THE CONSISTENCY OF PLANE HYPERBOLIC GEOMETRY

515

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CHAPTER I

THE PROBLEM AND DEFINITIONS OF THE TERMS USED

Consistency is one of the most desirable properties of any axiomatic system. The relative consistency of plane Hyperbolic geometry is established by a comparative proof. A proof showing Hyperbolic geometry to be consistent was developed by Henri Poincaré.[1] This proof will be used in this paper.

I. THE PROBLEM

Statement of the problem. The purpose of this study is to show that the set of axioms of plane Hyperbolic geometry are as consistent as the set of axioms of plane Euclidean geometry.

Importance of the study. A basic pattern is followed in establishing an axiomatic system. In every axiomatic system there is a set of undefined terms and a set of undefined relations between the undefined terms. It is impossible to define every term without being cyclic. That is, defining a word in terms of other words, which in turn have definitions which use the original word

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[1] Eves and Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics, (New York: Holt, Rinehart and Winston, 1965), 101.

being defined. All other technical terms are defined using the undefined terms. A set of statements about the undefined terms, technical terms, and the undefined relations is accepted without proof. These statements are called the axioms of the system. Statements which may be derived from the set of axioms by a system of logic such such as Classical (Aristotelian) logic are called theorems. In this study, the system of logic used is Aristotelian logic.

A fundamental property of any axiomatic system is that of consistency. Without this property, contradictory statements may be derived from the set of axioms. The set of axioms would then be somewhat useless, at least for purposes of application. Therefore, it is of importance to demonstrate that the system which establishes plane Hyperbolic geometry is consistent.

II. A DISCUSSION ON CONSISTENCY

Consistency. A set of axioms is said to be consistent if and only if there are no contradictions among the axioms and theorems which can be derived from the axioms. That is, there exists no statement and its negation that are both true. Not all axiomatic systems may be proved to be consistent by directly showing that there are no contradictions. If no contradictions are found in the axioms and the known theorems, then there is still the possibility of having contradictions between two "undiscovered" theorems. The usual method of establishing the consistency of a set of axioms is by the development of a model of the set of axioms.

Model. Let S be a mathematical system consisting of sets of undefined terms S_1, S_2, ..., S_n together with the undefined relations R_1, R_2, ..., R_m between them. Let M consist of sets S_1', S_2', ..., S_n' of abstract or physical elements with the undefined or physical relations R_1', R_2', ..., R_m' between them. M is said to be a model of S if and only if there exists a one to one correspondence between S_i and S_i' for all $i = 1, 2, \ldots, n$ such that for any relation R_i indicated by the axioms between certain elements of S_1, S_2, ..., S_n; R_i' holds between the corresponding elements of S_1', S_2', ..., S_n'.

Concrete Models. A model M is said to be concrete if and only if M consists of objects and relations of the real world.

Ideal Models. A model M is said to be ideal if and only if the sets S_1', S_2', ..., S_n' and relations R_1', R_2', ..., R_m' of M are the undefined terms and relations of another axiomatic system.

With the two types of models, there are associated two types of consistency.

Absolute Consistency. An axiomatic system is said to have absolute consistency if and only if the axiomatic system has a concrete model. The word absolute is used since any inconsistency in the set of axioms would appear as a corresponding inconsistency in the real world, which is considered to be impossible.

It is not always possible to prove that an axiomatic system has absolute consistency. Some axiomatic systems have an infinite number of elements. A concrete model of such a system would be impossible, since the real world does not contain an infinite number of objects, at least that is known. Therefore, a second type of consistency is needed.

Relative Consistency. An axiomatic system is said to have relative consistency if and only if the axiomatic system has an ideal model. Relative consistency establishes that one axiomatic system is as consistent as the axiomatic system the model is based on. This does not resolve the question of the consistency of the axiomatic system. It just puts the burden of proof on the axiomatic system of the model.

III. A FAMILIAR EXAMPLE OF A MODEL

Analytic geometry as a model of plane Euclidean geometry.[2] To establish an algebraic model of plane Euclidean geometry, the following must be shown: (1)

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[2] Ibid., 107.

that there exists one to one correspondences between the sets of undefined terms of plane Euclidean geometry and the defined terms of analytic geometry, (2) that there exist one to one correspondences between the undefined relations of plane Euclidean geometry and the relations of analytic geometry, and (3) R_i is a relation between certain undefined elements $S_1, S_2, \ldots, S_n$ of plane Euclidean geometry if and only if R_i' is a corresponding relation between corresponding elements of $S_1', S_2', \ldots, S_n'$ for all $i = 1, 2, \ldots, m$ of analytic geometry. The undefined elements of plane Euclidean geometry based on Hilbert's axioms are point and line. The undefined relations are on, between, and congruent. The following definitions will establish an analytic model for Euclidean geometry.

<u>Definition</u> <u>I.1.</u> A point is any ordered pair (x,y) of real numbers. The real numbers are called the coordinates of the point.

<u>Definition</u> <u>I.2</u>. A line is any equation in two variables x and y of the form $ax + by + c = 0$, where a, b, and c are real numbers and a and b are not both zero. If two or more linear equations in x and y have coefficients such that when the coefficients of one linear equation are multiplied by a constant nonzero factor, they equal the

coefficients of the other linear equation, then the equations represent the same line.

Definition I.3. A point is on a line if and only if the coordinates of the point satisfy the equation of the line.

Definition I.4. A point (x,y) is between the points (x_1, y_1) and (x_2, y_2) if and only if there exists a real number t, greater than zero and less than one, such that

$$x = (1 - t) \cdot x_1 + t \cdot x_2 \text{ and } y = (1 - t) \cdot y_1 + t \cdot y_2.$$

Definition I.5. A pair of points (x_1, y_1), (x_2, y_2) is congruent to the pair of points (x_3, y_3), (x_4, y_4) if and only if

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = (x_4 - x_3)^2 + (y_4 - y_3)^2.$$

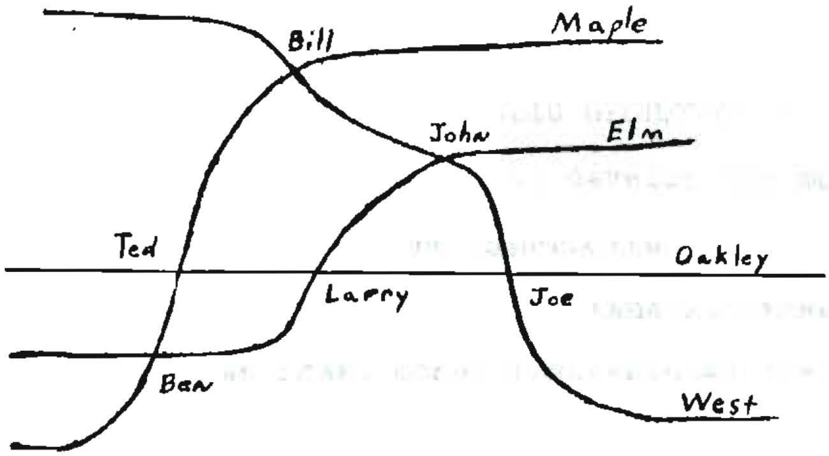
The value of $(x_2 - x_1)^2 + (y_2 - y_1)^2$ is the square of the distance between the points (x_1, y_1) and (x_2, y_2).

Definition I.6. An angle denoted by (x_2, y_2), (x_1, y_1), (x_3, y_3) is congruent to an angle (x_2', y_2'), (x_1', y_1'), (x_3', y_3') if and only if

$$\frac{(x_2 - x_1)\cdot(x_3 - x_1) + (y_2 - y_1)\cdot(y_3 - y_1)}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}} =$$

$$\frac{(x_2' - x_1')\cdot(x_3' - x_1') + (y_2' - y_1')\cdot(y_3' - y_1')}{\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2}\sqrt{(x_3' - x_1')^2 + (y_3' - y_1')^2}}.$$

Figure 1



It is easily seen that the set of axioms are satisfied. The model is an example of a concrete model. Therefore, the axiomatic system $\mathcal{T}$ has absolute consistency.

In Chapter Two, Poincaré's model of plane Hyperbolic geometry will be developed to show that the set of axioms of plane Hyperbolic geometry has relative consistency.

CHAPTER II

POINCARÉ'S MODEL OF HYPERBOLIC GEOMETRY

The purpose of this chapter is to develop the model devised by Henri Poincaré of plane Hyperbolic geometry.[4] The model uses objects and relations of plane Euclidean geometry. Therefore, an ideal model is developed for the axioms.

1. THE GEOMETRY OF THE CIRCLES ORTHOGONAL TO A FIXED CIRCLE

Consider any fixed circle Σ in the Euclidean plane and call it the fundamental circle. The following definitions will be used to interprete a concept of plane Hyperbolic geometry into the terms of plane Euclidean geometry.

Definition II.1. A point of the Hyperbolic plane is represented in the model by a point in the interior of Σ .

Definition II.2. A line of the Hyperbolic plane is represented in the model by the arc of any circle orthogonal to Σ which is interior to Σ . Any diameter of Σ is orthogonal to Σ and will also represent a line of Hyperbolic geometry.

Definition II.3. A point on a line in the Hyperbolic plane is represented in the model by a point interior to Σ and on an arc of a circle orthogonal to Σ , where the relation "on" has the usual Euclidean interpretation.

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[4] Eves, op. cit., p. 101.

Definition II.4. A line through or containing a point of the Hyperbolic plane is represented in the model by an arc of a circle orthogonal and interior to Σ through or containing a point, where the relations "through and containing" have the usual Euclidean meaning.

Definition II.5. A point between two points in the Hyperbolic plane is represented in the model by a point between two points on an arc of a circle orthogonal and interior to Σ , where the relation "between" has the usual Euclidean meaning of a point between two points on an arc in the Euclidean plane.

Definition II.6. A segment AB in the Hyperbolic plane is represented in the model by the points A and B and all points which are between A and B on the same arc of a circle orthogonal and interior to Σ . Points A and B are called the endpoints of the segment. A point C is said to be on the segment AB if it is A or B or some point between A and B.

Definition II.7. The length of a segment AB in the Hyperbolic plane is defined in the model as the $\log \left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right)$, where S and T are the points in which the arc containing arc AB intersects Σ and A is between S and B. The cross ratio is greater than one. Therefore, $\log \left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right)$ is greater than zero.

Definition II.8. A pair of points (A, B) congruent to a pair of points (C, D) in the Hyperbolic plane is represented in the model by pairs of points (A, B) and (C, D) such that the pairs of points are endpoints of segments and

$$\log\left(\frac{\widehat{AT}}{\widehat{BT}}\cdot\frac{\widehat{BS}}{\widehat{AS}}\right) = \log\left(\frac{\widehat{CV}}{\widehat{DV}}\cdot\frac{\widehat{DU}}{\widehat{CU}}\right).$$

Definition II.9. A segment AB congruent to segment CD in the Hyperbolic plane is represented in the model by two arcs AB and CD such that

$$\log\left(\frac{\widehat{AT}}{\widehat{BT}}\cdot\frac{\widehat{BS}}{\widehat{AS}}\right) = \log\left(\frac{\widehat{CV}}{\widehat{DV}}\cdot\frac{\widehat{DU}}{\widehat{CU}}\right).$$

Definition II.10. Intersecting lines or line segments in the Hyperbolic plane are represented in the model by two arcs, a diameter, or an arc and a diameter, which are said to be intersecting if there is a point which is on both of them.

Definition II.11. A ray AB in the Hyperbolic plane is represented in the model by the set of all points consisting of those which are between A and B, the point B itself, and all points C such that B is between A and C. The ray AB is said to emanate from point A.

Definition II.12. An angle in the Hyperbolic plane is represented in the model by a point (called the vertex of the angle) and two rays (called the sides of the angle) emanating from the point.

Definition II.13. The measure of an angle between two intersecting lines in the Hyperbolic plane is defined in the model as the measure of the angle between the tangents to the intersecting arcs.

Definition II.14. Angle ABC congruent to angle DEF in the Hyperbolic plane is represented in the model by angles ABC and DEF, where the measure of the angle ABC is equal to the measure of angle DEF.

Definition II.15. Let A, B, and C be three points not on the same arc in Σ. Then the segments of triangle ABC of the Hyperbolic plane are represented in the model by the three segments AB, BC, and CA called the sides of the triangle and the points A, B, and C called the vertices of the triangle.

Definition II.16. The angles of triangle ABC of the Hyperbolic plane are represented in the model by the three angles BAC, CBA, and ACB which are called the angles of triangle ABC. An angle BAC is said to be included by the sides AB and AC of the triangle.

Definition II.17. A triangle ABC congruent to triangle A'B'C' in the Hyperbolic plane is represented in the model by triangles ABC and A'B'C' such that, sides AB, AC, and BC are congruent to sides A'B', A'C', and B'C', respectively; and the angles ABC, ACB, and BAC are congruent to angles A'B'C', A'C'B', and B'A'C', respectively.

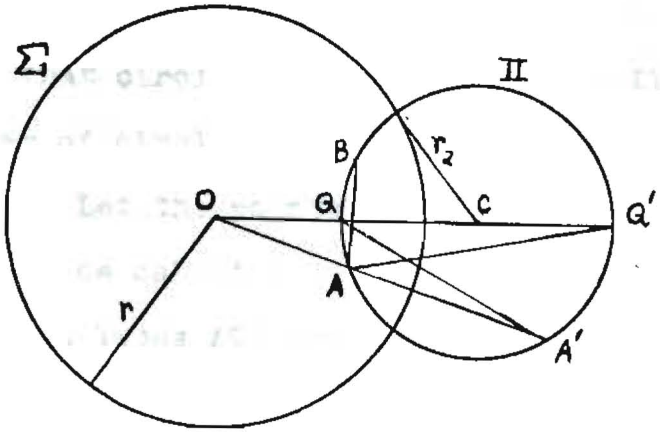
Capital letters will denote a concept of plane Hyperbolic geometry which is being represented in the model.

II. PROOFS SHOWING THAT THE POINCARÉ MODEL SATISFIES THE AXIOMS OF PLANE HYPERBOLIC GEOMETRY

Group 1: Postulates of Connection. Hilbert's axioms of plane Hyperbolic geometry may be found in the appendix.

To verify Postulate I-1 (See appendix.), it must be shown that given any two POINTS in Σ, there exists at least one LINE through the two given POINTS A and B and that this LINE is unique.

Figure 2



Proof:

Let O be the center of Σ with r the radius. (See figure 2.) Let POINTS A and B be any two distinct POINTS in Σ. Suppose that POINTS A, B, and O are collinear. Since Σ is in the Euclidean plane, there exists a unique diameter of Σ which contains POINTS A and B. This diameter is orthogonal to Σ and is, therefore, a LINE containing the POINTS A and B.

Suppose that POINTS A, B, and O are not collinear. Construct line OA. Using POINT O as the center of inversion and Σ as the circle of inversion, there exists a point A' on line OA such that $OA \cdot OA' = r^2$, where A' is the inverse of A. Suppose that the point A' is in Σ, then (in the Euclidean sense) the length of OA' is less than r. Likewise, the length of OA is less than r. Thus $OA \cdot OA'$ is less than r^2, which is a contradiction. Therefore, A' is not in Σ. Through the points A, A', and B there passes one and only one circle denoted by circle II.

To show that circle II is orthogonal to Σ, let C be the center of circle II with r_2 the radius. Construct line OC. Let the points of the intersection of OC and Circle II be called Q and Q' where Q is between O and C. Construct lines AQ' and A'Q. Consider the triangles OAQ' and OQA'. Angle AOQ' is congruent to angle A'OQ. The measure of angle $AA'Q = \frac{1}{2}$ m (arc AQ) and the measure of angle $QQ'A = \frac{1}{2}$ m (arc AQ), since they are inscribed angles subtended by the same arc of circle II. Then angle OA'Q is congruent to angle OQ'A. Therefore, triangle OAQ' is similar to triangle OQA'. Thus $\frac{OA'}{OQ'} = \frac{OQ}{OA}$

and $OA \cdot OA' = OQ \cdot OQ'$. Since $OA \cdot OA' = r^2$, then $OQ \cdot OQ' = r^2$. Segment $OQ = OC - r_2$ and segment $OQ' = OC + r_2$. Thus $r^2 = OA \cdot OA' = OQ \cdot OQ' = (OC - r_2) \cdot (OC + r_2) = OC^2 - r_2^2$. Therefore, $r^2 + r_2^2 = OC^2$ and circle II is orthogonal to Σ. There is at least one LINE contained in Σ passing through POINTS A and B.

Let circle III be any circle which passes through POINTS A and B and is orthogonal to Σ. Let circle III have center D and radius r_3. Construct line OA. Call the point of the intersection of line OA and circle III, A". Since circle III is orthogonal to Σ, then $OA \cdot OA'' = r^2$. Construct line OD and call the points of the intersection of line OD and circle III, P and P', with P between O and D.

Since circle III is orthogonal to Σ, $r^2 + r_3^2 = OD^2$. Then $r^2 = OD^2 - r_3^2 = (OD - r_3) \cdot (OD + r_3) = OP \cdot OP' = OA \cdot OA''$. But $OA \cdot OA' = r^2$ and $OA \cdot OA' = OA \cdot OA''$. Thus $OA' = OA''$ and A" concides with A'. Therefore, circle III coincides with circle II constructed earlier and there is one and only one circle through A and B orthogonal to Σ. Postulate I-1 has been verified.

Every LINE in Σ is an arc of a circle in the Euclidean plane which contains an infinite number of distinct points. Therefore, every LINE of Σ contains at least two distinct POINTS. Since an arc of a circle does not contain all the interior points of the circle, there is at least one point interior to Σ not on an arc of a circle orthogonal to Σ. Thus there is at least one POINT NOT ON A LINE. (See appendix.) This verified Postulate I-2.

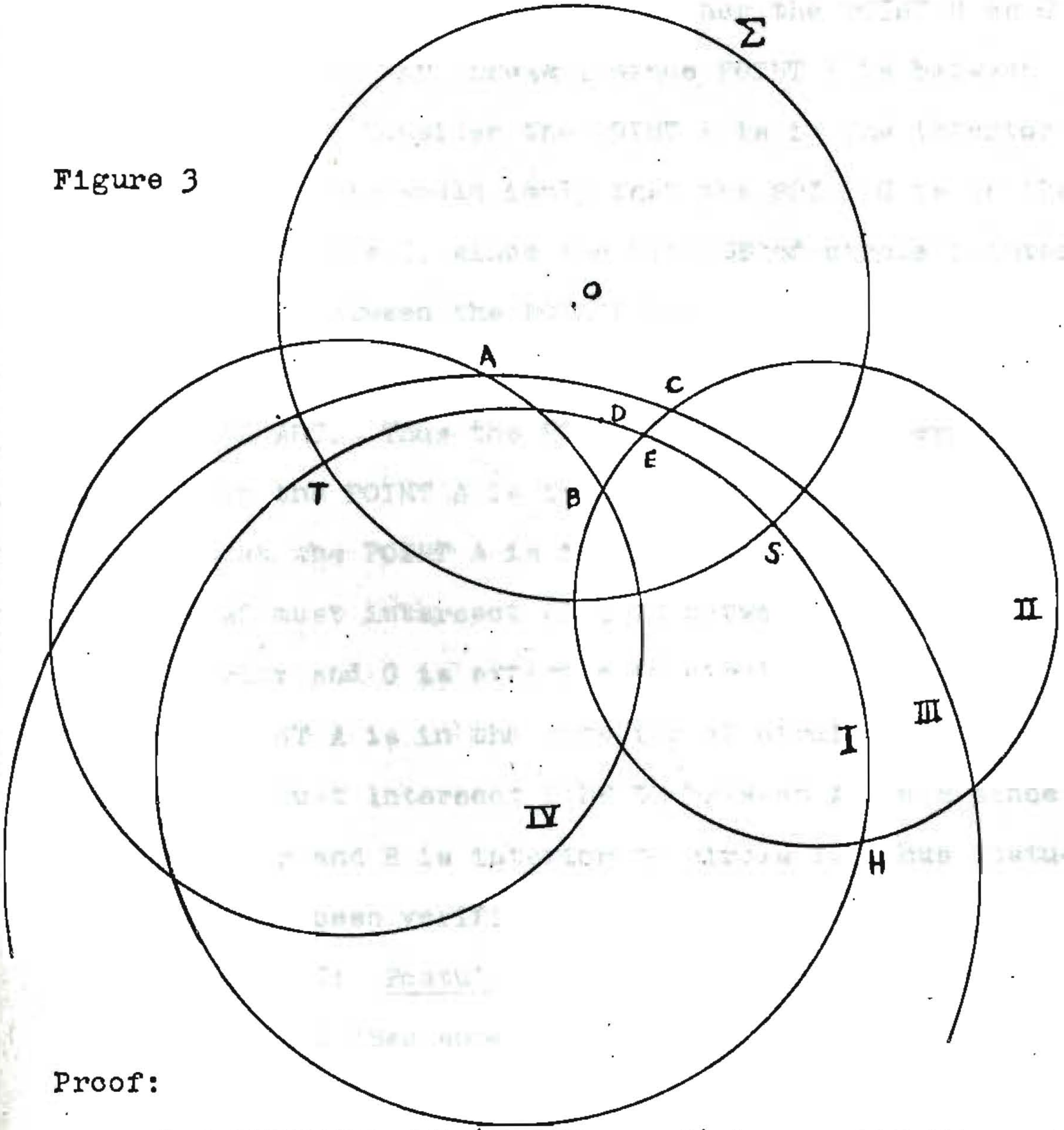
Group II: <u>Postulates of order</u>. Postulate II-1 (See appendix.) is proved directly by the interpretation of a POINT BETWEEN TWO POINTS, since the Euclidean interpretation is the order of points on an arc. This interpretation implies that if a point C is between A and B, then C is between B and A; B is not between A and C; and A is not between C and B.

To verify Postulate II-2, (See appendix.) consider any two distinct POINTS A and B in Σ. By Postulate I-1, there exists a LINE through A and B. Since every LINE is an arc of a Euclidean circle, there exists a POINT C between A and B and a POINT D such that B is between A and D.

To prove Postulate II-3 (See appendix.), consider any three POINTS A, B, and C on the same LINE. Since this LINE is an arc of an Euclidean circle, then one of the points A, B, or C must be between the other two.

To verify Postulate II-4, (Pasch's Postulate), it must be shown that a distinct LINE which intersects one side of a TRIANGLE but does not pass through any of the vertices of the TRIANGLE must intersect one other side of the TRIANGLE.

Figure 3

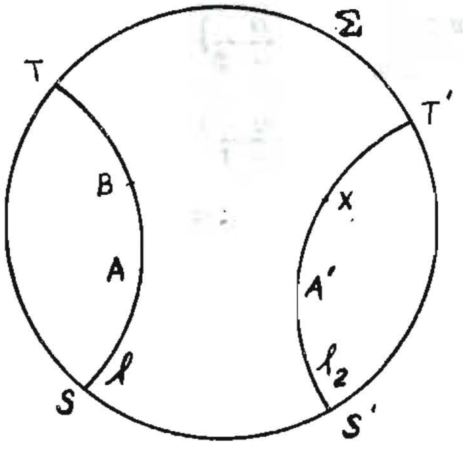


Proof:

Let TRIANGLE ABC (See figure 3.) be any TRIANGLE of Σ with DE intersecting side BC at POINT E. Let LINE DE be an arc of circle I which intersects Σ at points S and T. Let LINE BC be an arc of circle II. Since the two circles I and II are orthogonal to a third circle, Σ , and intersecting in Σ , then the second point of intersection H must be in the exterior of Σ . Either the POINT B or C is in the interior of circle I since POINT E is between POINTS B and C. Consider the POINT B is in the interior of circle I. This would imply that the POINT C is in the exterior of circle I, since the LINE DE of circle I intersects LINE BC between the POINTS B and C. The POINT A is not on circle I since DE does not pass thru a vertex of TRIANGLE ABC. Thus the POINT A is in the interior of circle I or the POINT A is in the exterior of circle I. Suppose that the POINT A is in the interior of circle I. The LINE AC must intersect LINE DE between A and C since A is interior and C is exterior to circle I. Suppose that the POINT A is in the exterior of circle I. Then the LINE AB must intersect LINE DE between A and B since A is exterior and B is interior to circle I. Thus Postulate II-4 has been verified.

Group III: <u>Postulates of Congruence</u>. The proof of Postulate III-1 (See appendix.) follows from the definition of the LENGTH OF A LINE SEGMENT.

Figure 4



Proof:

Let A and B (See figure 4.) be any two POINTS of Σ. Then by Postulate I-1, there exists a LINE ℓ, through POINTS A and B. Let points S and T denote the points of the intersection of LINE ℓ and Σ. The POINTS A, B, and points S, T are fixed points. Let LINE ℓ_2 be any LINE in Σ distinct from LINE ℓ. Choose any POINT A' on LINE ℓ_2. Let points S' and T' denote the points of the intersection of LINE ℓ_2 and Σ. The points A', S', and T' are fixed points.

It must be shown that there exists a POINT X on ℓ_2 such that the distance between A' and X on an arc can take on any value from 0 to ∞. Let $\widehat{A'T'} = c$, $\widehat{XS'} = d$, and $\widehat{S'T'} = q$. Consider the continuous function

$$f(X) = \log \left(\frac{\widehat{A'T'}}{\widehat{XT'}} \cdot \frac{\widehat{XS'}}{\widehat{A'S'}}\right)$$

$$= \log \left(\frac{c}{q-d} \cdot \frac{d}{q-c}\right).$$

Since log X is a continuous function, then the

$$\lim_{X \to T} f(X) = \lim_{d \to q} \left[\log \left(\frac{c}{q-c} \cdot \frac{d}{q-d}\right)\right]$$

$$= \log \left(\frac{c}{q-c} \cdot \left[\lim_{d \to q} \left(\frac{d}{q-d}\right)\right]\right)$$

$$= \log \left(\frac{c}{q-c} \cdot \infty\right)$$

$$= \log \infty$$

$$= \infty .$$

This shows that A'X can be made as large as necessary.

Likewise,

$$\lim_{X \to A} f(X) = \lim_{d \to (q-c)} \log \left[\left(\frac{c}{q-d} \cdot \frac{c}{q-c}\right)\right]$$

$$= \log \left[\frac{c}{q-c} \cdot \lim_{d \to (q-c)} \left(\frac{d}{q-d}\right)\right]$$

$$= \log \left(\frac{c}{q-c} \cdot \frac{q-c}{c}\right)$$

$$= \log 1$$

$$= 0.$$

Thus A'X can be made as small as necessary.

The LENGTH OF $\widehat{AB} = \log \left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right)$ by definition 11.7.

Let $r = \log \left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right)$, where r is a real number. A POINT B' can be found by solving the equation

$$r = \log \left(\frac{\widehat{A'T'}}{\widehat{XT'}} \cdot \frac{\widehat{XS'}}{\widehat{A'S'}}\right) \text{ for X.}$$

This POINT B' will between A' and T' on LINE ℓ_2. Similarly, a second POINT B" can be found by solving the equation

$$r = \log \left(\frac{\widehat{A'S'}}{\widehat{XS'}} \cdot \frac{\widehat{XT'}}{\widehat{A'T'}}\right) \text{ for X.}$$

The second POINT B" will between S' and A' on LINE ℓ_2. Since the POINT B' is between A' and T', the POINT B" is between S' and A', and the POINT A' is between S' and T', then A' is between B' and B".

Two unique POINTS B' and B" can now be found such that LENGTH $\widehat{AB}$ = LENGTH $\widehat{A'B'}$ and the LENGTH $\widehat{AB}$ = LENGTH $\widehat{A'B''}$. Therefore, by definition 11.8, the pair of POINTS

A', B' is congruent to the pair A, B. Likewise, the pair of POINTS A', B" is congruent to the pair A, B.

To prove Postulate III-2 (See appendix.), let the pair of POINTS A, B be congruent to the pair A', B'. Let the pair of POINTS A, B be congruent to the pair A", B". Then by definition II.8, the pairs of POINTS (A, B), (A', B'), and (A", B") are endpoints of segments and

$$\log\left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right) = \log\left(\frac{\widehat{A'T'}}{\widehat{B'T'}} \cdot \frac{\widehat{B'S'}}{\widehat{A'S'}}\right)$$

and

$$\log\left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right) = \log\left(\frac{\widehat{A''T''}}{\widehat{B''T''}} \cdot \frac{\widehat{B''S''}}{\widehat{A''S''}}\right).$$

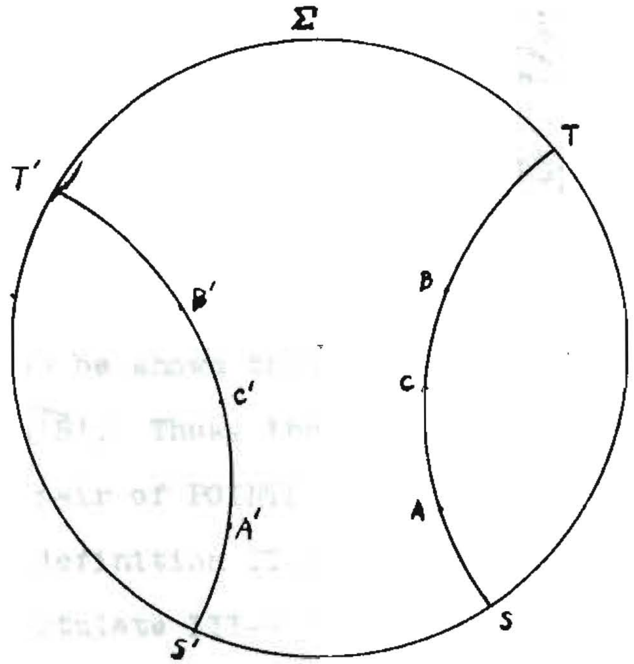
Then

$$\log\left(\frac{\widehat{A'T'}}{\widehat{B'T'}} \cdot \frac{\widehat{B'S'}}{\widehat{A'S'}}\right) = \log\left(\frac{\widehat{A''T''}}{\widehat{B''T''}} \cdot \frac{\widehat{B''S''}}{\widehat{A''S''}}\right).$$

Therefore, the pair of POINTS A', B', is congruent to the pair A", B" by definition II.8.

To verify Postulate III-3 (See appendix.), let AB be any SEGMENT IN Σ with POINT C any POINT between A and B. Let A'B' be a SEGMENT in Σ with C' between A' and B' such that the pair of POINTS A, C is CONGRUENT to the pair A', C', and the pair of POINTS C, B is CONGRUENT to the pair C', B'. (See figure 5.)

Figure 5



Proof:

By definition II.8,

$$\log \left(\frac{\widehat{AT}}{\widehat{CT}} \cdot \frac{\widehat{CS}}{\widehat{AS}} \right) = \log \left(\frac{\widehat{A'T'}}{\widehat{C'T'}} \cdot \frac{\widehat{C'S'}}{\widehat{A'S'}} \right) \text{ and}$$

$$\log \left(\frac{\widehat{CT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{CS}} \right) = \log \left(\frac{\widehat{C'T'}}{\widehat{B'T'}} \cdot \frac{\widehat{B'S'}}{\widehat{C'S'}} \right).$$

Then by definition II.7, the LENGTH $\widehat{AC}$ = LENGTH $\widehat{A'C'}$ and the LENGTH $\widehat{CB}$ = LENGTH $\widehat{C'B'}$. Therefore, the LENGTH $\widehat{AC}$ + LENGTH $\widehat{CB}$ = LENGTH $\widehat{A'C'}$ + LENGTH $\widehat{C'B'}$.

Then the

$$\text{LENGTH } \widehat{AC} + \text{LENGTH } \widehat{CB} = \log \left(\frac{\widehat{AT}}{\widehat{CT}} \cdot \frac{\widehat{CS}}{\widehat{AS}}\right) + \log \left(\frac{\widehat{CT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{CS}}\right)$$

$$= \log \left(\frac{\widehat{AT}}{\widehat{CT}} \cdot \frac{\widehat{CS}}{\widehat{AS}} \cdot \frac{\widehat{CT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{CS}}\right)$$

$$= \log \left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right)$$

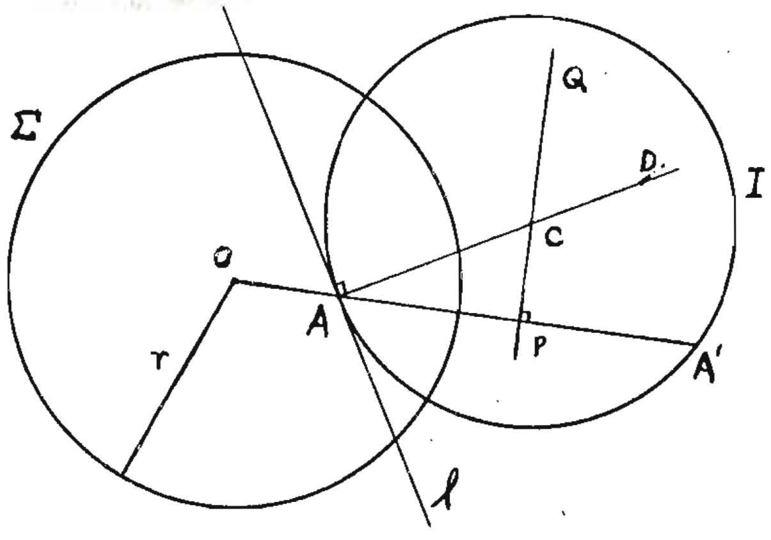
$$= \text{LENGTH } \widehat{AB}.$$

Likewise, it can be shown that the LENGTH $\widehat{A'C'}$ + LENGTH $\widehat{C'B'}$ = LENGTH $\widehat{A'B'}$. Thus, the LENGTH $\widehat{AB}$ = LENGTH $\widehat{A'B'}$. Therefore, the pair of POINTS A, B is CONGRUENT to the pair A', B' by definition II.8.

Before Postulate III-4 (See appendix.) can be verified, the following theorem must be proved.

<u>Theorem II.1</u>. There exists a unique circle orthogonal to Σ and tangent to a given line ℓ at a POINT A of ℓ not on Σ .

Figure 6



Proof:

Let O (See figure 6.) be the center of Σ. Let ℓ be any line passing through Σ with A any point on ℓ where A is in Σ. Construct on the line OA, a point A', the inverse of A. Therefore, OA $\cdot$ OA' $=$ r^2, where r is the radius of Σ. Construct the perpendicular bisector PQ of segment AA' with P on segment AA'. Construct line AD perpendicular to line ℓ. Let the point of intersection of line AD and line PQ be called C. Construct circle I with center C and radius equal to the length of AC. Circle I is tangent to line ℓ at A since the center of circle I is on the perpendicular line to ℓ at A. By using the same method which was established in proving Postulate I-1, circle I can be shown to be orthogonal to Σ. Circle I is unique since any other circle passing through A and A' could not be tangent to line ℓ and orthogonal to Σ on the same side of line ℓ.

Postulate III-4 can now be verified.

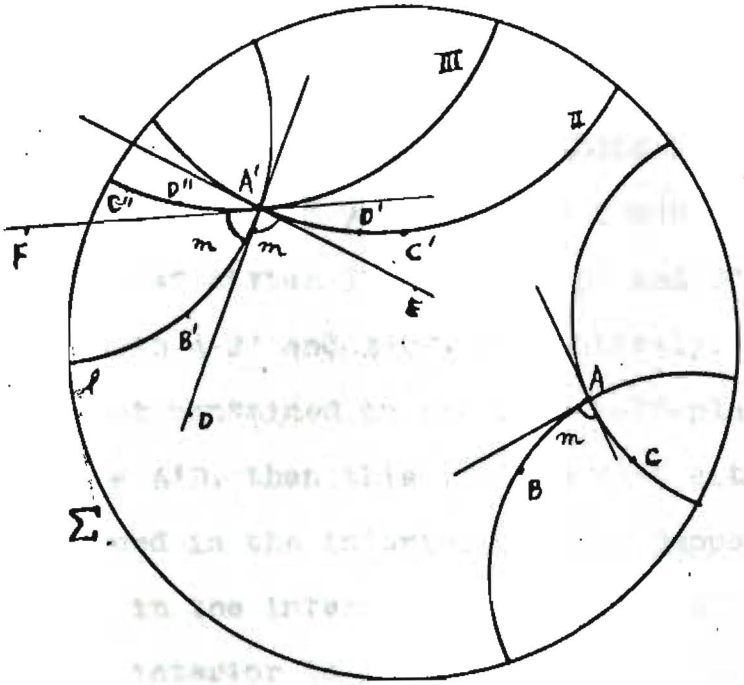


Figure 7

Proof:

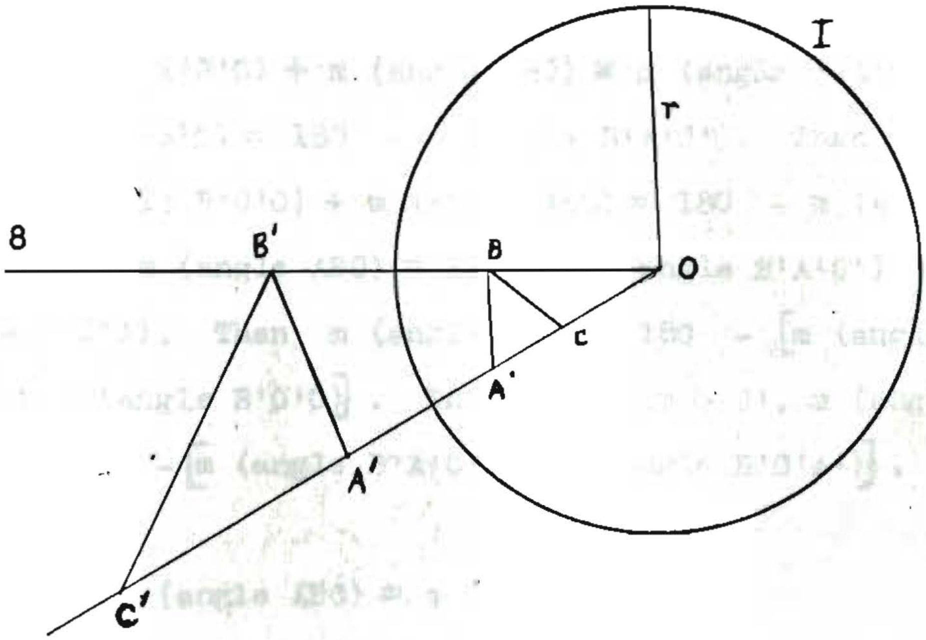
Consider any ANGLE BAC and any two distinct POINTS A' and B'. (See figure 7.) By Postulate I-1 (See appendix.), there exists a unique LINE ℓ through A' and B'. Let m equal the measure of ANGLE BAC. Construct the tangent A'D to ℓ at A'. Since Σ is in the Euclidean plane, then there exists two distinct lines A'E and A'F such that the measure of angle EA'D is equal to m and the measure of angle FA'D is equal to m. Then by theorem II.1, there exists a unique circle II orthogonal to Σ passing through A' and tangent to A'E. Likewise, there exists a unique circle III orthogonal to Σ and tangent to A'F at A'. Note that in the half-planes determined by line A'D, the points E and F are not contained in the same half-plane. Let C' and C" denote POINTS on circle II and circle III, respectively, such that the POINT C' is the same half-plane determined by A'D as E and the POINT C" is in the same half-plane as F. Then ANGLE BAC is CONGRUENT to ANGLE B'A'C' and ANGLE BAC is CONGRUENT to ANGLE B'A'C" by definition II.14. Let D' and D" be any two POINTS ON RAYS A'C' and A'C", respectively. Since C' and C" are not contained in the same half-plane determined by line A'D, then this implies that either C' or C" is contained in the interior of ℓ . Suppose that C" is contained in the interior of ℓ . Then all the POINTS on RAY A'C" are interior to ℓ . Likewise, C' is in the exterior of ℓ and all the POINTS on RAY A'C" are exterior to ℓ . By Postulate I-1, there exists a unique LINE ℓ_2 passing through D' and D" and orthogonal to Σ . Since D" is an interior POINT of ℓ and D' is an exterior POINT of ℓ , then LINE ℓ_2 must intersect ℓ which is LINE A'B'.

Postulate III-5 (See appendix.) is verified by definitions II.13 and II.14.

The following two theorems must be proved before Postulate III-6 can be verified.

<u>Theorem</u> <u>II.2</u>. Inversion is a conformal transformation, i.e., in an inversion the angle between two intersecting curves is equal to the corresponding angle between the two inverse curves.

Figure 8



Proof:

Consider any angle ABC and a point O (See figure 8.) such that the points A, C, and O are collinear. Let O be the center of circle I with radius r. Let angle ABC be contained in circle I. Using O as the center of inversion, there exists points A', B', and C' which are the inverse points of A, B, and C, respectively. Then $r^2 = OA \cdot OA' = OB \cdot OB' = OC \cdot OC'$. Construct lines A'B' and B'C'. It can easily be shown by using similar triangles that m (angle CBO) = m (angle B'C'O), m (angle BCO) = m (angle A'B'O) + m (angle A'B'C'), m (angle A'B'O) = m (angle BAO) and m (angle CBO) + m (angle ABC) = m (angle B'A'O), where m means the measure of the angle. Since, m (angle CBO) = m (angle B'C'O) and

$$m\ (\text{angle } CBO) + m\ (\text{angle } ABC) = m\ (\text{angle } B'A'O)$$

then

(1) $m\ (\text{angle } B'C'O) + m\ (\text{angle } ABC) = m\ (\text{angle } B'A'O)$.

The $m\ (\text{angle } B'A'O) = 180 - m\ (\text{angle } B'A'C')$. Then (1) becomes $m\ (\text{angle } B'C'O) + m\ (\text{angle } ABC) = 180 - m\ (\text{angle } B'A'C')$ and $m\ (\text{angle } ABC) = 180 - m\ (\text{angle } B'A'C') - m\ (\text{angle } B'C'O)$. Then $m\ (\text{angle } ABC) = 180 - [m\ (\text{angle } B'A'C') + m\ (\text{angle } B'C'O)]$. In triangle $A'B'C'$, $m\ (\text{angle } A'B'C') = 180 - [m\ (\text{angle } B'A'C') + m\ (\text{angle } B'C'A')]$.

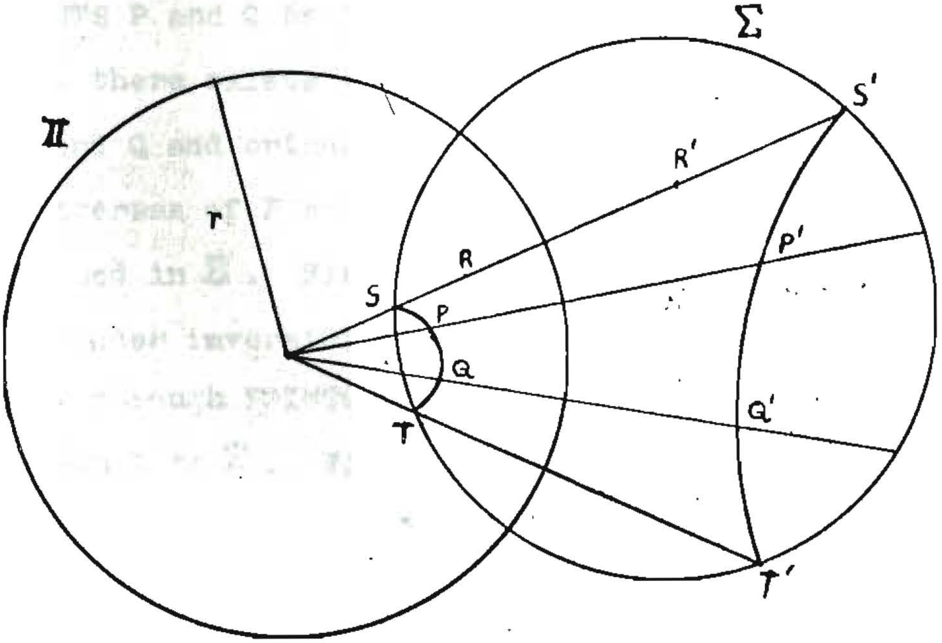
Thus,

$$m\ (\text{angle } ABC) = m\ (\text{angle } A'B'C').$$

The measure of an angle is invariant under inversion.

<u>Theorem</u> <u>II.3</u>. LENGTH $\widehat{PQ}$ is invariant under inversion in any circle orthogonal to Σ.

Figure 9



Proof:

Let circle II (See figure 9.) be the circle of inversion with center O and let circle II be orthogonal to Σ. Let r denote the radius of circle II. Let points S and T be on Σ and contained in circle II. Let S' and T' denote the inverses of S and T. Then $OS \cdot OS' = r^2$, since Σ is orthogonal to circle II. Consider any POINT R on line OS such that R is contained in the interior of Σ and circle II. Thus the length OR is greater than the length OS. Suppose that R', the inverse of R with respect to circle II, is not contained in Σ. This implies that the length OR' is greater than the length OS'. But since $OS \cdot OS' = r^2$, then $OR \cdot OR'$ is greater than r^2 which contradicts that R' is the inverse of R. Therefore, R' is contained in Σ.

Let POINTS P and Q be in the interior of Σ. By Postulate I-1, there exists a unique circle passing through the POINTS P and Q and orthogonal to Σ. The POINTS P' and Q', the inverses of P and Q with respect to circle II, are contained in Σ. Since the measures of angles are preserved under inversion by theorem II.2, then the circle passing through POINTS P', Q' and points S', T' will be orthogonal to Σ. First, it will be shown that

$$\widehat{PT} = \frac{\widehat{P'T'} \cdot r^2}{OP' \cdot OT'}, \quad \widehat{QT} = \frac{\widehat{Q'T'} \cdot r^2}{OQ' \cdot OT'},$$

$$\widehat{PS} = \frac{\widehat{P'S'} \cdot r^2}{OP' \cdot OS'}, \text{ and } \widehat{QS} = \frac{\widehat{Q'S'} \cdot r^2}{OQ' \cdot OS'}.$$

These relationships will later be used to prove LENGTH $\widehat{PQ}$ = LENGTH $\widehat{P'Q'}$.

In order to show this, suppose that points O, P', and T' are not collinear. Consider the triangles OPT and OP'T'. Since $r^2 = OP \cdot OP' = OT \cdot OT'$, then $\frac{OP}{OT} = \frac{OT'}{OP'}$.

Also since angle POT is congruent to angle P'OT', then triangle OPT is similar to triangle OP'T'.

Therefore,

$$\frac{\widehat{PT}}{P'T'} = \frac{OT}{OP'} = \frac{OT}{OP'} \cdot \frac{OT'}{OT'} = \frac{r^2}{OP' \cdot OT'}.$$

Then

$$\widehat{PT} = \frac{\widehat{P'T'} \cdot r^2}{OP' \cdot OT'}.$$

Suppose that the points O, P', and T' are collinear. Since $OP \cdot OP' = OT \cdot OT'$, then $(OT' - P'T') \cdot OP = OT' \cdot (OP - PT)$ and

$$(OT' \cdot OP) - (P'T' \cdot OP) = (OT' \cdot OP) - (OT' \cdot PT).$$

Thus

$$-(P'T' \cdot OP) = -(OT' \cdot PT).$$

Multiplying by -1,

$$P'T' \cdot OP = OT' \cdot PT$$

and

$$\frac{PT}{P'T'} = \frac{OP}{OT'} = \frac{OP}{OT'} \cdot \frac{OP'}{OP'} = \frac{r^2}{OT' \cdot OP'}.$$

Since the measure of angles are preserved under inversion, then m (angle QTP) = m (angle Q'T'P'), m (angle QPT) = m (angle Q'P'T').

Thus

$$\frac{\widehat{PT}}{\widehat{P'T'}} = \frac{PT}{P'T'} .$$

Then

$$\frac{\widehat{PT}}{\widehat{P'T'}} = \frac{r^2}{OT' \cdot OP'} \quad \text{and } \widehat{PT} = \frac{\widehat{P'T'} \cdot r^2}{OT' \cdot OP'} .$$

Likewise, it can be shown that

$$\widehat{QT} = \frac{\widehat{Q'T'} \cdot r^2}{OQ' \cdot OT'} , \; \widehat{PS} = \frac{\widehat{P'S'} \cdot r^2}{OP' \cdot OS'} , \text{ and } \widehat{QS} = \frac{\widehat{Q'S'} \cdot r^2}{OQ' \cdot OS'} .$$

By definition II.7, the

$$\text{LENGTH } \widehat{PQ} = \log \left(\frac{\widehat{PT}}{\widehat{QT}} \cdot \frac{\widehat{QS}}{\widehat{PS}}\right).$$

Since

$$\widehat{PT} = \frac{\widehat{P'T'} \cdot r^2}{OT' \cdot OP'} , \; \widehat{QT} = \frac{\widehat{Q'T'} \cdot r^2}{OQ' \cdot OT'} ,$$

$$\widehat{QS} = \frac{\widehat{Q'S'} \cdot r^2}{OQ' \cdot OS'} , \text{ and } \widehat{PS} = \frac{\widehat{P'S'} \cdot r^2}{OP' \cdot OS'} ,$$

the LENGTH $\widehat{PQ}$ becomes

$$\log \left(\frac{\widehat{PT}}{\widehat{QT}} \cdot \frac{\widehat{QS}}{\widehat{PS}}\right)$$

$$= \log \left(\left[\frac{\widehat{P'T'} \cdot r^2}{OP' \cdot OT'} \Big/ \frac{\widehat{Q'T'} \cdot r^2}{OQ' \cdot OT'}\right] \cdot \left[\frac{\widehat{Q'S'} \cdot r^2}{OQ' \cdot OS'} \Big/ \frac{\widehat{P'S'} \cdot r^2}{OP' \cdot OS'}\right]\right)$$

$$= \log \left(\frac{\widehat{P'T'} \cdot r^2}{OP' \cdot OT'} \cdot \frac{OQ' \cdot OT'}{\widehat{Q'T'} \cdot r^2} \cdot \frac{\widehat{Q'S'} \cdot r^2}{OQ' \cdot OS'} \cdot \frac{OP' \cdot OS'}{\widehat{P'S'} \cdot r^2}\right)$$

$$= \log \left(\frac{\widehat{P'T'}}{\widehat{Q'T'}} \cdot \frac{\widehat{Q'S'}}{\widehat{P'S'}}\right)$$

$$= \text{LENGTH } \widehat{P'Q'}$$

by definition II.7. Thus the LENGTH $\widehat{PQ}$ is invariant under inversion in any circle orthogonal to Σ .

Postulate III-6 (See appendix.) can now be proved.

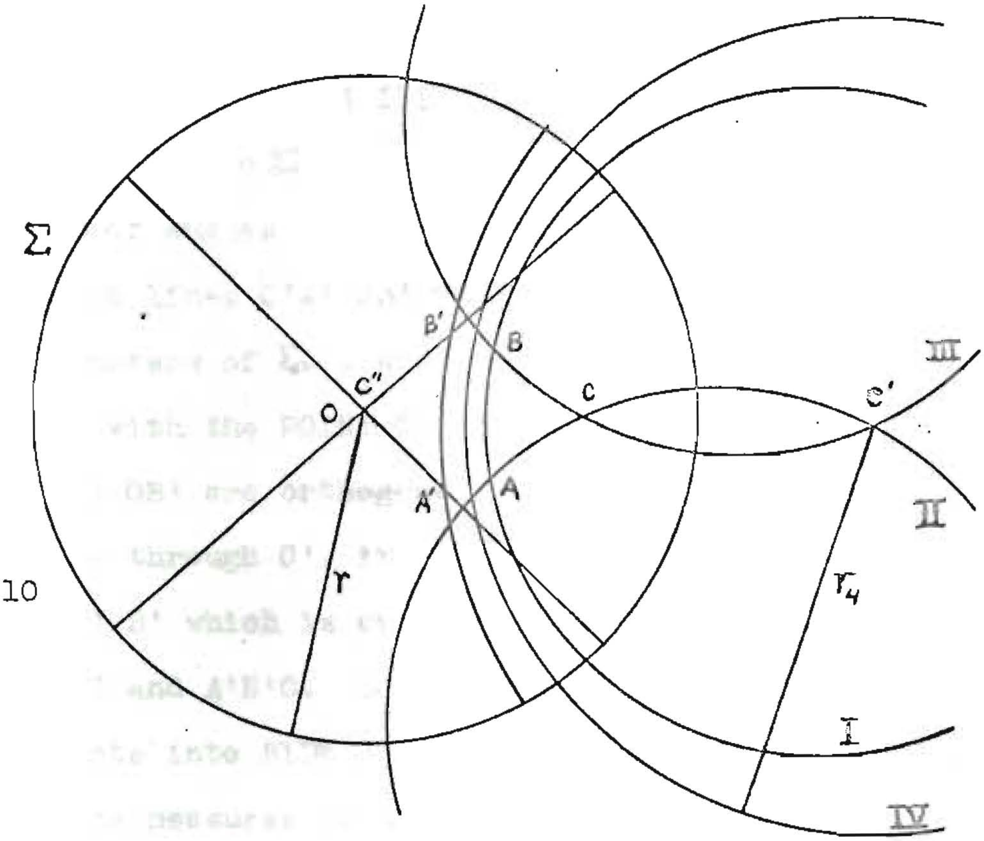


Figure 10

Proof:

Consider any TRIANGLE ABC in Σ . (See figure 10.) Let O be the center of Σ and let r denote the radius of Σ . Let SIDE AB be on circle I, SIDE AC on circle II, and SIDE BC on circle III. Construct line OC. Since circle II and III are orthogonal to Σ and intersect in Σ , then their second point of intersection C' is in the exterior of Σ. C' is the inverse of C with respect to Σ . Construct circle IV with center C' and radius r_4, where $r_4^2 = (OC')^2 - r^2$. Then Σ is orthogonal to circle IV. The point C' can be used to invert Σ into itself. Let C" be the inverse of C with respect to IV. Since circles II and III pass through C' the center of inversion, then they invert into straight lines C"A' and C"B'. The inverses of circles II and III with respect to circle IV will be orthogonal to Σ , since Σ inverts into itself and the measures of angles are preserved under inversion. Since straight lines C"A' and C"B' are orthogonal to Σ , they are diameters of Σ , and pass through O. Therefore, C" coincides with the POINT O. Then by definition II.2, LINES OA' and OB' are orthogonal to Σ . Since circle I does not pass through C', it inverts into a circle through POINTS A' and B' which is orthogonal to Σ . Consider the TRIANGLES ABC and A'B'O. SIDE AB inverts into SIDE A'B', SIDE AC inverts into SIDE A'O, and SIDE BC inverts into SIDE B'O. The measures of ANGLES and the LENGTH OF SEGMENTS are preserved under inversion by theorems II.2 and II.3, respectively. Therefore, TRIANGLE ABC is CONGRUENT to TRIANGLE A'B'O by definition II.17.

Consider TRIANGLES ABC and $A_1B_1C_1$, (See figure 11.) where SIDE AC is CONGRUENT to SIDE A_1C_1, ANGLE ACB is CONGRUENT to ANGLE $A_1C_1B_1$, and SIDE BC is CONGRUENT to SIDE B_1C_1.

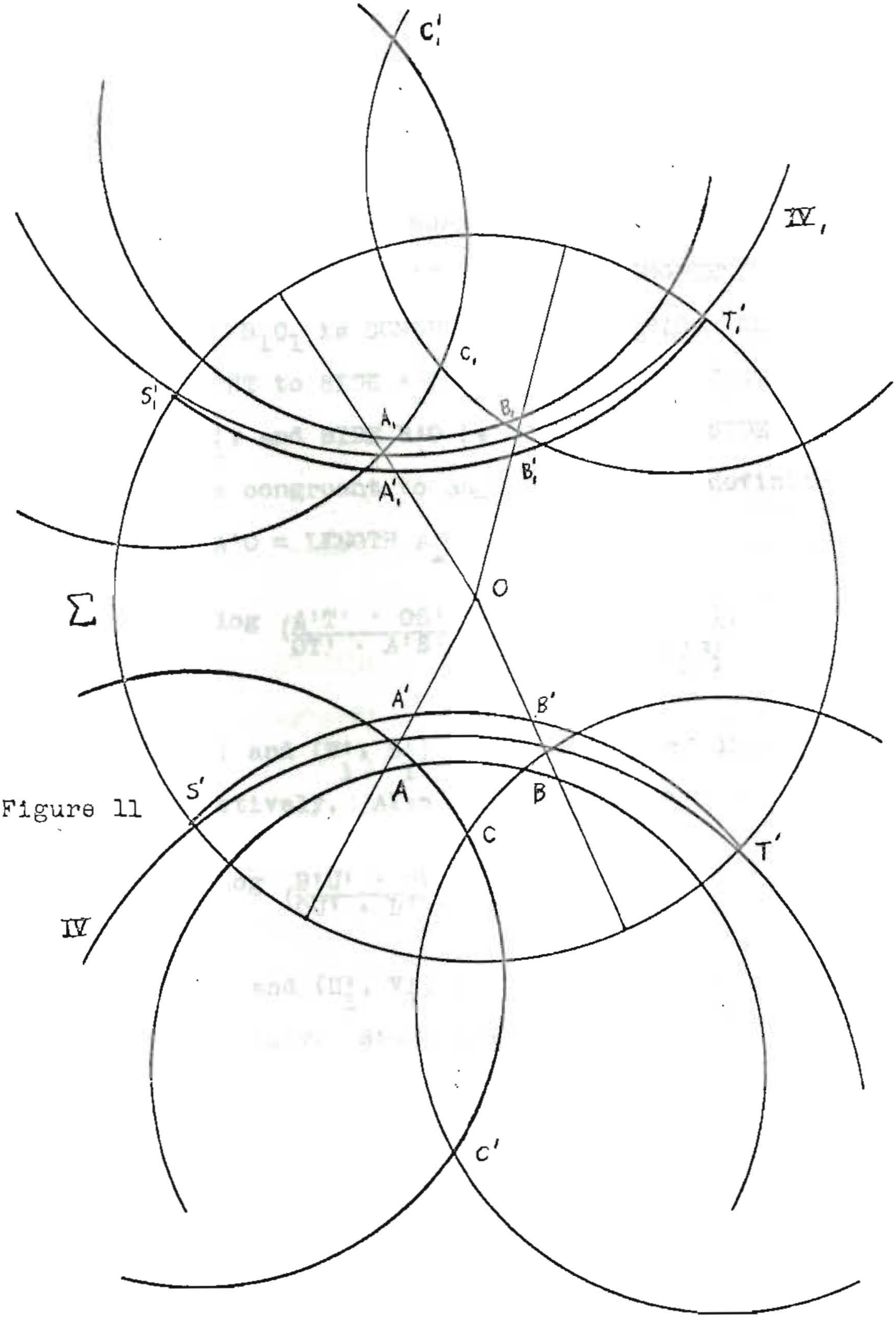


Figure 11

TRIANGLE ABC is CONGRUENT to a TRIANGLE A'B'O by the above paragraph. Likewise, it can be shown that TRIANGLE $A_1B_1C_1$ is CONGRUENT to a TRIANGLE $A_1'B_1'O$ by the same method. Thus SIDE AC is CONGRUENT to SIDE A'O, ANGLE ACB is CONGRUENT to ANGLE A'OB', SIDE BC is CONGRUENT to SIDE B'O, SIDE A_1C_1 is CONGRUENT to SIDE $A_1'O$, ANGLE $A_1C_1B_1$ is CONGRUENT to ANGLE $A_1'OB_1'$, and SIDE B_1C_1 is CONGRUENT to SIDE $B_1'O$. Then SIDE A'O is CONGRUENT to SIDE $A_1'O$, ANGLE A'OB' is CONGRUENT to ANGLE $A_1'OB_1'$, and SIDE B'O is CONGRUENT to SIDE $B_1'O$. Angle A'OB' is congruent to angle $A_1'OB_1'$. By definition II.9, LENGTH A'O = LENGTH $A_1'O$ and

$$\log \left(\frac{A'T' \cdot OS'}{OT' \cdot A'S'}\right) = \log \left(\frac{A_1'T_1' \cdot OS_1'}{OT_1' \cdot A_1'S_1'}\right)$$

where (T', S') and (T_1', S_1') are endpoints of diameters of Σ, respectively. Also, LENGTH B'O = LENGTH $B_1'O$ and

$$\log \left(\frac{B'U' \cdot OV'}{OU' \cdot B'V'}\right) = \log \left(\frac{B_1'U_1' \cdot OV_1'}{OU_1' \cdot B_1'V_1'}\right)$$

where (U', V') and (U_1', V_1') are endpoints of diameters of Σ, respectively. Since the diameters of Σ are of equal length, this implies that length A'O = length $A_1'O$ and length B'O = length $B_1'O$. There exists a unique circle passing through POINTS A' and B' which is orthogonal to Σ. Likewise, there exists a unique circle passing through POINTS A_1' and B_1' which is orthogonal to Σ.

Consider the rigid motion (rotation and line reflection) that maps $A' \to A_1'$, $O \to O$, and $B' \to B_1'$. Then SIDE $A'O$ coincides with SIDE $A_1'O$ and SIDE $B'O$ coincides with SIDE $B_1'O$. Suppose that $\widehat{A'B'}$ does not coincide under the rigid motion with $\widehat{A_1'B_1'}$. Since $\widehat{A'B'}$ and $\widehat{A_1'B_1'}$ are on circles orthogonal to Σ, this implies that there exists two distinct circles passing through POINTS A_1' and B_1', which are orthogonal to Σ. This is a contradiction of Postulate I-1. Thus $\widehat{A'B'}$ coincides with $\widehat{A_1'B_1'}$ and $\widehat{A'B'}$ is congruent to $\widehat{A_1'B_1'}$. Then the figure $A'B'O$ is congruent to figure $A_1'B_1'O$. Thus the Euclidean angle $OA'B'$ at A' is congruent to the angle $OA_1'B_1'$ at A_1' and from definition II.14, they are CONGRUENT. Likewise, the Euclidean angle $OB'A'$ at B' is congruent to the angle $OB_1'A_1'$ at B_1', and from definition II.14, they are CONGRUENT.

It remains to show the LENGTH $\widehat{A_1'B_1'}$ = LENGTH $\widehat{A'B'}$. Observe that under the rigid motion described above that length $\widehat{S_1'A_1'}$ = length $\widehat{S'A'}$ and length $\widehat{B_1'T_1'}$ = length $\widehat{B'T'}$. Thus it can be shown that LENGTH $\widehat{A_1'B_1'}$ = LENGTH $\widehat{A'B'}$. Then TRIANGLE $A'B'O$ is CONGRUENT to TRIANGLE $A_1'B_1'O$. Since TRIANGLE ABC is CONGRUENT to TRIANGLE $A'B'O$ which is CONGRUENT to TRIANGLE $A_1'B_1'O$ and TRIANGLE $A_1'B_1'O$ is CONGRUENT to TRIANGLE $A_1B_1C_1$, then TRIANGLE ABC is CONGRUENT to TRIANGLE $A_1B_1C_1$.

Group IV: <u>Postulate of Parallels.</u>

The Lobachevskian parallel postulate, Postulate IV'-1, can now be proved.

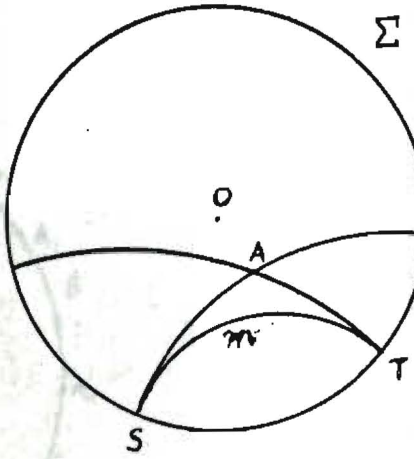


Figure 12

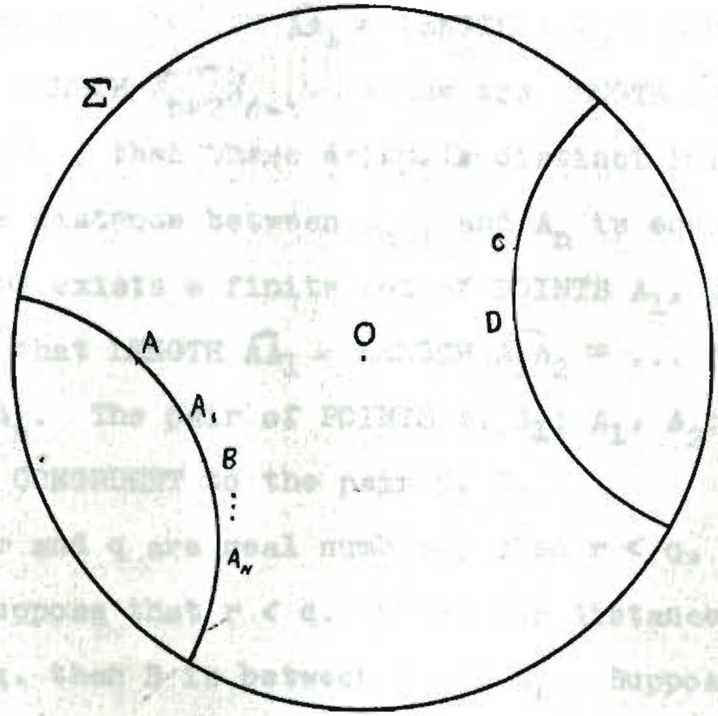
Consider any LINE m (See figure 12.) and any POINT A not on LINE m in Σ. Let S and T denote the points where circle m intersects Σ. There exists a unique circle I passing through points S and A which is orthogonal to Σ. Likewise, there exists a unique circle II passing through points T and A which is orthogonal to Σ. The two circle I and II are tangent to circle m at S and T, respectively. Thus the circles I and II are distinct. Since the points S and T are on Σ, they are not considered to be points of the model. Thus through a given POINT A not on a given LINE m, there pass at least two distinct LINES which do not intersect LINE m.

Group V: <u>Postulate of Continuity</u>

The proof of Postulate V-1 (See appendix.) follows from the definition of the LENGTH of a SEGMENT and

Postulate III-1.

Figure 13



Proof:

Consider any four distinct POINTS A, B, C, and D in Σ. (See figure 13) By Postulate I-1, there exists unique LINES AB and CD orthogonal to Σ. The LENGTH $\widehat{AB} = \log\left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right)$ and LENGTH $\widehat{CD} = \log\left(\frac{\widehat{CU}}{\widehat{DU}} \cdot \frac{\widehat{DV}}{\widehat{CV}}\right)$ by definition II.7. Let

$$r = \log\left(\frac{\widehat{AT}}{\widehat{BT}} \cdot \frac{\widehat{BS}}{\widehat{AS}}\right) \text{ and } q = \log\left(\frac{\widehat{CU}}{\widehat{DU}} \cdot \frac{\widehat{DV}}{\widehat{CV}}\right),$$

where r and q are real numbers. Then from Postulate III-1, there exists a distinct POINT A_1 on RAY AB such that the distance between A and A_1 is equal to q. By Postulate III-1, there exists a distinct POINT A_2 on RAY A_1B such that the distance between A_1 and A_2 is equal to

q. Suppose that there exists distinct POINTS A_1, A_2, ..., A_{n-1} such that LENGTH $\widehat{AA_1}$ = LENGTH $\widehat{A_1A_2}$ = LENGTH $\widehat{A_2A_3}$ = ... = LENGTH $\widehat{A_{n-2}A_{n-1}}$. Since the LENGTH $\widehat{AA_{n-1}} = (n-1) \cdot q \neq \infty$, then there exists a distinct POINT A_n such that the distance between A_{n-1} and A_n is equal to q. Thus there exists a finite set of POINTS A_1, A_2, ..., A_n such that LENGTH $\widehat{AA_1}$ = LENGTH $\widehat{A_1A_2}$ = ... = LENGTH $\widehat{A_{n-1}A_n}$. The pair of POINTS A, A_1; A_1, A_2; ... ; A_{n-1}, A_n are CONGRUENT to the pair C, D.

Since r and q are real numbers, then $r < q$, $r = q$, or $r > q$. Suppose that $r < q$. Since the distance between A and A_1 is q, then B is between A and A_1. Suppose that $r = q$. Then the POINT A_1 is the POINT B, since the distance between A and A_1 is q. Thus the POINT B will be between the POINTS A and A_2. Suppose that $r > q$, then there exists by the Archimedes principle of the real number system, a positive integer n such that $q \cdot n > r$. Thus there exists a POINT A_n such that the distance between A and A_n is greater than r. Therefore, the POINT B is between A and A_n.

III. SUMMARY

It has been shown that the model of Poincaré satisfies the axioms of plane Hyperbolic geometry. Thus plane Hyperbolic geometry is as consistent as plane Euclidean geometry. For, should there be any inconsistency in plane Hyperbolic geometry, there would have been a corresponding inconsistency in the plane Euclidean geometry of the Poincaré model. Therefore, plane Hyperbolic geometry has relative consistency.

It can be noted that, since plane Euclidean geometry is as consistent as the real number system, plane Hyperbolic geometry is as consistent as the real number system.

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A, B, C, are all on the same line, and C is between B and A, and B is not between C and A, and A is not between C and B.

II-2. For any two distinct points A and B there is always a point C which is between A and B, and a point D such that B is between A and D.

II-3. If A, B, and C are three distinct points on the same line, then one of the points is between the other two.

Definition 2-1. By the segment AB is meant the points A and B and all points which are between A and B. Points A and B are called the end points of the segment. A point C is said to be on the segment AB if it is A or B or some point between A and B.

Definition 2-2. Two lines, a line segment, or two segments, are said to intersect if there is a point which is on both of them.

Definition 2-3. Let A, B, C be three points not on the same line. Then by the triangle ABC is meant the three segments AB, BC, CA are called the sides of the triangle, and the points A, B, C are called the vertices of the triangle.

II-4. (Pasch's Postulate) A line which intersects one side of a triangle but does not pass through any of the vertices of the triangle must also intersect another side of the triangle.

<u>GROUP III</u>: Postulates of congruence

<u>III-1</u>. If A and B are distinct points and if A' is a point on a line m, then there are two and only two distinct points B' and B" on m such that the pair of points A', B' is congruent to A, B and the pair of points A', B" is congruent to the pair A, B; moreover, A' is between B' and B".

<u>III-2</u>. If two pair of points are congruent to the same pair of points, then they are congruent to each other.

<u>III-3</u>. If point C is between A and B and point C' is between points A' and B', and if the pair of points A, C is congruent to the pair A', C' and the pair of points C, B is congruent to the pair C', B', then the pair of points A, B is congruent to the pair A', B'.

<u>Definition 2-4</u>. Two segments are said to be congruent if the end points of the segments are <u>congruent</u> pairs of points.

<u>Definition 2-5</u>. By the <u>ray</u> AB is meant the set of all points consisting of those which are between A and B, the point B itself, and all points C such that B is between A and C. The ray AB is said to <u>emanate</u> from the point A.

<u>Definition 2-6</u>. By <u>angle</u> is meant a point (called the <u>vertex</u> of the angle) and two rays (called the <u>sides</u> of the angle) eminating from the point. If the vertex

of the angle is A and if B and C are any two points other than A of the two sides of the angle, then angle BAC $\cong$ angle CAB.

<u>Definition 2-7</u>. If ABC is a triangle, then the three angles BAC, CBA, ACB are called <u>angles</u> of the triangle. Angle BAC is said to be <u>included</u> by the sides AB and AC of the triangle.

<u>III-4</u>. If BAC is an angle whose sides do not lie in the same line, and if A' and B' are two distinct points, then there are two and only two distinct rays, A'C' and A'C", such that angle B'A'C' is congruent to angle BAC and angle B'A'C" is congruent to angle BAC; moreover, if D' is any point on the ray A'C' and D" is any point on ray A'C", then the segment D'D" intersects the line determined by A' and B'.

<u>III-5</u>. Every angle is congruent to itself.

<u>III-6</u>. If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then each of the remaining angles of the first triangle is congruent to the corresponding angle of the second triangle.

<u>GROUP IV</u>: Postulate of parallels

<u>IV'-1</u>. Through a given point not on a line m there passes at least two lines which do not intersect line m.

<u>GROUP V</u>: Postulate of continuity

V-1. (Postulate of Archimedes) If A, B, C, D are four distinct points, then there is, on the ray AB, a finite set of distinct points A_1, A_2, ..., A_n such that (1) each of the pairs A, A_1; A_1, A_2; ...; A_{n-1}, A_n is congruent to the pair C, D and (2) B is between A and A_n.

All theorems of plane Euclidean geometry which are not based on the Fifth Postulate of Euclid's Elements, may be used as theorems in plane Hyperbolic geometry. The following list of theorems are all equivalent to postulate IV'-1 of plane Hyperbolic geometry.

Theorem 2-1. The sum of the three angles of a triangle is always less than two right angles.

Theorem 2-2. There does not exist a pair of similar noncongruent triangles.

Theorem 2-3. There exists a quadrilateral in which a pair of opposite sides are equal and the angles adjacent to a third side are right angles and the other two angles are equal and acute.

Theorem 2-4. In a quadrilateral with three right angles the fourth angle is acute.

Theorem 2-5. There exists three noncollinear points such that no circle can pass through the three points.

Theorem 2-6. There is an upper limit to the area of a triangle.

Theorem 2-7. There exists two straight lines which are parallel and they are not symmetrical to one another with respect to the midpoints of all their transversal segments.