# MOBIUS TRANSFORMATIONS ON THE COMPLEX PLANE

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS AND THE GRADUATE COUNCIL OF THE KANSAS STATE TEACHERS COLLEGE OF EMPORIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF ARTS

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John E. French

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# CHAPTER I

## INTRODUCTION

The most important characteristic of the complex number system is the realization of the systematic structure developed by the extension of the real number system, so that it is possible to solve any polynomial equation. A one-to-one correspondence between two sets of all complex numbers can be established by a relationship called a mapping or transformation. The class of mappings to be considered is the Mobius transformation.

Until recently the Mobius transformation in its entirety was included in an elementary course of complex variables. Since new developments have warranted emphasis, this mapping has received only secondary consideration. The problem encountered by the author in studying the Mobius transformation was either the material was not presented in contemporary notation or the discussion was not complete.

Therefore the purpose of the thesis is to systematically develop the Mobius transformation in its entirety using the language of modern day mathematics. This complete study will be beneficial to any student of complex variables, especially when considering regions of the complex plane.

It should be mentioned that the reader of this thesis should have a working knowledge of calculus and an introduction to abstract algebra. All the topics not included in these fields or their prerequisites will be discussed in the material presented.

The topic of Mobius transformation lends itself to a systematic development. The complex number system will be presented in Chapter II, in such a manner as to acquaint the reader with all the representations that are encountered throughout the paper. Mobius transformations are defined and developed in Chapter III. Chapter IV discusses the properties of the Mobius transformation, while Chapter V classifies the various types of mappings that have been considered. The final chapter summarizes the material presented and suggests other topics that might be considered by a student pursuing the topic of Mobius transformations further.

# CHAPTER II

## THE SYSTEM OF COMPLEX NUMBERS

The concepts and properties of the complex number system are essential to an understanding of the topics to be considered. Complex numbers are most commonly introduced in terms of ordered pairs of real numbers. Other representations include: (1) geometrical, (2) vectorial, (3) polar, (4) exponential, and (5) spherical. A brief discussion of each of these will be given after a formal definition of the system of complex numbers has been established. It is often easier to explain certain operations in a more convenient form.

### Ordered Pairs of Real Numbers (2.1)

The system of complex numbers is a field C having as its elements the ordered pairs $[a,b]$ of real numbers, along with an equality and two binary operations. The operations will be called addition, $+$, and multiplication, $\circ$. Given two complex numbers $\alpha$ and $\beta$ such that $\alpha = [a,b]$ and $\beta = [c,d]$, where a, b, c and d are elements of the real number system, then the complex numbers $\alpha$ and $\beta$ are said to be equal if and only if $a = c$ and $b = d$. Addition and multiplication are defined as follows:

$$\alpha + \beta = [a,b] + [c,d] = [a + c, b + d] \qquad (2.1.1)$$

$$\alpha \cdot \beta = [a,b] \cdot [c,d] = [ac - bd, ad + bc]. \qquad (2.1.2)$$

It should be noted that there exists a zero element of C, namely $0 = [0,0]$, such that

$$\alpha + 0 = 0 + \alpha = \alpha \qquad (2.1.3)$$

Also there exists an element $\mu$ of C, such that

$$\alpha \circ \mu = \mu \circ \alpha = \alpha \qquad (2.1.4)$$

The letter $\mu$ is called the unit element of the complex numbers and is denoted by $[1,0]$, it is most commonly given by the symbol 1.

In order to verify that with this definition the complex numbers do form a field C, it is necessary to show the field postulates for addition and multiplication. The following addition postulates can easily be proved by use of the definition of complex numbers and the properties of the real number system.

Given $\alpha$, $\beta$ and $\gamma$ elements of C, where $\gamma = [f,g]$, the addition postulates are:

$A_1$ Closure- To every pair $\alpha$ and $\beta$ elements of C, there exists a uniquely defined element of C, known as the sum of $\alpha$ and $\beta$.

$A_2$ Commutative Law- $\alpha + \beta = \beta + \alpha$

$A_3$ Associative Law- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$A_4$ Cancellation Law- $\alpha + \gamma = \beta + \gamma$ implies $\alpha = \beta$

$A_5$ For every pair $\alpha$ and $\beta$ elements of C, there exists a unique element $\delta$, such that $\alpha + \delta = \beta$

Postulate $A_5$ implies the existence of the inverse operation of addition. This operation is referred to as subtraction.

Thus,

$$\delta = \beta - \alpha \tag{2.1.5}$$

Each element $\alpha = [a, b]$ of the field C has an inverse given by

$$-\alpha = [-a, -b].^{1} \tag{2.1.6}$$

The postulates of multiplication are given as for addition.

$M_1$ Closure- To every pair $\alpha$ and $\beta$ elements of C, there exists a uniquely defined element of C, known as the product of $\alpha$ and $\beta$.

$M_2$ Commutative Law- $\alpha \circ \beta = \beta \circ \alpha$

$M_3$ Associative Law- $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$

$M_4$ Cancellation Law- $\alpha \circ \gamma = \beta \circ \gamma$ with $\gamma \neq 0$ implies $\alpha = \beta$

$M_5$ For every pair $\alpha$ and $\beta$ elements of C, with $\alpha \neq [0,0]$ and $\beta \neq [0,0]$, there exists a unique element $\lambda$, such that $\alpha \circ \lambda = \beta$.[2]

Postulate $M_5$ implies the inverse operation of multiplication, which is called division. Therefore,

$$\lambda = \beta / \alpha \tag{2.1.7}$$

where $\beta/\alpha$ is called the quotient. The value of $\lambda$ is determined by

$$\lambda = \beta/\alpha = \left[\frac{ac + bd}{c^2 + d^2}, \frac{bc - ad}{c^2 + d^2}\right] \tag{2.1.8}$$

The distributive law is the final postulate necessary to prove the system of complex numbers form a field C. Hence,

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[1] Einar Hille, Analytic Function Theory (Boston: Ginn and Company, 1959), I, 2.

[2] Ibid., p. 3.

$D_1$ For any three $\alpha$, $\beta$ and $\gamma$ elements of C,

$$\alpha(\beta + \gamma) = \alpha \circ \beta + \alpha \circ \gamma \tag{2.1.9}$$

and therefore multiplication is distributive with respect to addition.[3] It has now been established that C forms a field.

The set of complex numbers of the form $[a,0]$ can be placed in a one-to-one correspondence with the field of real numbers,

$$[a,0] \longleftrightarrow a. \tag{2.1.10}$$

In particular,

$$[0,0] \longrightarrow 0 \tag{2.1.11}$$

$$[-a,0] \longrightarrow -a \tag{2.1.12}$$

$$[1,0] \longrightarrow 1. \tag{2.1.13}$$

Thus by definition, an isomorphism between the set of ordered pairs of the form $[a,0]$ and the set of real numbers exists.[4] Frequently this isomorphism is expressed as saying the field of real numbers is embedded in the complex field. Symbolically, it is denoted by

$$R \longleftrightarrow R' \subset C, \tag{2.1.14}$$

where R represents the field of real numbers and R' is the set of complex numbers of the form $[a,0]$.[5] Since this one-

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[3] Ibid.

[4] Neal H. McCoy, Introduction to Modern Algebra (Boston: Allyn and Bacon, Inc., 1960), p. 33.

[5] Hille, op. cit., p. 15.

to-one correspondence does exist between the reals and the complex numbers, all complex numbers of the form $[a,0]$ will simply be denoted by the symbol a.

The complex number $[0,1]$ may be given the symbol i, such that,

$$i^2 = [0,1] \cdot [0,1] = [-1,0] = -1 \qquad (2.1.15)$$

and i is the square root of -1. The symbol i is often referred to as the imaginary number.

Since,

$$[a,b] = [a,0] + [0,b]$$

then,

$$[a,b] = [a,0] + [b,0] \circ [0,1]$$

thus $[a,b]$ can be denoted by $a + bi$. In this notation it should be remembered that $a \rightarrow [a,0]$, $b \rightarrow [b,0]$ and $i \rightarrow [0,1]$. This form of complex numbers is the most commonly used.

Geometrical Representation (2.2)

The idea of mapping the complex numbers on the points of a plane was a decisive step forward in the theory of complex numbers. A geometrical representation derives its usefulness from the vivid mental pictures associated with a geometric language. This concept occurred almost simultaneously to three great mathematicians- Wessel, Argand and Gauss. It is generally agreed that Gauss was responsible for establishing this idea universally. The main reasons that Gauss receives this credit is that his development was very systematic and, of course, his authority had been universally accepted.[6]

This geometrical idea was brought about by the assumption that there is a one-to-one correspondence between the points on a straight line and the totality of real numbers. With this in mind, it should be possible to represent complex numbers in some geometrical fashion. In order to make this representation it is necessary to use a plane and introduce a system of rectangular coordinates similar to those used in analytical geometry. The plane will be referred to as the complex plane or z-plane. Sometimes this plane is called the Argand or Gauss plane. On this plane designate any point O as the origin. Through point O a pair of orthogonal axes are drawn and the Cartesian coordinates are introduced in the usual manner. The abcissa or distance along the x-axis is the first number of the number pair $[a,b]$. The unit element on the x-axis is $[1,0]$ or 1. All complex numbers of the form $[a,0]$ lie on the x-axis, so consequently, it is called the real axis. The ordinate or distance along y-axis is the second number of the number pair $[a,b]$. The unit element on this axis is the complex number $[0,1]$ or i. Since all complex numbers of the form $[0,b]$ are points on the y-axis, it is customary to call this axis the imaginary axis.

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[6] *Ibid.*, p. 18.

It is sometimes necessary to obtain the value of the real axis coordinate of a complex number z, where $z = [a,b]$ or $a+bi$. The symbol $R(z)$ is used to denote this coordinate. Likewise, the symbol $I(z)$ is used to symbolize the imaginary axis coordinate of the complex number z. In other words, given a complex number $z = a+bi$, $R(z)$ denotes the real part, a, and $I(z)$ designates the imaginary part, b. If the complex number is of the form $0+bi$, it is said to be pure imaginary.

Every complex number $a+bi$ can now be mapped onto a point of the complex plane whose coordinates are $[a,b]$. Conversely, every point on the complex plane is associated with a unique $a+bi$. For example, let $z = [x,y]$ or $z = x+yi$, then z would be represented on the complex plane as shown in figure 1. It should be noted that no distinction will be made for the symbols that designate points and numbers.

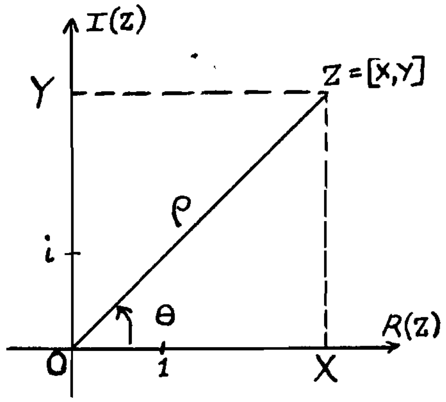


FIGURE 1

REPRESENTATION OF A POINT ON THE COMPLEX PLANE

Vector Representation (2.3)

Each point different from the origin of the complex plane determines a vector from the origin to the point z. Therefore a complex number can be represented as a vector. If the notation $z = a + bi$ is used, addition can be represented as the sum of the real components R(z) of two complex numbers and the sum of the imaginary parts of the same numbers. With this in mind it can be seen that addition of complex numbers corresponds to vector addition in the complex plane according to the parallelogram method.

Let $Z_1$ and $Z_2$ be any two points on the complex plane. The addition can then be performed by drawing through $Z_1$ a line segment equal and parallel to $OZ_2$. Call this line $Z_1Z_3$. Point $Z_3$ has the coordinates $X_1 + X_2$, $Y_1 + Y_2$, therefore $Z_3$ represents $Z_1 + Z_2$. The following figure shows the parallelogram method of addition.

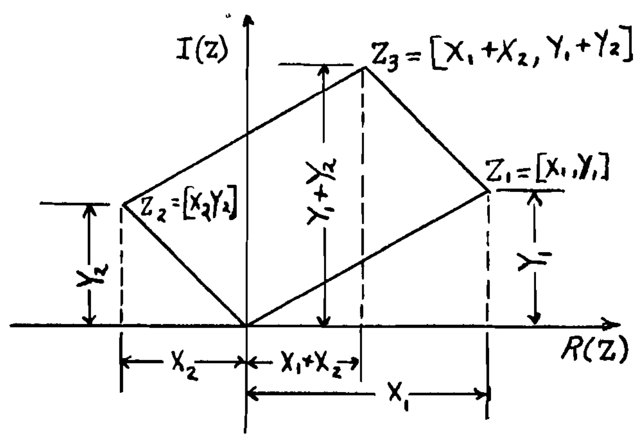


FIGURE 2

ADDITION OF COMPLEX NUMBERS

Subtraction of complex numbers can just as easily be carried out vectorially by use of the parallelogram.[7]

It is often necessary to determine the length of a vector associated with a complex number. In reference to figure 1, the distance of $\overline{OZ}$ or $\rho$ is the vector length. The value of $\rho$ is a real non-negative number called the modulus or absolute value. This value is equal to zero only if the complex number in question is the origin or $z = [0,0]$. The absolute value of $z = a + bi$ is denoted by $|z|$. By use of the Pythagorean theorem

$$|z| = \sqrt{a^2 + b^2}\ . \tag{2.3.1}$$

The square of the absolute value is called the norm of the complex number.

The concept of absolute value of a complex number is beneficial to the study of areas and regions of the complex plane. For example, let r be any positive real number and from the definition of absolute value, all the points z such that,

$$|z| = r \tag{2.3.2}$$

defines a circle C' of radius r whose center is at the origin. If the inequality signs $<$ or $>$ replace the equal

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[7]Louis L. Pennisi, Elements of Complex Variables (New York, Holt, Rinehart, and Winston, 1963), p. 10.

sign, the region described is the interior or the exterior of the circle C' respectively.

To obtain a circle with the center at some point other than the origin, say $z_0$, (2.3.2) becomes

$$|z - z_0| = r \tag{2.3.3}$$

Again the inequality signs can be used to designate the interior or exterior regions of the given circle.

The annulus ring is a region bounded by two concentric circles. The equation of the annulus is

$$r' < |z-z_0| < r'', \tag{2.3.4}$$

where r' and r'' are the radii of two circles with center $z_0$. If the boundaries are to be included it is necessary to have the equality signs.

Certain families of circles are very helpful in describing topics to be considered later. Take as an example the equation

$$\Omega(\xi_1, \xi_2): \quad \left|\frac{z - \xi_1}{z - \xi_2}\right| = C, \tag{2.3.5}$$

where $\xi_1$ and $\xi_2$ are fixed points and C is any positive real number. Equation (2.3.5) defines a circle except when $C = 1$. In this particular case the equation is the perpendicular bisector of line $\xi_1\xi_2$. It can be seen from the equation that these circles are the locus of points whose distance from $\xi_1$ and $\xi_2$ have a constant ratio. More information concerning this family of circles will be developed later. There also exist other families of circles and curves which can be described by using the concept of absolute value.[8]

Polar Representation (2.4)

Each point z of the complex plane can be represented by two polar coordinates $\rho$ and $\theta$. Coordinate $\rho$ is the non-negative length of vector $\overrightarrow{Oz}$, or simply $|z|$, while $\theta$ is the angle the vector makes with the real axis. This angle is called the argument or amplitude of z. It is denoted by

$$\theta = \arg z. \tag{2.4.1}$$

The polar coordinates $[\rho, \theta]$ of a complex number $z = x + iy$ can be determined from figure 1 by using the following equations:

$$\rho = |z| = \sqrt{x^2 + y^2} \tag{2.4.2}$$

$$x = \rho \cos \theta \tag{2.4.3}$$

$$y = \rho \sin \theta \tag{2.4.4}$$

$$\tan \theta = y/x. \tag{2.4.5}$$

By substitution $z = x + iy$ becomes

$$z = \rho \cos \theta + i \rho \sin \theta$$

$$\text{or } z = \rho(\cos \theta + i \sin \theta) \tag{2.4.6}$$

It should be noted that the argument of z gives the direction of $\overrightarrow{Oz}$. The argument of a complex number z is infinitely multivalued, hence all of its values differ only by integral multiples of $2\pi$, and are congruent to each other mod $2\pi$. The principal value of $\theta$ is that value of $\theta$ that satisfies

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[8] Hille, op. cit., pp. 26-28.

the condition $-\pi < \theta \leq \pi$. Therefore, two complex numbers are congruent if their principal values coincide and their absolute values are equal. For the null vector, or zero, the amplitude is regarded as undefined or indeterminate.

If two complex numbers differ only in the sign of the imaginary coordinate, they are said to be complex conjugates. That is, if $z = x + iy$ is a complex number, then the conjugate is given by $z = x - iy$. The conjugate is denoted by the symbol $\bar{z}$. In polar coordinates where $z = \rho(\cos\theta + i\sin\theta)$, the conjugate becomes $\bar{z} = \rho(\cos\theta - i\sin\theta)$. When number pairs are used, the conjugate of $z = [\rho, \theta]$ becomes $\bar{z} = [\rho, -\theta]$. In a geometrical representation the conjugate of z is the corresponding point symmetric to the real axis as shown in the following figure.

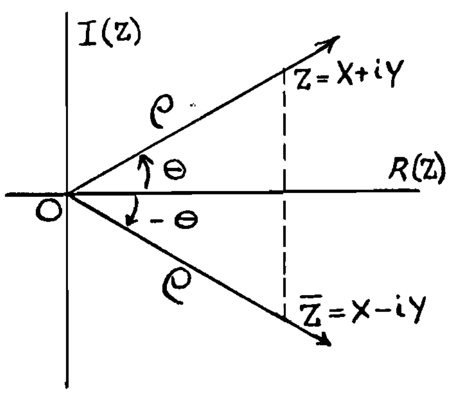


FIGURE 3

COMPLEX CONJUGATE

The important application of the polar form of complex numbers is the convenience of multiplication. Consider any two complex numbers written in polar form, say

$$z_1 = \rho_1(\cos\theta_1 + i\sin\theta_1)$$

$$z_2 = \rho_2(\cos\theta_2 + i\sin\theta_2),$$

then the product of $z_1 z_2$ can be written in the form

$$z_1 z_2 = \rho_1(\cos\theta_1 + i\sin\theta_1)\cdot\rho_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = \rho_1\rho_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)]$$

$$z_1 z_2 = \rho_1\rho_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \qquad (2.4.7)$$

The length of vector $z_1 z_2$ is equal to the product of the lengths of $z_1$ and $z_2$, that is $\rho_1\rho_2$. The angle of inclination of the vector $z_1 z_2$ is the sum of the angles $\theta_1$ and $\theta_2$. The following figure shows geometrically the multiplication of two complex numbers.

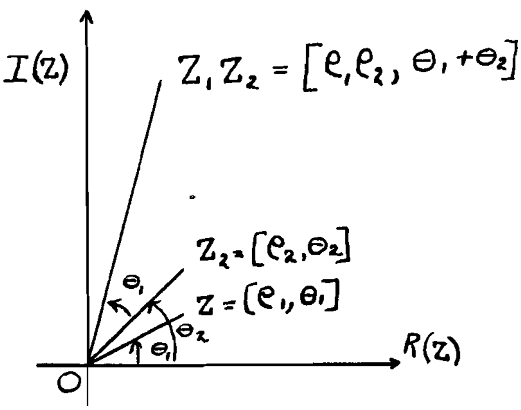


FIGURE 4

MULTIPLICATION OF COMPLEX NUMBERS

Another important use of the polar coordinates and complex numbers is in the representation of a special family of circles. This family has the equation

$$\Gamma(\theta): \quad \arg \frac{z - \xi_1}{z - \xi_1} = \theta, \tag{2.4.8}$$

where $\xi_1$ and $\xi_2$ are any two fixed points in the complex plane and $-\infty < \theta < +\infty$. In essence, these circles are actually circular arcs from $\xi_1$ to $\xi_2$ and the complementary arc is $\Gamma(\theta + 2\pi)$, which of course, is also a member of $\Gamma(\theta)$.

The families $\Gamma(\theta)$ and $\Omega(\xi_1, \xi_2)$ form a family of orthogonal circles if both have the same fixed points. This configuration is often referred to as the circular net or Steiner circles.[9]

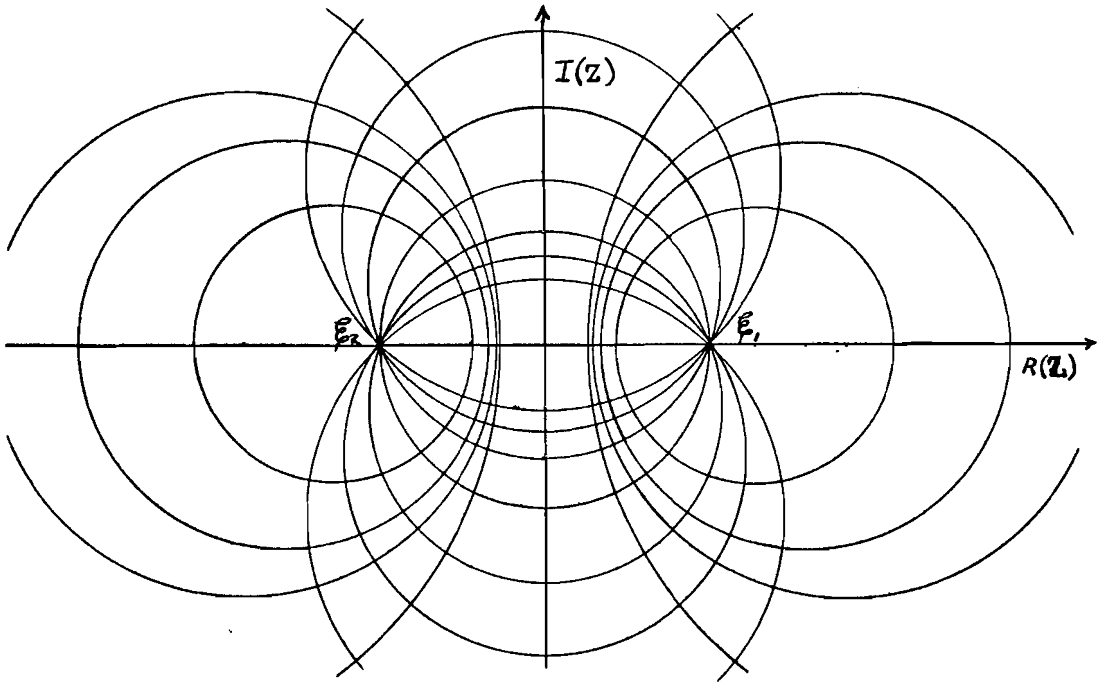


FIGURE 5

STEINER CIRCLES

[9]Lars. V. Ahlfors, Complex Analysis (New York: McGraw Hill Book Company, Inc., 1953), pp. 31-32.

The Steiner circles have some interesting properties that should be mentioned for later use. For example, every point of the complex plane except the fixed points has one and only one element of each of the families $\Upsilon(\theta)$ and $\Omega(\xi_1, \xi_2)$ passing through it. Also, every circle of the family $\Upsilon(\theta)$ meets every circle of $\Omega(\xi_1, \xi_2)$ at right angles, and conversely.[10]

Exponential Representation (2.5)

Complex numbers can be represented in a form of the exponential function. The derivation of the exponential representation of z will be presented as developed by Townsend.[11] It is important to remember that all the properties for $e^x$ must be valid, since the real numbers are embedded in the field of complex numbers.

The definition of the exponential function of the complex number $z = x + iy$ is given by the equation

$$e^z = \lim_{n \to \infty} (1 + z/n)^n, \tag{2.5.1}$$

where n is a positive whole number. For this definition to be true the limit must exist.

It should be noted that

$$1 + z/n = 1 + x/n + iy/n. \tag{2.5.2}$$

[10] Ibid. p. 32.

[11] E. J. Townsend, Functions of a Complex Variable (New York, Henry Holt and Company, 1915), p. 122.

Now let

$$1 + x/n = \rho \cos \theta \quad \text{and} \tag{2.5.3a}$$

$$y/n = \rho \sin \theta. \tag{2.5.3b}$$

By substitution $(1+z/n)^n$ becomes

$$(\rho \cos \theta + i \rho \sin \theta)^n,$$

which is simplified, by use of basic multiplication operations of complex numbers,[12] to the form

$$\rho^n(\cos n\theta + i\sin n\theta). \tag{2.5.4}$$

Since n can be taken so large that $1+x/n$ is always positive, $\cos \theta$ will always be greater than zero. From (2.5.3) the principal value of $\theta$ is the arctan $y/(n+x)$ or

$$\theta = \arctan y/(n+x). \tag{2.5.5}$$

The value of $\rho$ can easily be expressed in terms of x and y by use of the equation (2.4.2), thus

$$\rho = \left[(1+x/n)^2 + (y/n)^2\right]^{\frac{1}{2}},$$

Therefore $\rho^n$ becomes

$$\rho^n = \left[(1+x/n)^2 + (y/n)^2\right]^{n/2}.$$

$$\rho^n = (1+x/n)^n\left[1 + y^2/(n+x)^2\right]^{n/2}. \tag{2.5.6}$$

The limit (2.5.2) can now be simplified by use of (2.5.4), (2.5.5) and (2.5.6).[13] Thus,

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[12] Ruel V. Churchill, <u>Introduction to Complex Variables and Applications</u> (New York, McGraw-Hill Book Company, Inc., 1948), pp. 11-12.

[13] Townsend, <u>op. cit.</u>, pp. 122-3.

$$\lim_{n\to\infty} (1+z/n)^n = \lim_{n\to\infty} e^h(\cos n\theta + i\sin n\theta)$$

$$= \lim_{n\to\infty} (1+x/n)^n\left[1+y^2/(n+x)^2\right]^{n/2}.$$

$$(\cos n\theta + i\sin n\theta)$$

$$= \lim_{n\to\infty} (1+x/n)^n. \qquad (2.5.7a)$$

$$\lim_{n\to\infty} \left[1+y^2/(n+x)^2\right]^{n/2}. \qquad (2.5.7b)$$

$$\left[\lim_{n\to\infty} \cos n \arctan y/(n+x) + \right.$$

$$\left. i \lim_{n\to\infty} \sin n \arctan y/(n+x)\right]$$

provided these limits exist. These limits do exist and can be easily evaluated.

By the definition of $e^x$, (2.5.7a) becomes

$$\lim_{n\to\infty} (1+x/n)^n = e^x. \qquad (2.5.8)$$

To evaluate (2.5.7b), two situations must be examined. That is, when $y=0$, then the limit becomes one. If $y\neq 0$, then it is necessary to rewrite (2.5.7b) in the form

$$\left\{\lim_{n\to\infty} \left[1+y^2/(n+x)^2\right]^n\right\}^{\frac{1}{2}}. \qquad (2.5.9)$$

Now $\lim_{n\to\infty} \left[1+y^2/(n+x)^2\right]^n$

$$= \left\{\lim_{n\to\infty}\left[1+\frac{1/(n+x)^2}{y^2}\right]^{\frac{(n+x)^2}{y^2}}\right\}^{\lim_{n\to\infty} \frac{n\cdot y^2}{(n+x)^2}}. \quad 14$$

Separating the exponent and applying limit rules

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[14] *Ibid.*, p. 123.

$$\lim_{n\to\infty}\left[1+\frac{1}{\frac{(n+x)^2}{y^2}}\right]^{\frac{(n+x)^2}{y^2}} = e^1 \text{ and}$$

$$\lim_{n\to\infty} n\left[y^2/(n+x)^2\right] = 0.$$

Hence,

$$\lim_{n\to\infty}\left[1+\frac{y^2}{(n+x)^2}\right]^n = e^{0\circ 1} = 1. \qquad (2.5.10)$$

By substituting (2.5.10) into (2.5.9), it can be seen that

$$\left\{\lim_{n\to\infty}\left[1+y^2/(n+x)^2\right]^n\right\}^{\frac{1}{2}} = (e^{0\circ 1})^{\frac{1}{2}} = 1. \qquad (2.5.11)$$

It should be noted that the cosine is a continuous function and the

$$\lim_{n\to\infty} \cos n \arctan y/(n+x)$$

$$= \cos \lim_{n\to\infty} n \arctan y/(n+x)$$

$$= \cos \lim_{n\to\infty} ny/(n+x) \cdot \frac{\arctan y/(n+x)}{y/(n+x)}$$

$$= \cos y. \qquad (2.5.12a)$$

Similarly,

$$\lim_{n\to\infty} \sin n \arctan y/(n+x) = \sin y. \qquad (2.5.12b)$$

Using the limits (2.5.8), (2.5.11) and (2.5.12) equation (2.5.7) now becomes

$e^z = e^x(\cos y + i\sin y)$.[15]

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[15] Ibid.

If $x = 0$ in (2.5.13), the equation becomes

$$e^{iy} = \cos y + i\sin y$$

or writing the above in conventional notation

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (2.5.14)$$

Therefore, the complex number z in polar coordinates,

$$z = \rho\,(\cos\theta + i\sin\theta),$$

can also be written as

$$z = \rho\, e^{i\theta}. \qquad (2.5.15)$$

Complex numbers of the form (2.5.15) can easily be multiplied. For example, the product of any two complex numbers say, $z_1 = \rho_1 e^{i\theta_1}$ and $z_2 = \rho_2 e^{i\theta_2}$ would be

$$z_1 z_2 = (\rho_1 e^{i\theta_1})\,(\rho_2\, e^{i\theta_2})$$

$$z_1 z_2 = \rho_1 \rho_2 e^{i}(\theta_1 + \theta_2).$$

Again the vector length of $z_1 z_2$ is equal to $\rho_1\rho_2$ and the angle of inclination is equivalent to the sum of the arguments of the two numbers being multiplied or $\theta_1 + \theta_2$.

Spherical Representation (2.6)

The complex numbers have been represented as points on a plane. For some purposes it is important to employ a surface in three dimensions, specifically the sphere. If the sphere is to perform the same function as the plane, then, there must exist a one-to-one mapping of the sphere onto the plane and vice versa.

The common procedure for performing this one-to-one correspondence is to use the complex plane and to place a sphere of radius one-half unit, tangent to the plane at the origin. The point of tangency is called the south pole, S, and the point diametrically opposite it is called the north pole, N.

It is possible to systematically project every point of the plane onto the sphere if the north pole is the center of projection. This is accomplished by considering the rays that emit from N and intersect the plane at point P. Each ray will also intersect the sphere at some point called the image or P'. A mapping of this kind is referred to as a stereographic projection.

The stereographic projection sets up a one-to-one correspondence between the points on the complex plane and the sphere. In a similar manner all the points on the sphere correspond to one point on the plane, with the exception of N. If a new complex number-$\infty$, called infinity-is introduced, then this one exception is removed because N now maps to $\infty$. This new number performs as any other number, except that it cannot be used in the combinational operations. The point $\infty$ is considered to be infinitely distant and therefore the length of its vector is not defined. Whereas, all the other complex numbers are considered to be finite. Considering only the ordered pairs of real numbers the complex plane will be called finite or open. By including the point $\infty$, the plane will be referred to as the infinite or closed plane.

In working with the complex plane it is beneficial to be able to map a given configuration of the sphere onto the plane and then map it back onto the sphere after performing certain operations. The procedure for determining this mapping and the properties of the stereographic transformation are discussed in detail by Townsend.[16]

[16]Townsend, Ibid., pp. 184-90.

CHAPTER III

MOBIUS TRANSFORMATIONS

The properties of certain elementary mappings of the complex plane onto itself are essential to the development of the topic to be considered. It is often convenient to use two planes to visulize the mappings. These planes will be referred to as the z-plane and the w-plane. The w-plane is used to represent the image of the z-plane under a given transformation.

The general form of the basic transformation to be considered is defined by the fraction

$$T(z) = w = \frac{az+b}{cz+d}, \qquad (3)$$

where z is the independent complex variable, w is the image of the mapping and a, b, c and d are predetermined complex constants. The mapping is known as the Mobius transformation, because A. F. Mobius began the study of an equivalent class of geometrical transformations. He called this set of mappings "Kreisverwandtschaft", which means circle relationships. This same mapping can be referred to as the bilinear transformation, fractional linear transformation, homographic transformation or homography.[1]

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[1]Hille, op. cit., p. 46.

To complete the definition (3), it is necessary to include the following restriction, that is

$$ad - bc \neq 0. \tag{3a}$$

The reason this stipulation must be included is that the equality produces a degenerate case. In other words, the image points will be identical to the points in the z-plane, the image points are undefined, or the image points are a constant or zero. The extended definition now allows it to be said that all Mobius transformations map the points in the z-plane to unique points in the w-plane, except when $z = -d/c$. Let the point $\infty$ of the w-plane be assigned to correspond with $z = -d/c$ and when $z = \infty$ let $w = a/c$. Now the transformations set up a one-to-one correspondence between all the points in the two planes.

The study of the Mobius transformations can be approached by discussing the properties of those mappings that have specially assigned complex constants. It will then be possible to discuss the general transformation in relation to the simpler mappings.

Translation (3.1)

When the complex constants are assigned the values $a = d$, $c = 0$ and b any arbitrary complex number, the relationship (3) becomes

$$w = z + B, \tag{3.1.1}$$

where $B = b/d$. The images of z are obtained by adding the vector B to each value of z being considered. This is a rigid motion transformation that carries each point z of the z-plane the distance $|B|$ in the direction $\Theta$, where $\Theta = \arg B$. For example, consider the region $0 \leq R(z) \leq 2$ and $0 \leq I(z) \leq 3$ of the z-plane. The following figure shows the region and its image under the mapping $w = z + (2+3i)$.

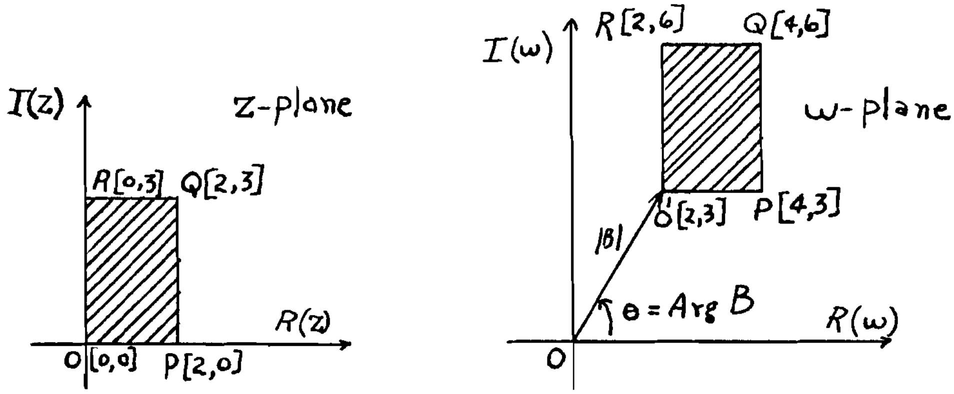


FIGURE 6

TRANSLATION

It should be noted that the image is congruent to the given region. Generally speaking, the whole plane is translated parallel to the line joining the origin to the point B. Each point is moved except the invariant point $\infty$. A line parallel to $\overrightarrow{OB}$ is translated into itself. Any other line is transformed into a line parallel to the given line.

Rotation and/or Dilation (3.2)

If the constants of (3) are designated as $b = c = 0$, a and d to be any arbitrary constants, the Mobius transformation becomes

$$w = Az, \tag{3.2.1}$$

where $A = a/d$. Since $A$ is a complex number, it can be written as

$$A = \rho e^{i\theta},$$

where $\rho = |A|$ and $\theta = \arg A$. Assuming $z = re^{i\phi}$, (3.2.1) becomes

$$w = (\rho e^{i\theta})(re^{i\phi})$$

$$w = \rho re^{i(\theta+\phi)} \ . \tag{3.2.2}$$

When a value of $A$ is chosen such that $|A| = \rho = 1$, (3.2.2) takes the form

$$w = re^{i(\theta+\phi)}, \tag{3.2.3}$$

which is another rigid motion mapping known as rotation. Geometrically speaking (3.2.3) is a rotation of the radius vector about the origin through the angle $\theta$. For example, the region $0 \leq R(z) \leq 2$, $0 \leq I(z) \leq 3$ under the Mobius transformation $w = \left(\frac{\sqrt{2}+\sqrt{2}\,i}{2}\right) z$ is shown in figure 7.

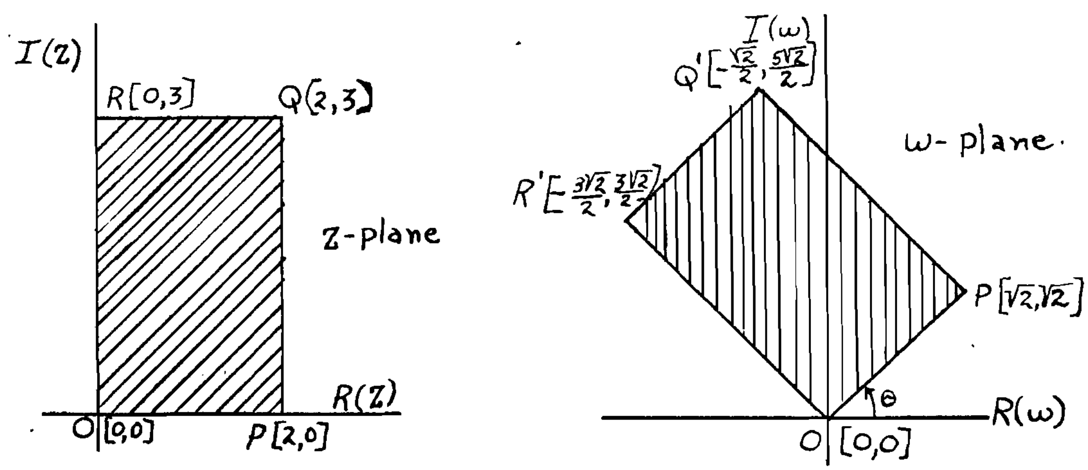


FIGURE 7

ROTATION

It should be noted that from the definition of polar coordinates the Arctan $\Theta = \frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = 1$ or $\Theta = 45^\circ$. Therefore the w-plane shows the region of the z-plane rotated $45^\circ$ around the origin. The image is congruent to the region under consideration.

The origin is always an invariant point of the rotation transformation. If the extended plane is being used, then $\infty$ is mapped into itself. Also, circles with centers at the origin are mapped into themselves, with the interior of each circle transformed into itself. A line through the origin is carried into another line through the origin making an angle $\Theta$ with the first line.

When a value for A is chosen such that $|A| = \rho \neq 1$ and arg A or $\Theta = 0$, then (3.2.2) would be

$$w = \rho r e^{i\phi}. \tag{3.2.4}$$

This transformation is known as dilation or stretching. It is often classified by the relationship of $\rho$ to 1. That is, if $\rho > 1$, the mapping is known as a magnification, and if $\rho < 1$ it is referred to as a contraction.

In this mapping the radius vector is magnified or contracted by a ratio of $\rho$: 1. For example the region $0 \leq R(z) \leq 2$ and $0 \leq I(z) \leq 3$ is magnified by the Mobius transformation

$$w_1 = 2z$$

and contracted by

$$w_2 = z/2$$

as shown in figure 8.

It should be noted that the image is similar to the given region. The invariant points of the dilation mappings are the same as for rotation. In (3.2.4) some properties should be observed. These are: (1) all straight lines through the origin are transformed into themselves, (2) a half plane on one side of a line through the origin is mapped into itself, and (3) any circle with center at the origin is carried into another circle with center at the origin.

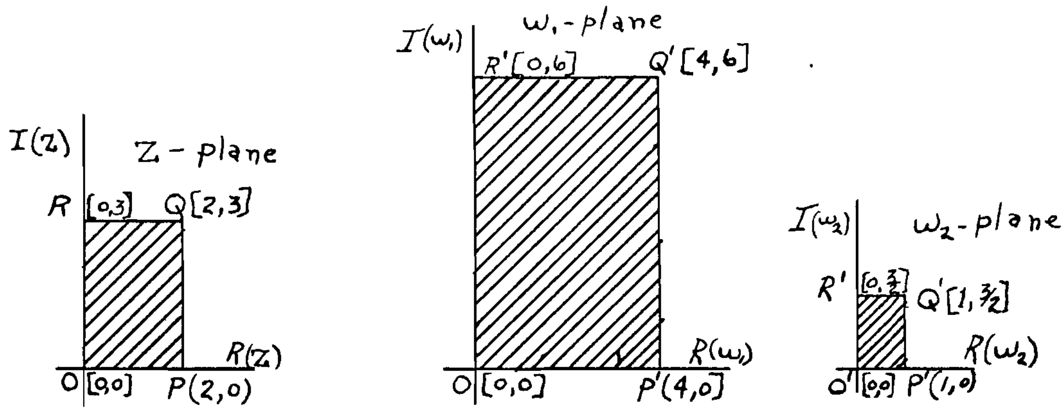


FIGURE 8

DILATION

When a value for A is chosen such that $|A| \neq 1$ and $\arg A \neq 0$, the Mobius transformation takes the form given in (3.2.2). This mapping can be imagined as a combination of (3.2.3) and (3.2.4). Obviously, the invariant points are 0 and $\infty$. It should be mentioned that all circles with center at the origin are carried into other circles having the same center. Each half line through the origin is mapped into a half line making an angle $\theta$ with the given line.

Linear Transformation (3.3)

A linear mapping is a Mobius transformation that is a combination of the mappings discussed in (3.1) and (3.2). The complex constant $c = 0$, while the other constants are any complex numbers. The general form of the linear transformation is

$$w = Az + B \quad , \tag{3.3.1}$$

where $A = a/d$ and $B = b/d$.

The linear transformation is the product of two mappings. Let,

$Z = Az$ and

$w = Z + B$,

then (3.3.1) is a rotation and/or dilation followed by a translation. It is important to perform the mappings in the indicated order, because these transformations are not commutative.

Generally speaking, under a linear transformation the image is similar to the given region. The invariant points are $\infty$ and $b/(d - a)$. The determination of the invariant points will be described in 4.5.

Reciprocal Transformation (3.4)

The Möbius transformation

$$w = 1/z \tag{3.4.1}$$

is obtained by letting $a = d = 0$ and $b = c$. This mapping is referred to as the reciprocal transformation. It should be mentioned that this particular mapping is an involution. That is, repeated application of this transformation gives the original region as an image. By using polar coordinates in exponential form, (3.4.1) can be written as

$$w = 1/\, re^{i\theta} \text{ or}$$

$$w = \rho\, e^{i\phi}, \tag{3.4.2}$$

where $\rho = 1/r$ and $\phi = -\theta$.

It can be seen that (3.4.2) is the product of two consecutive transformations. The mappings are

$$z' = \frac{1}{r}\, e^{i\theta} \tag{3.4.3a}$$

$$w = \bar{z}'. \tag{3.4.3b}$$

The point $z'$ is obtained by inversion with respect to the unit circle. Geometrically speaking, $z'$ is the point on the half ray originating at the origin and passing through z such that $|z| \circ |z'| = 1$. The final image or w is the conjugate of $z'$, which is the rotating of the point $z'$ about the real axis. This operation is often called reflection. The following figure shows the image of any point z under the Mobius transformation $w = 1/z$.

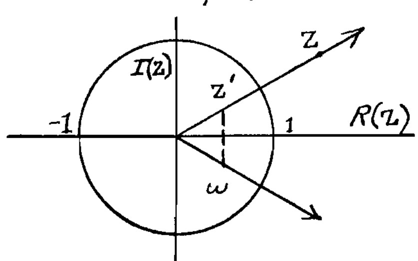


FIGURE 9

RECIPROCATION

It should be noted that the points outside the unit circle are carried to points inside the unit circle and vice versa. Those points on the unit circle are mapped to other points on the unit circle. The points 1 and -1 are left invariant under this mapping.

To consider the point at infinity it should be observed when z increases without bound, z' decreases correspondingly. This is stated symbolically by

if $z \to \infty$, then $z' \to 0$.

Therefore the image of $z = \infty$ is 0 under the mapping $w = 1/z$. Similarly, when $z = 0$, then $w = \infty$.

Generally speaking, the reciprocal transformation maps circles or straight lines into circles or straight lines. This is shown by using the general equation for a circle of analytical geometry, that is,

$$A(x^2 + y^2) + b_1 x + b_2 y + c = 0. \tag{3.4.4}$$

Let

$B = \frac{1}{2A}(b_1 - ib_2)$ and since

$z\bar{z} = x^2 + y^2$,

the relationship (3.4.4) can be rewritten with these substitutions. The new general equation of a circle becomes

$$Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0, \tag{3.4.5}$$

where $A \neq 0$ and C are real constants.

When $C = 0$, (3.4.5) becomes the equation of a circle that passes through the origin, $z = 0$. When $A = 0$, the general relationship reduces to a straight line.

The reciprocal mapping maps the circle given by (3.4.5) into another circle given by

$$C w\bar{w} + B\bar{w} + \bar{B}w + A = 0, \text{ if } C \neq 0. \tag{3.4.6}$$

The above equation is obtained by substitution. If $C = 0$, the general circle maps into a straight line not through the origin.

When $A = 0$ and $C \neq 0$, the equation (3.4.5) would represent straight lines not through the origin. Applying $w = 1/z$, the image would take the form

$$C w\bar{w} + B\bar{w} + \bar{B}w = 0,$$

which are circles through the origin.

Finally, when $A = 0$ and $C = 0$, (3.4.5) becomes

$$Bz + \bar{B}\bar{z} = 0,$$

or straight lines through the origin. The reciprocal transformation would transform these lines into

$$B\bar{w} + \bar{B}w = 0,$$

which are obviously straight lines through the origin.

As an example of the preceeding discussion, consider the lines $x = L_1 \neq 0$ and $y = L_2 \neq 0$. The images of these lines under the mapping (3.4.1) would be circles which are tangent to the axes at the origin of the w-plane. If

$$w = u + iv \text{ and } z = x + iy$$

then the mapping $w = 1/z$ would be

$$u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}.$$

Therefore,

$$\frac{x}{x^2+y^2} \longrightarrow u \quad \text{and} \quad \frac{-y}{x^2+y^2} \longrightarrow v.$$

Also,

$$u^2+v^2 = \frac{1}{x^2+y^2}$$

hence,

$$\frac{u}{u^2+v^2} \longrightarrow x \quad \text{and} \quad \frac{-v}{u^2+v^2} \longrightarrow y.$$

So when $x = L_1$ and $y = L_2$, then

$$\frac{u}{u^2+v^2} = L_1 \quad \text{and} \quad \frac{-v}{u^2+v^2} = L_2. \tag{3.4.7}$$

Upon completing the squares of the equations (3.4.7), the following relationships develop. That is,

$$u^2+v^2 - \frac{u}{L_1} = 0 \qquad\qquad u^2+v^2+\frac{v}{L_2} = 0$$

$$\left(u - \frac{1}{2L_1}\right)^2 + v^2 = \left(\frac{1}{2L_1}\right)^2 \qquad\qquad u^2 + \left(v+\frac{1}{2L_2}\right)^2 = \left(\frac{1}{2L_2}\right)^2$$

which are the images of the given lines.[2] The given set of lines and their images are shown in figure 10.

Mobius Transformation (3.5)

The Mobius transformation as defined in (3) can be

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[2]Churchill, *op. cit.*, p. 55.

shown to be composed of the three transformations already presented in this chapter. There are two cases according as $c=0$ or $c\neq 0$. In the first case,

$$w = \frac{az+b}{d}$$

which reduces to

$$w = \frac{az}{d} + \frac{b}{d}. \qquad (3.5.1)$$

Obviously, (3.5.1) is a dilation and/or rotation followed by a translation.

If $c=0$ the derivation is as follows,

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ &= \frac{a}{c}\left[\frac{z+b/a}{z+d/c}\right] \\ &= \frac{a}{c}\left[\frac{z+b/a+d/c\ -\ d/c}{z+d/c}\right] \\ &= \frac{a}{c}\left[1+\frac{b/a\ -\ d/c}{z+d/c}\right] \\ &= \frac{a}{c}\left[1+\frac{bc\ -\ ad}{ac(z+d/c)}\right] \\ &= \frac{a}{c}+\frac{bc\ -\ ad}{c^2}\left(\frac{1}{z\quad d/c}\right). \end{aligned} \qquad (3.5.2)$$

The equation (3.5.2) should be considered as a product of the simpler mappings. That is, let

$z_1 = z + d/c$ (translation) (3.5.2a)

$z_2 = 1/z_1$ (reciprocation) (3.5.2b)

$$z_3 = \left(\frac{bc - ad}{c^2}\right) z_2 \quad \text{(dilation and/or rotation)} \qquad (3.5.2c)$$

then

$$w = z_3 + a/c. \qquad \text{(translation)} \qquad (3.5.2d)$$

The order of performing these mappings is important because the elementary mappings are not commutative.

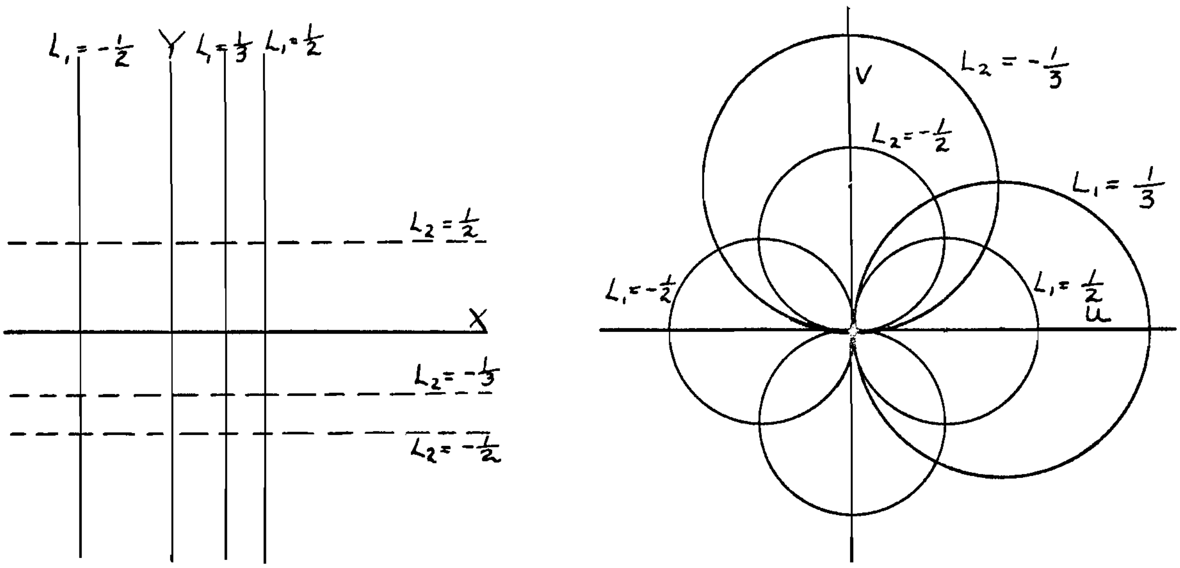


FIGURE 10

RECIPROCATION

## CHAPTER IV

## PROPERTIES OF MOBIUS TRANSFORMATIONS

In order to fully appreciate the usefulness of the Mobius transformation, its properties must be considered. It has been established that this mapping transforms the z-plane onto the w-plane in a one-to-one correspondence.

Group (4.1)

The set of Mobius transformations forms a group M. It should be remembered that a group is an algebraic structure consisting of a set G of elements and a binary law of combination (*) on G having the following properties:

A) There exists an identity element E in G, such that $a * E = E * a = a$, where a is an element of G.

B) The operation * is associative in G. That is, $(a * b) * c = a * (b * c)$, where a, b and c are elements of G.

C) Every q in G has an inverse in G, denoted by $q^{-1}$, such that, $q * q^{-1} = q^{-1} * q = E$.

A subgroup of G is a subset of G that is also a group with respect to the operation of G restricted to the subset, and having the same identity element.[1]

The set M of elements used to define a transformation

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[1]McCoy, *op. cit.*, p. 167.

group will consist of all the transformations of the form

$$T(z) = w = \frac{az+b}{cz+d}, \text{ where } ad - bc \neq 0.$$

The functional notation is used as a convenience in verifying the group properties. When the argument of the function is clearly understood, it is usually omitted. For example, in the above relationship, it would be satisfactory to use

$$T = \frac{az+b}{cz+d}.$$

If R is another transformation of M, such that

$$w' = R(w),$$

then the definition for the binary operation, called multiplication is

$$w' = R\left[T(z)\right] = RT(z) = RT. \tag{4.1.1}$$

It should be noted that RT is a single linear transformation resulting from the application of two transformations. The order of performance of the operation is from right to left. In other words, first perform the transformation T and then operate with R on the image of T.

The identity element E of M is

$$T(z) = w = z \tag{4.1.2}$$

If three transformations of M are given, say R, S, and T, then

$$R(ST) = (RS)T. \tag{4.1.3}$$

The law of association is verified by direct substitution.

Finally, the inverse of T or $T^{-1}$ is given by the transformation

$$T^{-1}(z) = \frac{-dw+b}{cw-a} \quad . \tag{4.1.4}$$

It has now been established that the set of all Mobius transformations forms a group M.

There exists various subgroups of M. Obviously, the elementary transformations of rotation, dilation, and translation are subgroups of M. The transformations of the form $w = 1/z$ do not form a subgroup of M because the set does not contain the identity element. A special subgroup of M is composed of all the Mobius transformations where the constants have the relationship, $ad-bc=1$. These transformations are said to be of normal form and are referred to as the unimodular group.[2]

Circle Preserving (4.2)

The straight line is sometimes considered to be a special case of a circle. That is, a straight line is a circle considered to have an infinite radius. It can then be said that the Mobius transformation maps circles into circles. This circle preserving property can be established from the fact that the three simple transformations that compose the Mobius transformation carry circles and straight

[2]Constantin Caratheodory, Theory of Functions of a Complex Variable (New York: Chelsea Publishing Company, 1954), p. 5.

lines into circles and straight lines. To fully verify this property it is necessary to develop the concept of cross ratio.

Cross Ratio (4.3)

Consider four points A, B, C, and D on a given line L as shown below.

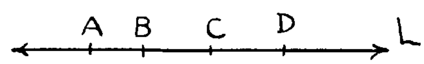


FIGURE 11

LINE L

Suppose the line segment $\overline{AB}$ is divided by C to give the ratio

$$\overline{AC} : \overline{CB}, \qquad (4.2.1)$$

then divided by D to obtain the corresponding ratio,

$$\overline{AD} : \overline{DB} . \qquad (4.2.2)$$

Cross ratio, sometimes called anharmonic ratio, is the ratio of (4.2.1) to (4.2.2). That is

$$(\overline{AC} : \overline{CB}) : (\overline{AD} : \overline{DB}).$$

The above relationship can also be written in the following manner

$$\frac{\overline{AC} \circ \overline{DB}}{\overline{AD} \circ \overline{CB}} \qquad (4.2.3)$$

Symbolically (4.2.3) is denoted by (AB,CD). Notice grouping and order are essential. When the cross ratio is to be applied it is usually rewritten in the form

$$\frac{(c-a)\circ(b-d)}{(d-a)\circ(b-c)}, \qquad (4.2.4)$$

where the small letters are the coordinates of the points.

The four points given in figure 11 have twenty-four permutations, that is, different arrangements. Therefore, there would be twenty-four cross ratios of four points. Some of the permutations have the same cross ratio values. In fact, there are only six distinct values of the cross ratios. For example, all of the following have the same value:

$(\overline{AB},\overline{CD})$; $(\overline{BA},\overline{DC})$; $(\overline{CD},\overline{AB})$; $(\overline{DC},\overline{BA})$.

Relationship (4.2.3) is the ratio of products of line segments. That is, the cross ratio is the ratio of product $\overline{AC}\circ\overline{DB}$ to the product $\overline{AD}\circ\overline{CB}$. Since there exists four points, these determine six different line segments. The line segments can be paired as above to form the products

$$\overline{AB}\circ\overline{CD};\ \overline{AC}\cdot\overline{DB};\ \overline{AD}\circ\overline{BC}\ . \qquad (4.2.5)$$

The cross ratio of four points now becomes the ratio of two products (4.2.5) along with a negative sign. If the values of (4.2.5) are r, s, and t respectively, then the six distinct cross ratios are

$$\begin{array}{ccc} -\frac{r}{s} & -\frac{s}{t} & -\frac{t}{r} \\ -\frac{s}{r} & -\frac{t}{s} & -\frac{r}{t} \end{array} \qquad (4.2.6)$$

The above ratios are not independent but rather in reciprocal pairs and the product of each row it -1. Also,

$r+s+t=0$. That is,

$$r+s+t=\overline{AB}\cdot\overline{CD}+\overline{AC}\cdot\overline{DB}+\overline{AD}\cdot\overline{BC}$$

$$=\overline{AB}\cdot\overline{CD}+\overline{AC}\cdot\overline{DB}+(\overline{AC}+\overline{CD})(\overline{BD}+\overline{DC})$$

$$=\overline{AB}\cdot\overline{CD}+\overline{AC}\cdot\overline{DB}+\overline{AC}\cdot\overline{BD}+\overline{AC}\cdot\overline{DC}+\overline{CD}\cdot\overline{BD}+\overline{CD}\cdot\overline{DC}$$

$$=\overline{AB}\cdot\overline{CD}+\overline{AC}\cdot\overline{DB}-(\overline{AC}\cdot\overline{DB}+\overline{AC}\cdot\overline{CD}+\overline{CD}\cdot\overline{DB}+\overline{CD}\cdot\overline{CD})$$

$$=\overline{AB}\cdot\overline{CD}+\overline{AC}\cdot\overline{DB}-\overline{AC}\cdot\overline{DB}-\overline{AC}\cdot\overline{CD}-\overline{CD}\cdot\overline{CD}-\overline{CD}\cdot\overline{DB}$$

$$=\overline{AB}\cdot\overline{CD}-\overline{CD}(\overline{AC}+\overline{CD}+\overline{DB})$$

$$=\overline{AB}\cdot\overline{CD}-\overline{CD}\cdot\overline{AB}$$

$$r+s+t=0.$$

It can now be shown that if $-r/s=K$, then $1-K$ is also a cross ratio. Hence,

$$-\frac{r}{s}-\frac{t}{s}-\frac{s}{s}=0$$

$$-\frac{r}{s}-\frac{t}{s}=1$$

$$-\frac{t}{s}=1+\frac{r}{s}$$

$$-\frac{t}{s}=1-K \qquad (4.2.7)$$

If $-r/s=K$, then $(K-1)/K$ is another cross ratio. This relationship is developed from

$$(-r/s)(-s/t)(-t/r)=-1. \qquad (4.2.8)$$

The value for $-s/t$ in (4.2.7) becomes $s/(s+r)$, therefore (4.2.8) can be rewritten as

$$(-r/s)\left[s/(s+r)\right](-t/r)=-1$$

$$-t/r=(s+r)/r$$

$$-t/r=s/r+1$$

$$-t/r = -1/K + 1$$

$$-t/r = (K - 1)/K \qquad (4.2.9)$$

The reciprocals for (4.2.7) and (4.2.9) are easily found. Therefore if $-r/s = K$, then the cross ratios of (4.2.5) can be represented as:

$$\begin{array}{ccc} K & \frac{1}{1-K} & \frac{K-1}{K} \\ \frac{1}{K} & 1-K & \frac{K}{K-1} \end{array} \qquad (4.2.10)$$

The cross ratios under consideration have been for distinct finite points. If the value of one of the points becomes infinite the cross ratio (4.2.3) reduces to a simple ratio. For example, let D become infinite, then the cross ratio

$$\frac{\overline{AC}}{\overline{CB}} : \frac{\overline{AD}}{\overline{DB}}$$

becomes

$$\frac{\overline{AC}}{\overline{CB}} : -1$$

since $\overline{AD}$ and $\overline{DB}$ are equal in magnitude but opposite in direction. Thus, $(AB, C\infty)$ becomes $\overline{AC} : \overline{BC}$.

From (4.2.4) it can be seen that a line segment is represented as the difference of two coordinates. If two images of a Mobius transformation are $w_i$ and $w_j$, then the line segment used in the cross ratio would be $w_i - w_j$. Obviously,

$$w_i - w_j = \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d}$$

$$= \frac{(az_i + b)(cz_j + d) - (az_j + b)(cz_i + d)}{(cz_i + d)(cz_j + d)}$$

$$= \frac{acz_i z_j + adz_i + bcz_j + bd - (acz_i z_j + adz_j + bcz_i + bd)}{(cz_i + d)(cz_j + d)}$$

$$= \frac{acz_i z_j + adz_i + bcz_j + bd - acz_i z_j - adz_j - bcz_i - bd}{(cz_i + d)(cz_j + d)}.$$

$$= \frac{adz_i + bcz_j - adz_j - bcz_i}{(cz_i + d)(cz_j + d)}$$

$$= \frac{adz_i - adz_j - bcz_i + bcz_j}{(cz_i + d)(cz_j + d)}$$

$$= \frac{ad(z_i - z_j) - bc(z_i - z_j)}{(cz_i + d)(cz_j + d)}$$

$$= \frac{ad - bc}{(cz_i + d)(cz_j + d)} (z_i - z_j) \qquad (4.2.11)$$

Let $i = 3$, $j = 1$ and $i = 2$, $j = 4$ in (4.2.11) and then form the product. Thus,

$$(w_3 - w_1)(w_2 - w_4) = Q(z_3 - z_1)(z_2 - z_4), \qquad (4.2.12)$$

where

$$Q = \frac{(ad-bc)^2}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}.$$

Since Q is a symmetrical relationship with respect to $z_K$, where ($K = 1,2,3,4$), other values for i and j will obtain the same Q. For example, let $i = 4$, $j = 1$ and $i = 2$, $j = 3$, then form the product. Thus

$$(w_4 - w_1)(w_2 - w_3) = Q(z_4 - z_1)(z_2 - z_3). \qquad (4.2.13)$$

If the values for $z_K$ and $w_K$, where ($K = 1,2,3,4$) are finite and distinct, the quotient of (4.2.12) and (4.2.13) is

$$\frac{(w_3 - w_1)(w_2 - w_4)}{(w_4 - w_1)(w_2 - w_3)} = \frac{(z_3 - z_1)(z_2 - z_4)}{(z_4 - z_1)(z_2 - z_3)} \tag{4.2.14}$$

Each side of the relationship (4.2.14) is the cross ratio of four finite distinct points. The right hand side can be designated by

$$(z_1, z_2; z_3, z_4). \tag{4.2.15}$$

Again there are twenty-four cross ratios of the four points, but there exists only six distinct values. That is, all of the following have the same value:

$$(z_1, z_2; z_3, z_4),\ (z_3, z_4; z_1, z_2),\ (z_2, z_1; z_4, z_3),\ (z_4, z_3; z_2, z_1).$$

If (4.2.14) exists then this relationship asserts that under a Mobius transformation the cross ratio of four distinct finite points is invariant if these points map into four distinct finite points. The proof of the invariance property is obtained by substituting $w_i = \frac{az_i + b}{cz_i + d}$, as ($i$=1,2,3,4) in (4.2.14) and reducing both sides of the equation to identical expressions.

If $(z_1, z_2; z_3, z_4) = \lambda$, then all the possible cross ratio values can be obtained with respect to $\lambda$ as in (4.2.10). The values are:

$$\begin{aligned}
(z_1, z_2; z_3, z_4) &= \lambda \\
(z_1, z_2; z_4, z_3) &= 1/\lambda \\
(z_1, z_3; z_2, z_4) &= 1 - \lambda \\
(z_1, z_3; z_4, z_2) &= 1/(1 - \lambda) \\
(z_1, z_4; z_3, z_2) &= \lambda/(\lambda - 1) \\
(z_1, z_4; z_2, z_3) &= (\lambda - 1)/\lambda
\end{aligned} \tag{4.2.16}$$

The values for $\lambda$ are usually complex numbers as seen from the definition, hence the cross ratios are points on the complex plane. If $\lambda$ corresponds to circles in the complex plane, then the values for the various ratios also correspond to circles. If $\lambda$ is the points in a region bounded by circles, then the other ratios are similarly described.[3]

The definition of cross ratio can be extended to include infinite points. If $z_1 = \infty$, the ratio is denoted by $(\infty, z_2; z_3, z_4)$. When limit and continuity definitions are used the value of $\lambda$ is obtained as shown below.

$$\begin{aligned}
\lambda &= \lim_{z_1 \to \infty} \frac{(z_3 - z_1)(z_2 - z_4)}{(z_4 - z_1)(z_2 - z_3)} \\
&= \lim_{z_1 \to \infty} \frac{(z_3/z_1 \quad - 1)(z_2 - z_4)}{(z_4/z_1 \quad - 1)(z_2 - z_3)} \\
&= \lim_{z_1 \to \infty} \frac{(0 - 1)(z_2 - z_4)}{(0 - 1)(z_2 - z_3)} \\
&= \frac{z_2 - z_4}{z_2 - z_3} \qquad (4.2.17)
\end{aligned}$$

In a similar manner the value for $\lambda$ can be obtained for $z_2$, $z_3$ or $z_4$ as these values approach infinity. Therefore, the cross ratio of four distinct points is invariant under a Mobius transformation.

The cross ratio can be expressed as a Mobius transformation, that is

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[3]Townsend, _op. cit._, pp. 180-2.

$$\lambda = \frac{Az_1 + B}{Cz_1 + D},$$

where, $A = z_4 - z_2$, $B = z_3(z_2 - z_4)$, $C = z_3 - z_2$, and $D = z_4(z_3 - z_2)$. It can now be seen that three distinct points determine one and only one Mobius transformation, because the fourth point $z_1$ assumes an arbitrary value in such a way that the cross ratio equals $\lambda$. Hence, for the cross ratio of four points, $(z_1, z_2; z_3, z_4)$, to have a unique value at least three of the four points must be distinct.

If one of the points $z_1$, $z_2$ or $z_3$ coincides with $z_4$, the values for $\lambda$ become $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$, respectively. For a given cross ratio $\lambda$, there exists a unique fourth point determined by three points, it is then easily verified that

$$(1, \infty; 0, \lambda) = \lambda. \tag{4.2.18}$$

The three points $1, \infty$, and $0$ all lie on the real axis. If $\lambda$ is a real number, the fourth point also lies on the real axis. It can now be said that four points $z_1$, $z_2$, $z_3$ and $z_4$ lie on the same circle $C^*$, if and only if the value of the cross ratio $(z_1, z_2; z_3, z_4)$ is real.[4]

It was mentioned earlier that as $z_1 \longrightarrow z_3$ the cross ratio approached zero, consequently, $w_1 \longrightarrow w_3$. Similarly as $z_1 \longrightarrow z_2$ and $z_1 \longrightarrow z_4$ the images of these points as obtained from the cross ratio are $w_2$ and $w_4$ respectively.

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[4]Caratheodory, op. cit., p. 30.

If $C_z$ and $C_w$ are circles on the z-plane and w-plane respectively, then there exists a unique Mobius transformation that will map $C_z$ onto $C_w$. The mapping is easily obtained by choosing any three points on $C_z$ and any three points on $C_w$, then applying the cross ratio. When the cross ratio is solved for w, the desired transformation is obtained.[5]

Conformality (4.4)

Another important property to be considered in this chapter is that the Mobius transformation is a conformal mapping. Simply, a mapping is said to be conformal if it preserves angles and sense of direction. It seems obvious that this would be a property of the Mobius transformations since rotation, translation, dilation and reciprocation are conformal.

To prove that the Mobius transformation is a conformal mapping it is necessary to use the following theorem, at each point where a function $f(z)$ is analytic and $f'(z) \neq 0$, the mapping $w = f(z)$ is conformal.[6]

The proof is made in two steps. First, it is necessary to show the Mobius transformation is an analytic function, and then show the derivative is not equal to zero.

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[5]Churchill, op. cit., p. 60.

[6]Ibid., p. 136.

One way to show that a function is analytic is to derive the derivative directly from the definition. If the derivative exists in a two dimensional open region, then the function is said to be analytic at all points of the region.

The definition of a derivative is

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z_o + \Delta z) - f(z_o)}{\Delta z}, \quad \text{where } \Delta z = z_o - z.$$

The derivative of the Mobius transformation then becomes

$$f'(z) = \lim_{\Delta z \to 0} \frac{\left[\dfrac{a(z_o+\Delta z)+b}{c(z_o+\Delta z)+d}\right] - \left[\dfrac{az_o+b}{cz_o+d}\right]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{[a(z_o+\Delta z)+b][cz_o+d] - [az_o+b][c(z_o+\Delta z)+d]}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]}$$

$$= \lim_{\Delta z \to 0} \frac{[az_o+a\Delta z+b][cz_o+d] - [az_o+b][cz_o+c\Delta z+d]}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]}$$

$$= \lim_{\Delta z \to 0} \frac{acz_o^2 + acz_o\Delta z + bcz_o + adz_o + ad\Delta z + bd}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]} - \frac{acz_o^2 + acz_o\Delta z + az_o d + bcz_o + bc\Delta z + bd}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]}$$

$$= \lim_{\Delta z \to 0} \frac{acz_o^2 + acz_o\Delta z + bcz_o + adz_o + ad\Delta z + bd}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]} + \frac{-acz_o^2 - acz_o\Delta z - adz_o - bcz_o - bc\Delta z - bd}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]}$$

$$= \lim_{\Delta z \to 0} \frac{ad\Delta z - bc\Delta z}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(ad - bc)}{\Delta z[c(z_o+\Delta z)+d][cz_o+d]}$$

$$= \lim_{\Delta z \to 0} \frac{ad - bc}{[c(z_o+\Delta z)+d][cz_o+d]}$$

$$f'(z) = \frac{ad - bc}{(cz+d)^2}$$

When $\Delta z = 0$, then $z = z_o$. Therefore,

$$f'(z) = \frac{ad - bc}{(cz+d)^2}$$

The derivative exists at all points except $z = -d/c$, where the function is undefined. hence, the transformation is analytic except at the undefined point. Also the derivative is not equal to zero because of the restriction $ab - bc \neq 0$ included in the definition of the Mobius transformation.

Invariant Points (4.5)

The final property to be considered is the coincidence of the points of the z-plane and the w-plane under the Mobius transformation. The fixed points occur when

$$w = z$$

hence,

$$z = \frac{az+b}{cz+d}$$

$$cz^2 + dz = az + b$$

$$cz^2 + dz - az - b = 0$$

$$cz^2 + (d-a)z - b = 0. \qquad (4.5.1)$$

It is immediately observed that there exists at most, two roots of (4.5.1). By applying the quadratic formula these two roots $\xi_1$, $\xi_2$ are obtained as

$$\xi_1 = \frac{a-d+D}{2c} \qquad (4.5.2a)$$

$$\xi_2 = \frac{a-d-D}{2c} \qquad (4.5.2b)$$

where $D = \left[(d-a)^2 + 4bc\right]^{\frac{1}{2}}$. When $D = 0$ the two roots coincide. When $c = 0$ and $a \neq d$ there exists one finite root and an infinite root. The finite root $\xi$, becomes

$$\xi = \frac{b}{d-a}, \tag{4.5.3}$$

as mentioned in (3.3). In the case of pure translation, where $c = 0$ and $a = d$, the invariant points coincide at infinity. This coincidence is usually referred to as the double root at infinity.

Therefore, every Mobius transformation, except the identity, has two and only two invariant points. Hence, if a Mobius transformation has more than two fixed points the mapping must be the identity.

## CHAPTER V

## CLASSIFICATION OF MOBIUS TRANSFORMATIONS

The number of distinct invariant points and the behavior of the Mobius transformation with respect to these points provides a useful basis for classification.

Introduction (5.1)

Any Mobius transformation can be classified to be loxodromic or parabolic according to the invariant points being distinct or coincident.[1]

Consider the case where the mapping has two finite fixed points $\xi_1$ and $\xi_2$, such that $\xi_1 \neq \xi_2$. Under the Mobius transformation,

$$G(z) = \mathfrak{z} = \frac{z - \xi_1}{z - \xi_2}, \tag{5.1.1}$$

$\xi_1$ is mapped to the origin of the $\mathfrak{z}$-plane and $\xi_2$ is carried to infinity. Therefore, the family of circles $\Upsilon(\theta)$, (2.4.8), through $\xi_1$ and $\xi_2$ is mapped into the family of straight lines through the origin. The family of circles $\Omega(\xi_1, \xi_2)$, (2.3.5), is transformed into the family of concentric circles with the origin as the center. If these families lie in the z-plane, then the corresponding family lies in the $\mathfrak{z}$-plane.

Similarly, if the families of circles are imagined

[1] Hille, op. cit., p. 52.

to be in the w-plane, then their corresponding families would lie in the $\mathcal{W}$-plane and

$$G(w) = \mathcal{W} = \frac{w - \xi_1}{w - \xi_2} \quad . \tag{5.1.2}$$

The relationship for $\mathcal{W}$, written in functional notation, is

$$\mathcal{W} = GTG^{-1}(\mathfrak{z}). \tag{5.1.3}$$

Since (5.1.3) has 0 and $\infty$ as invariant points, the mapping is a rotation and/or dilation as defined in (3.2). Thus, relation (5.1.3) reduces to the simpler form

$$\mathcal{W} = K\mathfrak{z}, \tag{5.1.4}$$

where K is a given complex number, commonly called the multiplier. Consequently, (5.1.4) can be written in a normal form

$$\frac{w - \xi_1}{w - \xi_2} = K\,\frac{z - \xi_1}{z - \xi_2} \tag{5.1.5}$$

The value for K is obtained by use of the cross ratio. Let $\xi_1$, $\xi_2$ and $\infty$ be carried into $\xi_1$, $\xi_2$ and a/c respectively. Then the cross ratio

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_1 - w_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$$

reduces to

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_1 - w_3)} = \frac{(z - z_1)}{(z - z_2)} \tag{5.1.6}$$

since $z_3 = \infty$. Upon substitution of the known constants (5.1.6) becomes

$$\frac{(w - \xi_1)(\xi_2 - a/c)}{(w - \xi_2)(\xi_1 - a/c)} = \frac{z - \xi_1}{z - \xi_2}$$

$$\frac{w-\xi_1}{w-\xi_2}=\left(\frac{a-c\,\xi_1}{a-c\,\xi_2}\right)\left(\frac{z-\xi_1}{z-\xi_2}\right) \tag{5.1.7}$$

The above relationship takes the normal form

$$\frac{w-\xi_1}{w-\xi_2}=K\,\frac{z-\xi_1}{z-\xi_2}, \tag{5.1.8}$$

where $K=\dfrac{a-c\,\xi_1}{a-c\,\xi_2}$.

It should be noted that $z=-d/c$ and $w=\infty$ can be used for points of the cross ratio, instead of $z=\infty$ and $w=a/c$. In this case $K=\dfrac{d+c\,\xi_2}{d+c\,\xi_1}$.

The value of K determines the character of the transformation. The importance of using K is shown when writing powers of K. That is, if (5.1.8) is to be repeated n times, the mapping is written

$$\frac{w-\xi_1}{w-\xi_2}=K^n\;\frac{z-\xi_1}{z-\xi_2}.$$

The loxodromic transformations are classified in terms of K. By writing K with a modulus A ($>0$) and amplitude $\theta$, the multiplier K becomes

$$K=Ae^{i\theta}.$$

<u>Hyperbolic Transformation (5.2)</u>

A transformation is called a hyperbolic transformation when $K=A$, $0<A\neq 1$, and $\theta=0$. Thus, (5.1.4) reduces to

$$W=Az. \tag{5.2.1}$$

The above relationship is a dilation from the origin as discussed in (3.2).

It should be remembered that: 1) a straight line through the origin is transformed into itself, 2) the half plane on one side of a line through the origin is mapped into itself, 3) a circle with center at the origin (the family of circles orthogonal to the family of fixed lines through the origin) is transformed into some other member of the family, and 4) the points 0 and $\infty$ are inverse points with respect to any circle with center at the origin.

The hyperbolic transformation has the following properties: 1) any circle through the fixed points is carried into itself, 2) the interior of a circle through the fixed points is transformed into itself, 3) any circle orthogonal to the circles through the fixed points is mapped into some other circle orthogonal to the family of circles through the fixed points, and 4) the fixed points are inverse points with respect to each circle of (3).

An example of a hyperbolic Mobius transformation is

$$w = \frac{3z - 4}{-z + 3} \ . \qquad (5.2.2)$$

The fixed points $\xi_1$, $\xi_2$ are 2 and -2. That is, when $w = z$, then

$$z = \frac{3z - 4}{-z + 3}$$

$$-z^2 + 3z = 3z - 4$$

$$z^2 - 4 = 0$$

therefore $\xi_1$, $\xi_2$ are 2 and -2 respectively.

The calculation for K is easily obtained since,

$$K = \frac{a - c\,\xi_1}{a - c\,\xi_2}$$

$$= \frac{3 - (-1)(2)}{3 - (-1)(-2)}$$

$$= 5$$

The multiplier is a positive real number and therefore is classified as a hyberbolic transformation.

To illustrate the hyperbolic transformation, or in fact any transformation that has two finite fixed points, it is convenient to use four different planes. Let the planes be z, $\mathfrak{z}$, $W$ and w. The region to be transformed is represented in the z-plane. The best examples for the hyperbolic transformations are elements of families of circles $\Upsilon(\Theta)$ and $\Omega(\xi_1, \xi_2)$. The circles $C_1$ and $C_2$ are elements of $\Upsilon(\Theta)$ and $C_3$ and $C_4$ are members of $\Omega(\xi_1, \xi_2)$. The region of the z-plane is shown below.

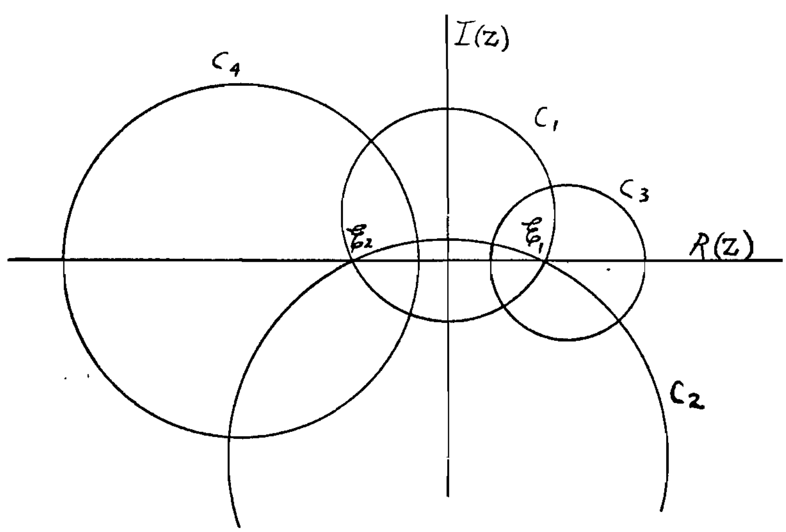


FIGURE 12
z-PLANE

The circles of the z-plane are now transformed into the $\mathfrak{z}$-plane by transformation (5.1.1). It should be remembered that this mapping carries $\xi_1$ to 0 and $\xi_2$ to $\infty$. Thus, it is readily seen that the family $\Upsilon(\theta)$ is transformed in the family of fixed lines passing through the origin. The family of circles $\Omega(\xi_1, \xi_2)$ maps into the family of concentric circles with the centers at the origin. The following figure shows the four circles of the z-plane after the mapping (5.2.3).

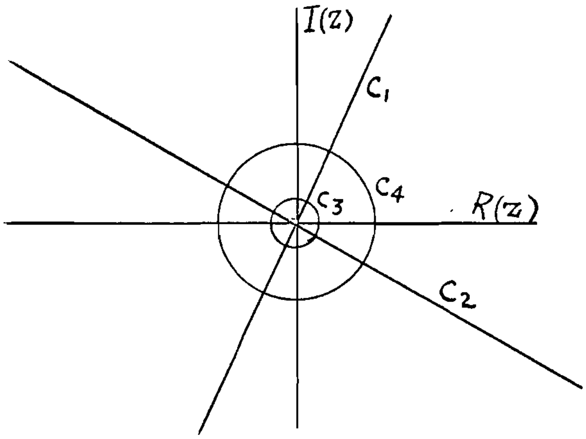


FIGURE 13

$\mathfrak{z}$-PLANE

In the hyperbolic transformation the multiplier is a positive real number, so the image of the $W$ plane is obtained from the $\mathfrak{z}$-plane by a dilation whose factor is K: 1. Thus, the transformation under consideration becomes

$$W = 5\mathfrak{z}. \tag{5.2.3}$$

This mapping carries straight lines through the origin into themselves. Concentric circles with center at the origin go into other circles with center at the origin. The image of the $\mathcal{W}$-plane under (5.2.3) is shown in figure 14.

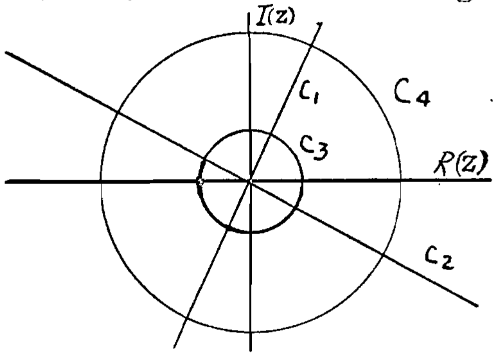


FIGURE 14

$\mathcal{W}$-PLANE

The final image plane w is obtained from transformation (5.1.2). Transformation (5.1.2) maps 0 and $\infty$ into $\xi_1$ and $\xi_2$ respectively. Obviously, the straight lines through the origin are carried into the original circles $\gamma(\theta)$. The concentric circles about the origin are mapped into elements of the $\Omega(\xi_1, \xi_2)$ family. The final image is shown below in the w-plane.

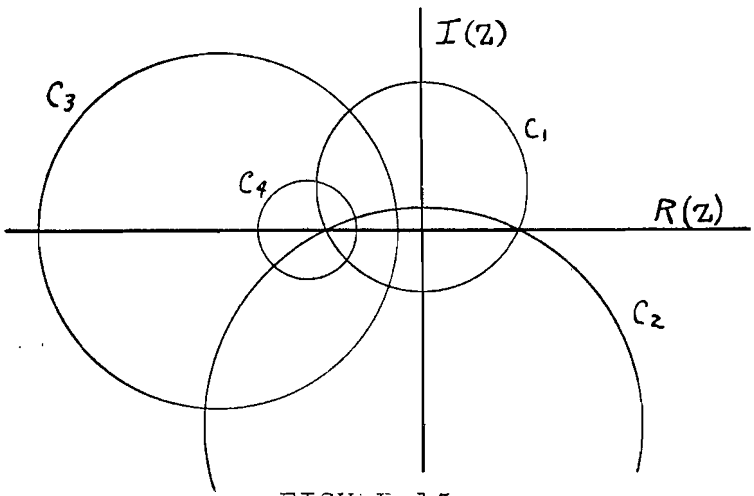


FIGURE 15

w-PLANE

All Mobius transformations have been graphically represented as a region in one plane and its image in another. Even though two planes have been used, it is often convenient to think of the mapping as effected in one plane. In this manner it is possible to find the path which any particular z-point makes in passing to its corresponding image or w-point. The following figure shows the hyperbolic transformation of $\Gamma(\theta)$ and $\Omega(\xi_1, \xi_2)$. The regions are transformed in the direction indicated by the arrows. The arrows indicate what is often called lines of flow.

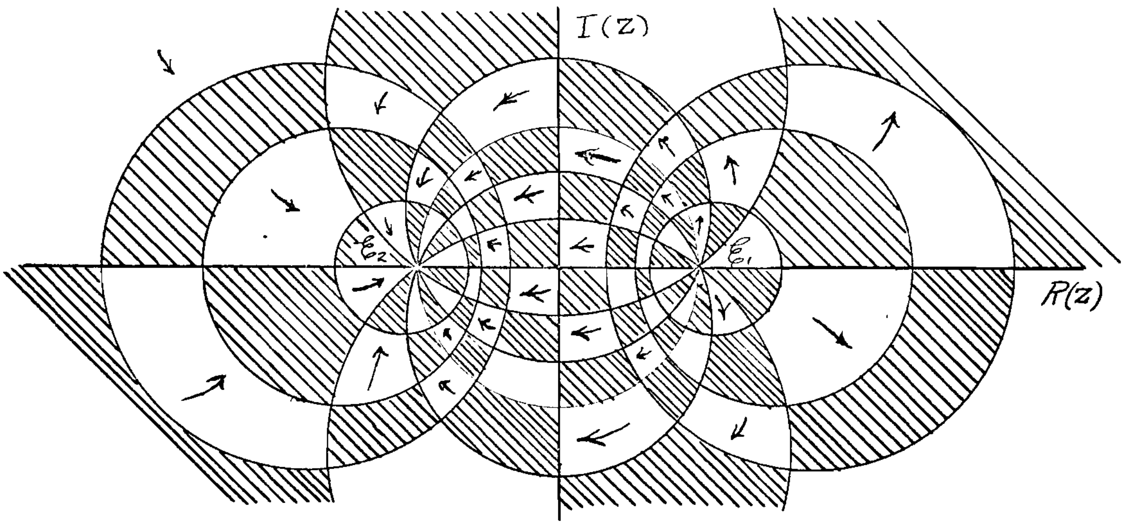


FIGURE 16

HYPERBOLIC TRANSFORMATION

It should be mentioned that $\xi_1$ is sometimes thought of as a repulsive fixed point, while $\xi_2$ is considered to be an attracting point.

Elliptic Transformation (5.3)

When $A = 1$ and $\theta \neq 2n\pi$, where n is any integer, the multiplier becomes

$$K = e^{i\theta}.$$

Thus the Mobius transformation becomes

$$\frac{w - \xi_1}{w - \xi_2} = e^{i\theta} \frac{z - \xi_1}{z - \xi_2} \quad . \tag{5.3.1}$$

By changing the variables relationship (5.1.4) becomes

$$\mathcal{W} = e^{i\theta} \mathcal{z}, \tag{5.3.2}$$

which is a rotation about the origin of the $\mathcal{z}$-plane. The mapping written in the form (5.3.1) is known as an elliptic transformation. In this mapping the role of straight lines and circles are just the opposite of the hyperbolic transformation.

The important properties of the rotation mapping as discussed in (3.2) are: 1) a circle with its center at the origin is mapped into itself, 2) the interior of the circles in (1) are transformed into themselves, 3) the points 0 and $\infty$ are inverse with respect to each fixed circle, 4) a line through the origin is transformed into a line through the origin that makes an angle $\theta$ with the first.

The elliptic transformation has the following properties: 1) an arc of a circle joining the fixed points is transformed into an arc of a circle making an angle $\theta$ with the first arc, 2) each circle of $\Omega(\xi_1, \xi_2)$ is carried into itself, 3) the interior of each circle of $\Omega(\xi_1, \xi_2)$ is mapped into itself, and 4) the fixed points are inverse points with respect to the circles of $\Omega(\xi_1, \xi_2)$.

The Mobius transformation

$$w \quad \frac{(2+\sqrt{2}+\sqrt{2}\,i)z + 4 - 2\sqrt{2} - 2\sqrt{2}\,i}{\frac{(2-\sqrt{2}-\sqrt{2}\,i)z}{2} \quad \frac{(2\sqrt{2}+2+2i)}{\sqrt{2}}} \tag{5.3.3}$$

is an elliptic transformation. Setting (5.3.3) equal to z, the fixed points $\xi_1$ and $\xi_2$ are 2 and -2 respectively. Also the multiplier takes the value

$$K = \frac{\sqrt{2}+\sqrt{2}\,i}{2},$$

which is a pure rotation of $45^\circ$ about the origin.

By using the same circles as in the hyperbolic transformation, it is possible to observe the motions under an elliptic transformation. The transformation (5.1.1) maps the regions into the same images as shown in figure 12. Transformation (5.1.4) becomes a rotation and the family of fixed lines is rotated about the origin through $45^\circ$. The family of orthogonal circles is left invariant. It can now be seen that in this particular mapping the roles of lines and circles are interchanged in relation to their roles in the hyperbolic transformation.

The transformation (5.1.2) maps 0 and $\infty$ back into the fixed points. It should be emphasized that the elliptic transformation transforms the family of circles orthogonal to the family of circles through the fixed points into itself and the elements of the family $\Gamma(\theta)$ permute among themselves.

The character of the transformation is shown in the following figure. The shaded regions are transferred in the direction shown by the flow of arrows. Notice that the fixed points are neither repulsive nor attractive.

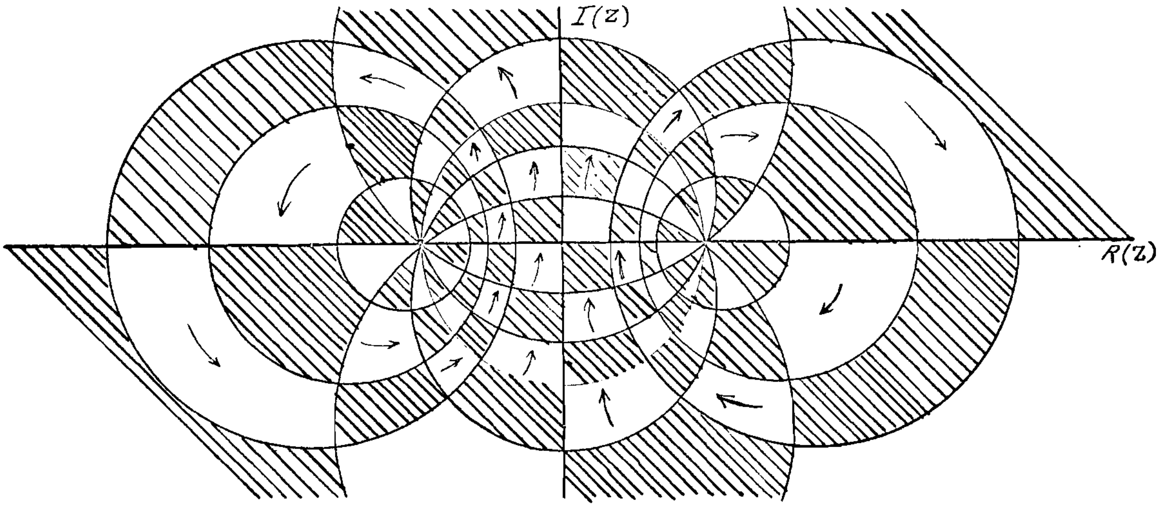


FIGURE 17

ELLIPTIC TRANSFORMATION

The elliptic transformations are the only Mobius transformations that can be periodic. It $\theta$ is commensurable with $2\pi$, then there will exist integers m and n, such that $n\theta = 2m\pi$. Therefore, when a transformation is applied n times the multiplier becomes $K = e^{2m\pi i} = 1$, and obviously the transformation takes each point into itself. The transformation is said to be of period n. For example if $\theta = \pi$, then the period of the transformation is 2.

Loxodromic Transformation (5.4)

The loxodromic transformation is a combination of (5.2) and (5.3). That is, the mapping is a Mobius transformation with two distinct invariant points that occur when the multiplier K has $0 < A \neq 1$ and $\theta \neq 2n\pi$. Thus the transformation takes the form:

$$W = Ae^{i\theta}z. \tag{5.4.1}$$

The relationship (5.4.1) can be thought of as two successive mappings. That is,

$$W_1 = Az \tag{5.4.2}$$

$$W = e^{i\theta}W_1. \tag{5.4.3}$$

The first transformation is a stretching from the origin. Then a rotation about the origin occurs on the image of (5.4.2). Obviously, this mapping is a combination of the hyperbolic and elliptic transformations.

The images in $W$ in relation to $z$ under the mapping (5.4.1) will be: 1) the family of concentric circles about the origin are carried into other circles of the same family, 2) each line through the origin will be transformed into a line through the origin making an angle $\theta$ with the first line.

It can then be seen that the loxodromic transformation takes each circular arc joining the fixed points and carries it into another such arc making angle $\theta$ with the first. The circles orthogonal to the circles through the fixed points are permuted among themselves.

The loxodromic mapping has no fixed circles. There is one exception that is sometimes classified as an improper hyperbolic.[2] The exception occurs when $\theta = \pi$ and any circular arc joining the fixed points is transformed into a circular arc making an angle $\theta$ with the first arc. Thus the circle formed by two arcs through the fixed points is carried into itself except that the interior is mapped to the exterior. The loxodromic transformation does have as fixed curves certain logrithmic spirals.[3] The spirals are sometimes referred to as loxodromes. A loxodrome is a navigational term used to denote a ship's course which cuts successive meridians at a constant angle.[4] It is from this usage that the transformation under consideration receives its name.

A combination of figures 16 and 17 can be imagined as the loxodromic transformation.

The linear transformation as discussed in (3.3) is a special case of the loxodromic transformation. A mapping,

$$w = \frac{a}{d} z + \frac{b}{d}\,, \qquad (5.4.4)$$

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[2]Hans Schwerdtfeger, Geometry of Complex Numbers (Toronto: University of Toronto Press, 1962), p. 65.

[3]Ibid., p. 73.

[4]Townsend, op. cit., p. 137.

has the invariant points $\xi$ and infinity, where

$$\xi = \frac{b}{d - a} .$$

The relationship (5.4.4) can be written in normal form. That is,

$$w - \xi = K(z - \xi), \qquad (5.4.5)$$

where $K = a/d$.

The variables of (5.4.5) can be changed to

$$G(z) = \mathfrak{z} = z - \xi \qquad (5.4.6)$$

$$G(w) = \mathcal{W} = w - \xi \qquad (5.4.7)$$

so that the relationship can be rewritten as

$$\mathcal{W} = K\mathfrak{z}. \qquad (5.4.8)$$

Now that the special case of the loxodromic transformation can be written in the form (5.4.8), it can be classified into a particular type depending on the value of the complex number K. The classification is the same as described for Mobius transformations with two finite roots. The characteristics of this mapping are essentially the same as discussed in (3.2).

Parabolic (5.5)

When the fixed points coincide, the Mobius transformation is considered to be parabolic. If $c \neq 0$, the fixed point is

$$\xi = \frac{a - d}{2c} . \qquad (5.5.1)$$

Obviously, the multiplier K is equal to 1. To obtain the normal form it is necessary to set up the cross ratio and map $\infty$, $\xi$ and $-d/c$ into $a/c$, $\xi$, and $\infty$ respectively. Since $z_1$ and $w_3$ become infinite in the cross ratio, the ratio simplifies to

$$\frac{w - w_1}{w - w_2} = \frac{z_3 - z_2}{z - z_2} . \tag{5.5.2}$$

By substituting the constants, (5.5.2) becomes

$$\frac{w - a/c}{w - \xi} = \frac{-d/c - \xi}{z - \xi} .$$

Subtract 1 from both sides, thus

$$\frac{w - a/c - w + \xi}{w - \xi} = \frac{-d/c - \xi}{z - \xi} - 1$$

$$\frac{\xi - a/c}{w - \xi} = \frac{-d/c - \xi}{z - \xi} - 1. \tag{5.5.3}$$

Since $\xi = -(a - d)/2c$,

$$\xi - a/c = -\frac{(a - d)}{2c} - \frac{a}{c}$$

$$\xi - a/c = -(a + d)/2c . \tag{5.5.4}$$

Likewise,

$$-d/c - \xi = -(a + d)/2c. \tag{5.5.5}$$

The relationship (5.5.3) now becomes,

$$\frac{-(a + d)/2c}{w - \xi} = \frac{-(a + d)/2c}{z - \xi},$$ which simplifies to

$$\frac{1}{w - \xi} = \frac{1}{z - \xi} + \frac{2c}{a + d} . \tag{5.5.6}$$

Sometimes the value $c/(a - c\xi)$ is used as an equivalent form of $2c/(a + d)$. The variables can be changed by letting,

$$\mathfrak{z} = 1/(z - \xi) \tag{5.5.7}$$

$$\mathcal{W} = 1/(w - \xi) \tag{5.5.8}$$

$$B = 2c/(a + d) \tag{5.5.9}$$

With the above change in variables (5.5.6) can be written as

$$\mathcal{W} = \mathcal{z} + B \tag{5.5.10}$$

The relationship (5.5.10) is a translation. The following facts about translation should be recalled from (3.1). That is, 1) The entire plane is transformed parallel to a vector B, 2) any line parallel to vector B is carried into itself, and 3) a line not parallel to vector B is carried into a line parallel to the given line.

To fully appreciate the parabolic transformation, it is helpful to study the transformations that constitute (5.5.10) one at a time. Consider the families of circles through $\xi$. In particular, the family of circles with a common tangent at $\xi$. The transformation (5.5.7) is a translation followed by a reciprocation mapping as discussed in (3.3). Therefore, $\xi$ is translated to the origin and then to its inverse point, $\infty$. Similarly, all of the circles through $\xi$ are mapped into a system of lines that intersect at $\infty$. Hence, the image of z in $\mathcal{z}$ is a system of parallel lines.

The transformation (5.5.10) carries the family of parallel lines into itself, since $\mathcal{W}$ and $\mathcal{z}$ are mapped linearly on each other by B. In other words, the elements permute among themselves. Finally, mapping (5.5.8) carries the family of parallel lines back into the family of tangent circles.

A parabolic transformation has the following properties that should be observed: 1) a circle through the fixed point is transformed into a tangent circle through the fixed point, 2) each family of circles tangent to $\xi$ is carried into itself, and 3) the interior of each transformed circle is mapped into itself.

The following figure shows the lines of flow of the parabolic transformation on the family of circles with a common tangent. This configuration is often called the degenerate Stiener circles.[5]

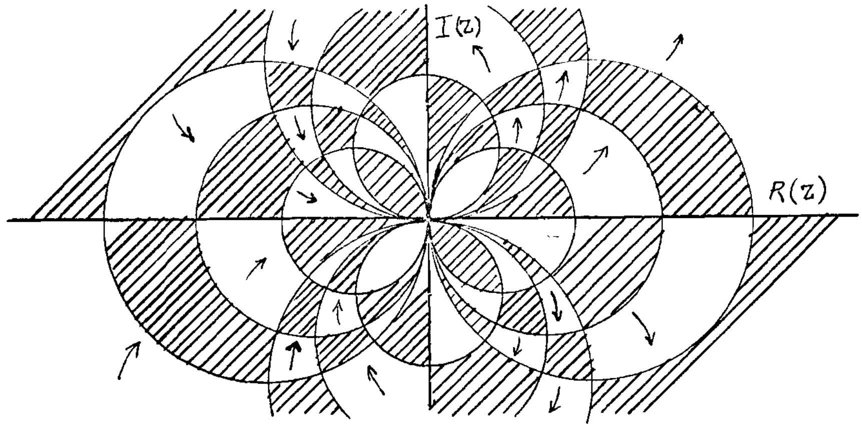


FIGURE 18

PARABOLIC TRANSFORMATION

Translation is considered to be a special case of the parabolic transformation, since its fixed points coincide at

[5]Ahlfors, op. cit., p. 34.

infinity.

It has now been established that every Mobius transformation can be classified to be parabolic or loxodromic as to whether its fixed points coincide or not.

Loxodromic Transformation Group (5.6)

The set of all loxodromic Mobius transformations having the same invariant points form a group as defined in (4.1). The loxodromic transformations have three essential parameters--K, any non-zero complex number, $\xi_1$ and $\xi_2$, the fixed points. Each transformation of the set can be denoted by

$$T(K: \xi_1 , \xi_2). \tag{5.6.1}$$

Consider any other element of the set, say

$$T(K_1: \xi_1 , \xi_2)$$

then the operational law of combination is defined as

$$T(K: \xi_1 , \xi_2) \circ T(K_1: \xi_1 , \xi_2) \circ T(K\,K_1: \xi_1, \xi_2). \tag{5.6.2}$$

The identity element would be

$$T(1: \xi_1 , \xi_2), \tag{5.6.3}$$

and the inverse element has the form

$$T(K^{-1}: \xi_1 , \xi_2). \tag{5.6.4}$$

The elements of the set are associative since the field of complex numbers is associative with respect to multiplication. Therefore, the set of loxodromic transformations forms group,

$$GL(K: \xi_1 , \xi_2). \tag{5.6.5}$$

This group can be considered a one-parameter group since $\xi_1$ and $\xi_2$ are fixed. It should also be mentioned that (5.6.5) is a subgroup of M.[6] The loxodromic transformation is a generalization of the rotation and dilation transformations discussed in (3.2).

The sets of hyperbolic and elliptic transformations with fixed points $\xi_1$ and $\xi_2$ form subgroups of $GL(K: \xi_1 , \xi_2 )$. The hyperbolic transformation is a generalization of the pure dilation transformations, and the elliptic transformation is a generalization of the pure rotation transformations.

Parabolic Transformation Group (5.7)

The set of all parabolic transformations with fixed point $\xi$ form a group denoted by

$$GP( \alpha , \xi ), \tag{5.7.1}$$

where $\alpha$ is any complex number and $\xi$ is the fixed point. When $T( \alpha , \xi )$ and $T( \beta , \xi )$ are any two elements of (5.7.1), the binary operation of combination is defined as

$$T( \alpha , \xi ) \circ T( \beta , \xi ) \quad T( \alpha + \beta , \xi ). \tag{5.7.2}$$

Since addition is associative in the field of complex numbers, the associative law of combination is applicable in this group. The identity is represented by

$$T(1, \xi ) \tag{5.7.3}$$

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[6]Hille, op. cit. pp. 52-53.

and the inverse element of any element $T(\alpha, \xi)$ would be

$$T(-\alpha, \xi). \tag{5.7.4}$$

Thus, the group properties have been established. It should also be mentioned that this group is a generalization of the translation transformation as presented in (3.1).

Commutative Property (5.8)

Theorem 5.8 -- Two Mobius transformations commute if they have the same fixed points.

The proof of the above theorem can be established by considering the general cases of the parabolic and loxodromic transformations. An alternative proof is presented below.

Let two Mobius transformations be

$$w = \frac{az+b}{cz+d}, \tag{5.8.1}$$

where $ad - bc \neq 0$, and

$$W = \frac{Az+B}{Cz+D}, \tag{5.8.2}$$

where $AD - BC \neq 0$. Given (5.8.1) and (5.8.2), then the following relations must exist if $w \circ W = W \circ w$,

$$aA + bC = Aa + Bc \tag{5.8.3a}$$

$$aB + bD = Ab + Bd \tag{5.8.3b}$$

$$cA + dC = Ca + Dc \tag{5.8.3c}$$

$$cB + dD = Cb + Dd. \tag{5.8.3d}$$

The equations used to obtain the invariant points of (5.8.1) and (5.8.2) are

$$cz^2 + (d - a)z - b = 0 \quad \text{and} \tag{5.8.4}$$

$$Cz^2 + (D - A)z - B = 0 \tag{5.8.5}$$

respectively.[7] If these two equations have the same roots then a relationship between (5.8.4) and (5.8.5) can be established using the quadratic formula. That is,

$$\frac{c}{C} = \frac{d - a}{D - A} = \frac{b}{B} \,. \tag{5.8.6}$$

Equations (5.8.3a) and (5.8.3d) reduce to $bC = Bc$. This relationship can be expressed in the form

$$\frac{b}{B} = \frac{c}{C} \,. \tag{5.8.7}$$

By use of elementary algebraic operations, equations (5.8.3b) and (5.8.3c) have the following relationships

$$\frac{b}{B} = \frac{d - a}{D - A} \tag{5.8.8}$$

$$\frac{c}{C} = \frac{d - a}{D - A} \tag{5.8.9}$$

respectively. It now becomes obvious that the equations of (5.8.3) can be related by the expression

$$\frac{c}{C} = \frac{d - a}{D - A} = \frac{b}{B} \,. \tag{5.8.10}$$

Therefore, two Mobius transformations commute if they have the same fixed points.

Theorem 5.8 is also a necessary condition when considering the parabolic transformation. The proof of the

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[7] Olga Taussky and John Todd, "Commuting Bilinear Transformations and Matrices," <u>Journal of the Washington Academy of Science</u>, 46:374, 1956.

necessity is obtained by direct calculation. That is, given any two parabolic transformations, $T_1(\alpha, \xi_1)$ and $T_2(\beta, \xi_2)$ such that $\xi_1 \neq \xi_2$, then

$$T_1 T_2 \neq T_2 T_1 .$$

Theorem 5.8 is only a sufficient condition for Mobius transformations of the loxodromic type. The theorem is not the necessity because there exists a set of involutionary transformations whose involution changes the order of the fixed points. The involutionary set with two fixed points $\xi_1$ and $\xi_2$ and the set of Mobius transformations, $(K: \xi_2, \xi_1)$ are commutative.

## CHAPTER VI

## SUMMARY

The Mobius transformation is a very important one-to-one conformal mapping of the complex plane onto itself. The set of all Mobius transformations is an abstract group and certain subsets of the mapping form subgroups. Since the mapping transforms circles and straight lines into circles and straight lines, it is considered to be a circle preserving transformation. That is, a transformation can be obtained by use of the cross ratio that will map a given circle into another specified circle. It should be remembered that straight lines are considered to be a special case of the circle.

Those points that remain fixed under a Mobius transformation are used as a basis for classification. The set of all mappings of one particular class forms a commutative group. Within each group of classified Mobius transformation there exists certain subgroups that are generalizations of the simple transformations that constitute the relationship defined by the Mobius transformation.

The Mobius transformation can be very beneficial to the investigation of regions of the complex plane. Other applications of this mapping are found in projective geometry and physical science.

The topic of Mobius transformations can be investigated further. For example, the mapping can be developed in an analogous manner using spherical coordinates and the sphere, rather that points and the complex plane. Also a similar relationship in the area of functions of more than one complex variable might be developed. Finally, an investigation of all the applications of the Mobius transformation is a possible area to consider.

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