gives

$$de = \frac{dr + (ae \sin E - 1)da - ae \sin E\, dM}{ae \sin^2 E - a \cos E}$$

or

$$\frac{de}{dV_o} = \frac{dr + (ae \sin E - 1)da - ae \sin E\, dM}{(ae \sin^2 E - a \cos E)dV_o} \tag{4.33}$$

Using the equation

$$n = \frac{k\sqrt{m_1 + m_2}}{a^{3/2}}$$

will give

$$dn = - \frac{k\sqrt{m_1 + m_2}}{a^{5/2}} \frac{3}{2} da. \qquad (4.34)$$

Equation (4.22) gives

$$dM = dn(t-T).$$

Substituting for dn from (4.34) yields

$$dM = - \frac{k\sqrt{m_1 + m_2}}{a^{5/2}} \frac{3}{2} da\ (t-T)$$

or

$$\frac{dM}{dV_o} = - \frac{k\sqrt{m_1 + m_2}}{a^{5/2}} \frac{3}{2} \frac{da}{dV_o} (t-T). \qquad (4.35)$$

Substituting $da/dV_o$ from equation (4.30) gives

$$\frac{dM}{dV_o} = - \frac{k\sqrt{m_1+m_2}}{a^{5/2}} \cdot \frac{3}{2} \cdot \frac{2V_o}{k^2(m_1+m_2)\ [\frac{1}{a^2} - \frac{2}{r^2}]} (t-T)$$

or

$$\frac{dM}{dV_o} = \frac{- 3V_o\ (t-T)}{a^{5/2}\ k\sqrt{m_1+m_2}\ \ [\frac{1}{a^2} - \frac{2}{r^2}]} . \qquad (4.36)$$

Now that $dM/dV_o$ is known, a solution for $de/dV_o$ can be found. Using equation (4.33) rewritten

$$\frac{de}{dV_o} = [\frac{dr}{dV_o} + (ae \sin E-1) \frac{da}{dV_o} - ae \sin E \frac{dm}{dV_o}]$$

$$\frac{1}{ae \sin^2 E - a \cos E} ,$$

with a great deal of substitution, finally gives

$$\frac{de}{dV_o} = \frac{2aeV_o \sin E \ (a^{5/2} + k\sqrt{m_1 + m_2}\ (t-T))}{k^2(m_1+m_2)[\frac{1}{a^2} - \frac{2}{r^2}][ae \sin^2 E - a \cos E]} . \tag{4.37}$$

The final element used in the comparison is P, period of revolution. From equation (4.23)

$$dP = - \frac{2\pi}{n^2} dn$$

or

$$\frac{dP}{dV_o} = - \frac{2\pi}{n^2} \frac{dn}{dV_o} . \tag{4.38}$$

Since

$$dn = - \frac{k\sqrt{m_1 + m_2}}{a^{5/2}} \ \frac{3}{2} da,$$

then

$$\frac{dn}{dV_o} = - \frac{k\sqrt{m_1 + m_2} \ \ 3V_o}{a^{5/2} \ k^2(m_1 + m_2) \ [\frac{1}{a^2} - \frac{2}{r^2}]} . \tag{4.39}$$

Substituting $dr/dV_o$ from (4.39) into (4.38) yields

$$\frac{dP}{dV_o} = \frac{3V_o}{a^{5/2} \ k\sqrt{m_1 + m_2} \ [\frac{1}{a^2} - \frac{2}{r^2}]} . \tag{4.40}$$

The astronaut's elements are the elements of the capsule plus any change due to $dV_o$. The following are the astronaut's orbital elements.

$$r^* = r + \frac{dr}{dV_o} = r + \frac{2V_o}{k^2(m_1 + m_2) \ [\frac{1}{a^2} - \frac{2}{r^2}]} \tag{4.41}$$

$$a^* = a + \frac{da}{dV_o} = a + \frac{2V_o}{k^2(m_1 + m_2)\ [\frac{1}{a^2} - \frac{2}{r^2}]} \tag{4.42}$$

$$e^* = e + \frac{de}{dV_o} = e + \frac{2aeV_o \sin E(a^{5/2} + \frac{1}{2}k\sqrt{m_1+m_2}\ (t-T))}{k^2(m_1+m_2)[\frac{1}{a^2} - \frac{2}{r^2}](ae \sin^2 E - a \cos E)}$$

(4.43)

$$P^* = P + \frac{dP}{dV_o} = P + \frac{3V_o}{a^{5/2}\ k\sqrt{m_1 + m_2}\ [\frac{1}{a^2} - \frac{2}{r^2}]}$$

(4.44)

CHAPTER V

# CONCLUSION

## I. ASTRONAUT'S POSITION

This final analysis will use the derived orbital elements for the astronaut from Chapter IV. A comparison can be made between the capsule's future position and the astronaut's future position by using the original elements and the derived elements. The solution is simplified because the orbit planes are the same. The elements which determine the satellite's position in the orbit are the only elements that have changed. The astronaut's elements are written in terms of the capsule elements.

The radius vector to the astronaut, the magnitude of which is given in equation (4.41), will have a change amounting to

$$\frac{2V_o}{k^2(m_1 + m_2)\ [\frac{1}{a^2} - \frac{2}{r^2}]} \tag{5.1}$$

from r. Where $V_o$ is the velocity of the capsule, $k^2(m_1 + m_2)$ is a constant factor defined in Chapter III, a the semi-major axis, and r the distance between the origin and the object. r* will be smaller than r if (5.1) is negative, and larger than r if (5.1) is positive. Both $V_o$ and $k^2(m_1 + m_2)$ are positive.

Another compromise in the analysis must be made; the eccentricity must be less than .14. This is not a serious compromise since most of the satellites which carry astronauts travel in rather circular orbits. If the orbit is circular, then $e = 0$.

Equation (4.9) suggests that

$$a\,(1 - e) \le r \le a\,(1 + e) \tag{5.2}$$

Since $e < .14$, (5.2) gives

$$r < 1.14\,a,$$

for all r, or

$$r^2 - 2a^2 < 0. \tag{5.3}$$

Dividing both sides of (5.3) by $a^2r^2$ yields

$$\frac{r^2 - 2a^2}{a^2r^2} = \frac{1}{a^2} - \frac{2}{r^2} < 0, \text{ for } e < .14.$$

Therefore $\frac{1}{a^2} - \frac{2}{r^2}$ is negative; then (5.1) will be negative, and $r^*$ will be less than r.

Since the equation for the change in the semi-major axis, a, is the same as that for r, the $a^*$ will be smaller than a.

The factor $\frac{1}{a^2} - \frac{2}{r^2}$ also appears in the equation (4.44) as indicated.

$$P^* = P + \frac{3V_o}{a^{5/2}\,k\sqrt{m_1 + m_2}\,[\frac{1}{a^2} - \frac{2}{r^2}]}$$

Since $V_0$, $a^{5/2}$ and $k\sqrt{m_1 + m_2}$ are all positive and $\frac{1}{a^2} - \frac{2}{r^2}$ is negative, the value of P* will be less than the value for P.

The eccentricity is so small that any change in it would not affect this analysis.

All three elements r*, a*, and P* for the astronaut will have a smaller magnitude than r, a, and P for the capsule.

The values of a* and r* indicate that the astronaut will be somewhere between the capsule and the earth. The value of P* indicates the position of the astronaut. The astronaut will either lead the capsule, trail the capsule, or be directly between the capsule and the earth. The time per revolution for the astronaut is less than the time per revolution for the capsule, since P* is less than P. This means that at any given moment after the separation of the astronaut from the capsule the astronaut will lead the capsule. After a great many revolutions the astronaut will be well in front of the capsule in an orbit somewhat smaller than that of the capsule's orbit.

## II. FUTURE STUDY

The preceding discussion does not close the problem or give a refined answer. It has been possible to predict a probable path of the released astronaut with the aid of a great many compromises.

The problem could be approached from a numerical point of view with the aid of a computer. The inaccuracies of the problem could be overcome by giving the capsule and the astronaut exact positions and velocities. Using an orbit prediction program and a high-speed computer, one could predict very accurate positions of the two objects at any future time.

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