ON THE ALGEBRA OF QUATERNIONS

A Thesis

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CHAPTER I

INTRODUCTION

1.1. Introduction. The quaternions are interesting in that they are a multiplicatively non-commutative system in which all of the other field postulates are valid. Since the quaternions were the first discovered non-commutative division algebra, an investigation of their properties and construction became the basis of this study.

1.2. Statement of the problem. It is the purpose of this study (1.) to give an insight on the history and construction of the quaternions; (2.) to give a complete proof that the quaternions form a division ring; (3.) to investigate the conditions for two quaternions to be commutative multiplicatively and their consequences, and (4.) to present some representations of quaternions.

1.3. Organization of the paper. The second chapter contains a brief history of what led to the discovery of quaternions and their construction as a four dimensional number.

Chapter three develops the algebra of quaternions and the proof that the quaternions form a division ring. A theorem attributed to Frobenius is presented and proved.

In chapter four the commutative quaternions are introduced and as a consequence of Frobenius' theorem an isomorphism between the commutative quaternions and the complex numbers is known to exist. The isomorphism is then presented.

Chapter five presents some theorems on representations of quaternions by means of isomorphisms.

The sixth and last chapter gives a summary and a suggestion for further study.

CHAPTER II

THE DEVELOPMENT OF QUATERNIONS

2.1. Introduction. The algebra of quaternions was born in a paper presented before the Royal Irish Academy on November 13, 1843, by the Irish mathematician Sir William Rowan Hamilton. Hamilton expanded this paper to include applications in the area of physics and in 1853 published *Lectures on Quaternions* and in 1866 *Elements of Quaternions*.

Hamilton's motivation in developing the quaternions came about through his investigations of the complex numbers. Instead of regarding a complex number as one number, Hamilton conceived it as an ordered pair. He then used these ordered pairs to represent directed line segments in the Cartesian plane. With this representation, the imaginary unit i was considered as an operator to rotate a directed line segment in the plane. The complex number system was then a very convenient number system for the study of directed line segments and rotations in a plane.

With this in mind, Hamilton attempted to devise an analagous system of numbers for application to directed line segments and rotations in three dimensional space. It seemed natural to Hamilton that since the complex numbers could be represented by ordered pairs the analagous number system for the study of directed line segments and rotations in space would require a representation by ordered triplets. That is, numbers of the form $a+bi+cj$ where a, b and c are real numbers and i and j are imaginary units. Hamilton had great difficulties for several years as he was unable to define a satisfactory multiplication operation on these ordered triplets. This difficulty was explained in 1878 when the algebraist Frobenius showed that no numbers exist beyond the ordinary complex numbers which could satisfy all the postulates of ordinary algebra. Finally, Hamilton realized that a number composed of a real part and two imaginary parts was not adequate for the required number system and that a number composed of a real part and three imaginary parts was required. Why this is the case is outlined in the following discussion of directed line segments in space. The outline of this discussion is taken from An Elementary Treatise on Quaternions by P.G. Tait.[1]

2.2. Construction of quaternions. Consider space to be coordinatized according to the ordinary Cartesian methods of geometry of three dimensions. Let $\overline{AB}$ denote a directed line segment. The relative position of point A to point B is given by the excesses of B's three coordinates over those of A. Denote these excesses by a number triplet to represent

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[1] P.G. Tait, An Elementary Treatise on Quaternions, London, 1867, pp. 32-49.

the directed line $\overline{AB}$. In this sense all directed line segments parallel and equal in length to AB will be denoted by the same number triplet. The usual procedures for adding directed line segments could also be employed so that $\overline{AB} + \overline{BC} = \overline{AC}$.

The three unit directed segments emanating from the origin where one each is contained by the x, y and z axes are denoted as i, j and k. (see Figure 2-1.) Any directed line segment in space parallel to one of the unit directed segments is written as a scalar multiple of that particular unit directed segment. Thus, any directed line segment in space may be resolved into three components, one of each of the components being parallel to and a scalar multiple of one of each of the unit directed segments i, j and k.

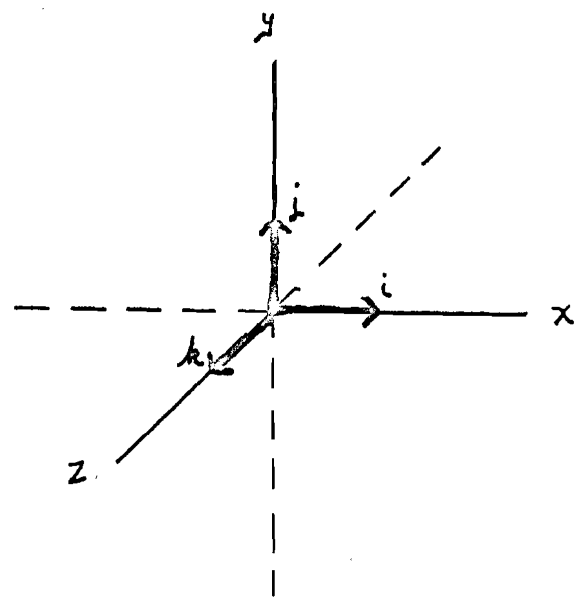

FIGURE 2-1

Define the operation of multiplication of one unit directed line segment b by another unit directed line segment a as giving the unit directed line segment c, which is in a counterclockwise rotation from b and mutually perpendicular to a and b. From Figure 2-1 this yields

$$i \cdot j = k, \quad k \cdot i = j, \quad j \cdot k = i$$

$$j \cdot i = -k, \quad i \cdot k = -j, \text{ and } k \cdot j = -i.$$

Note that this multiplication is non-commutative.

The quotient of two directed line segments is now considered to be a "number" which will act as an operator on one directed line segment to make it equivalent to another directed line segment. For example, $\frac{\overline{AB}}{\overline{CD}} = q$ so that $\overline{AB} = q \cdot \overline{CD}$. This "number" q was the type of number which Hamilton decided was to be composed of a real part and three imaginary parts. Hamilton used the following cases to arrive at this conclusion.

<u>Case</u> 1. If $\overline{AB}$ and $\overline{CD}$ are parallel it is well known in Cartesian space geometry that a scalar multiple of $\overline{CD}$ is equivalent to $\overline{AB}$. Thus, the "number" q in this case would operate as a real number to increase or decrease the length of $\overline{CD}$.

<u>Case</u> 2. If $\overline{AB}$ and $\overline{CD}$ are not parallel, take the equivalent of $\overline{CD}$ to be $\overline{AD'}$ so that $\overline{AB}$ and $\overline{AD'}$ emanate from point A. Now rotate $\overline{AD'}$ about A until its direction coincides with that of $\overline{AB}$. To specify this rotation operation the "number" q must be composed of three elements; two

elements to represent the angles which fix the plane in which the rotation takes place and one more element to represent the angle for the amount of this rotation. Also, this "number" q must contain an additional element to increase or decrease the length of $\overline{AD}'$ so that $\overline{AD}'$ would be equivalent to $\overline{AB}$ when operated on by q as in case 1.

With these cases in mind, Hamilton used the three imaginary unit directed line segments to establish the plane and the amount of rotation and a real number to increase or decrease the length of the given directed line segment in constructing the "number" q. These "numbers" q then act as operators to rotate a given directed line segment into another given directed line segment. It was the set of all these "numbers" q, given as the quotient of any two directed line segments and represented with a real and three imaginary parts, which Hamilton designated as quaternions.

2.3. Historical consequences. Although Hamilton expected his quaternions to prove to be a powerful tool for the advancement of physics, his expectation was never completely fulfilled. A large part of this is due perhaps, in the loss of naturalness in taking the square of a directed line segment to be a negative scalar and the fact that quaternions are rather bulky to work with.

It is interesting to note that the American physicist J.W. Gibbs, by simplifying and making more flexible the operations with quaternions, was able to develop a more applicable vector algebra to meet the demands of tensor calculus. In this respect, vector theory might be regarded as being latent in the theory of quaternions.

To the pure mathematician, a consequence of Hamilton's quaternion algebra is that it was the first example of a consistent algebra in which one of the fundamental postulates, the commutative law of multiplication, was deleted.[2] As a result, the door was opened for the study of structures of algebraic systems.

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[2] C. Lanczos, William Rowan Hamilton - An Appreciation, American Scientist, vol. 55 (1967), p. 137.

CHAPTER III

THE ALGEBRA OF QUATERNIONS

3.1. <u>Introduction</u>. The algebra of quaternions is the system consisting of the four basis units (1, i, j, k) over any field. However, unless otherwise stated, restriction will be made to the system of quaternions over the field of the real numbers. All properties of the real number system will be assumed for any operation performed on real numbers.

In this chapter, it will be shown that the quaternions over the field of real numbers form a division ring and that these quaternions are the only non-commutative division ring algebraic over the field of real numbers.

3.2. <u>Definitions and algebraic properties</u>. Let Q be the set of all numbers of the form $a+bi+cj+dk$ where a, b, c and d are real numbers and 1, i, j and k are basis units. Equality and the operations of addition and multiplication are defined on the elements of Q as follows.

<u>Definition</u>. 3.1. Equality. $(a+bi+cj+dk)=(x+yi+zj+wk)$ if and only if $a=x$, $b=y$, $c=z$ and $d=w$.

<u>Definition</u>. 3.2. Addition. $(a+bi+cj+dk)+(x+yi+zj+wk)=\left[(a+x)+(b+y)i+(c+z)j+(d+w)k\right]$.

<u>Definition</u>. 3.3. Multiplication of basis units.

$i^2 = j^2 = k^2 = -1$

$i \cdot j = k, \quad j \cdot k = i, \quad k \cdot i = j$

$j \cdot i = -k, \quad k \cdot j = -i$ and $i \cdot k = -j$.

Set Q, with the above operations defined on Q, will now be referred to as the quaternions.

In the algebra of quaternions, multiplication of a basis unit with a real number is assumed to be commutative. That is, for any real number x and the basis unit i, $x \cdot i = i \cdot x$ and similarly for the other basis units.

<u>Definition</u>. 3.4. Multiplication of quaternions.
$(a + bi + cj + dk) \cdot (x + yi + zj + wk) = \left[(ax - by - cz - dw) + (ay + bx + cw - dz)i + (az - bw + cx + dy)j + (aw + bz - cy + dx)k\right]$.

It is well to note here that multiplication of quaternions is very similar to multiplication of polynomials with the use of definition 3.4. and the commutativity of basis units with real numbers.

Investigation of the structure of quaternions as a mathematical system will now follow.

<u>Definition</u>. 3.5. A group G is a collection of objects for which a binary operation $*$ is defined where the operation is subject to the following laws.

(1.) If a and b are in G, then $a * b$ is in G.

(2.) If a, b and c are in G, then $(a*b)*c = a*(b*c)$.

(3.) There exists a unique identity element e in G such that for all a in G $a*e = a$.

(4.) For every a in G there exists a unique inverse element a' such that $a*a' = e$.[1]

If the group G is also commutative under the defined operation (i.e. $a*b = b*a$ for all a,b in G), then G is called an abelian group.

Theorem. 3.1. The quaternions form an additive abelian group.

Proof: By use of definition 3.2. and the closure property of addition for the real number system, it is seen that addition is closed in the quaternions. In a similar manner associativity and commutativity follow from definition 3.2. and the associative and commutative properties of the real numbers. The unique additive identity element is the zero quaternion $0 = 0 + 0i + 0j + 0k$. The unique additive inverse for any quaternion $a + bi + cj + dk$ is the quaternion $[(-a) + (-b)i + (-c)j + (-d)k]$. Thus, the quaternions form an additive abelian group.

Definition. 3.6. A ring R is an additive abelian group with the additional properties:

(1.) The group R is closed with respect to a second binary operation $\bullet$.

(2.) The operation $\bullet$ is associative for all elements in R.

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[1]Kenneth S. Miller, Elements of Modern Abstract Algebra, New York, 1958, p. 1.

(3.) The operation $\cdot$ is distributive with respect to $*$ on both the left and right for all elements in R.[2]

If a ring has a unity element u for each element x in R such that $x \cdot u = x$, then the ring is called a ring with unity.

Theorem. 3.2. The quaternions form a ring with unity.

Proof: Theorem 3.1. satisfies the first condition for quaternions to be a ring with unity. Closure under multiplication is seen from definition 3.4. and the closure properties of addition and multiplication for the real numbers. To show that multiplication is associative is a rather tedious task. To satisfy this condition the argument is used that it suffices to show that the basis units are associative for multiplication and since the real numbers are associative for multiplication this would imply that quaternions are also associative for multiplication. To show that the basis units are associative for multiplication definition 3.3. is used as follows:

$$(i \cdot j) \cdot i = k \cdot i = j = i \cdot (-k) = i \cdot (j \cdot i)$$

$$(i \cdot j) \cdot j = k \cdot j = -i = i \cdot (-1) = i \cdot (j \cdot j)$$

$$(i \cdot j) \cdot k = k \cdot k = -1 = i \cdot i = i \cdot (j \cdot k)$$

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[2] Ibid., p. 54.

$$(i \cdot i) \cdot i = -1 \cdot i = -i = i \cdot (-1) = i \cdot (i \cdot i)$$

$$(i \cdot i) \cdot j = -1 \cdot j = -j = i \cdot k = i \cdot (i \cdot j)$$

$$(i \cdot i) \cdot k = -1 \cdot k = -k = i \cdot (-j) = i \cdot (i \cdot k)$$

$$(i \cdot k) \cdot i = -j \cdot i = k = i \cdot j = i \cdot (k \cdot i)$$

$$(i \cdot k) \cdot j = -j \cdot j = 1 = i \cdot (-i) = i \cdot (k \cdot j)$$

$$(i \cdot k) \cdot k = -j \cdot k = -i = i \cdot (-1) = i \cdot (k \cdot k)$$

Since multiplication is unchanged under the substitution $i \to j$, $j \to k$, $k \to i$ it follows that the basis units are associative under multiplication.[3] The conclusion can now be made that the quaternions are multiplicatively associative. It must now be shown that the quaternions satisfy the distributive law of multiplication over addition. That is, for any quaternions Q_1, Q_2 and Q_3 it must be shown that $Q_1 \cdot (Q_2 + Q_3) = Q_1 \cdot Q_2 + Q_1 \cdot Q_3$ and $(Q_2 + Q_3) \cdot Q_1 = Q_2 \cdot Q_1 + Q_3 \cdot Q_1$. For the left hand distributive property, if the sum of Q_2 and Q_3 is found and the product of Q_1 with this sum is taken, then this product is identical to the sum of the products $Q_1 \cdot Q_2$ and $Q_1 \cdot Q_3$ due to the commutativity of the real numbers with the basis units and definition 3.2. A similar arguement holds for the right hand distributive property. The existence of a multiplicative unity element is verified since for $1 = 1 + oi + oj + ok$, $1 \cdot (a + bi + cj + dk) = a + bi + cj + dk$ for any quaternion $a + bi + cj + dk$. Thus, the quaternions form a ring with unity.

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[3]Cyrus C. MacDuffee, An Introduction to Abstract Algebra, New York, 1940, p. 251.

Definition. 3.7. If every element of a ring with unity, except the additive group identity, has an inverse, then the ring is called a division ring.[4]

To show that the quaternions form a division ring the following definitions will be needed.

Definition. 3.8. The conjugate of any quaternion $a+bi+cj+dk$ is the quaternion $a+(-b)i+(-c)j+(-d)k$.

The conjugate of any quaternion Q will be denoted as $\overline{Q}$.

Definition. 3.9. The norm of any quaternion Q is the product $Q\cdot\overline{Q}$.

The norm of any quaternion Q will be denoted as $|Q|$.

It should be noted that the norm of any quaternion is a real valued quaternion. That is, for any quaternion $Q=a+bi+cj+dk$, then $|Q|=\left[(a^2+b^2+c^2+d^2)+oi+oj+ok\right]$.

Theorem. 3.3. The quaternions form a division ring.

Proof: Theorem 3.2. satisfies the first condition for the quaternions to be a division ring. It remains to be shown that every quaternion Q, $Q\neq 0$ has an inverse. Since $Q\cdot\overline{Q}=\overline{Q}\cdot Q=|Q|\neq 0$ (if $Q\neq 0$) and $|Q|$ is real valued, $|Q|$ has an inverse $|Q|^{-1}$. Then, $Q(\overline{Q}\cdot|Q|^{-1})=(Q\cdot\overline{Q})\cdot|Q|^{-1}=|Q|\cdot|Q|^{-1}=1$. Thus, by closure of multiplication the inverse of any quaternion Q is the quaternion $(\overline{Q}\cdot|Q|^{-1})$ and the set of quaternions is a division ring.

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[4]Miller, op. cit., p. 57.

A division ring is said to be a field if the division ring is multiplicatively commutative. The quaternions are not a field as is evident in definition 3.3.

To conclude this chapter the proof of a theorem attributed to Frobenius will be constructed. This theorem contains an interesting characteristic of quaternions, namely that the quaternions constitute the only non-commutative division ring over the field of real numbers. To construct this proof the following facts must be recalled about the field of complex numbers. These facts are derived from the fundamental theorem of algebra.

Fact. 3.1. Every polynomial of degree n over the field of complex numbers has all its n roots in the field of complex numbers.

Fact. 3.2. The only irreducible polynomial over the field of real numbers are of degree one or two.

Definition. 3.10. The set C of all elements which commute with every element of group G is called the center of G.[5]

Definition. 3.11. A division ring D is said to be algebraic over a field F if:

(1.) F is contained in the center of D and

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[5]Ibid., p. 51.

(2.) for every a in D, a satisfies a nontrivial polynomial with coefficients in F.[6]

Lemma. 3.1. Let C be the field of complex numbers and suppose that the division ring D is algebraic over C. Then, $D = C$.

Proof: Suppose that a is any element of D. Since D is algebraic over C, $a^n + p_1 \cdot a^{n-1} + \ldots + p_{n-1} \cdot a + p_n = 0$ for some $p_1, p_2, \ldots p_n$ in C. By fact 3.1. the polynomial $p(x) = x^n + p_1 \cdot x^{n-1} + \ldots + p_{n-1} \cdot x + p_n$ can be factored so that $p(x) = (x-q_1) \cdot (x-q_2) \ldots (x-q_n)$ where $q_1, q_2 \ldots, q_n$ are all in C. Since C is in the center of D, every element of C commutes with a and with the hypothesis $p(a) = (a-q_1) \cdot (a-q_2) \ldots (a-q_n) = 0$. Since D is a division ring $a-q_k = 0$ and $a = q_k$ for some k. Thus, every element of D is in C and since every element of C is in D, then $D = C$.[7]

Theorem. 3.4. (Frobenius). Let D be a division ring algebraic over the field of real numbers F. Then D is isomorphic to one of:

(1.) the field of real numbers

(2.) the field of complex numbers, or

(3.) the division ring of quaternions over the real numbers.

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[6]I.N. Herstein, Topics in Algebra, New York, 1964, p. 326.

[7]Ibid., pp. 326-327.

Proof: Suppose $D \neq F$ and that a is in D but not in F. By hypothesis a satisfies some polynomial over F, hence some irreducible polynomial over F. By fact 3.2., a satisfies either a linear or quadratic equation over F. If this equation is linear, a is in F contrary to assumption. Therefore, suppose $a^2 + 2pa + q = 0$ where p and q are in F. Then, $(a+p)^2 = p^2-q$. Now $p^2-q < 0$ for otherwise $a+p = \pm d$ where d is the real number, $\sqrt{p^2-q}$ and so a would be in F; but, a is not in F. Since $p^2-q < 0$ we may write $p^2-q = -r^2$ where r is in F. Therefore, $(a+p)^2 = -r^2$ and $\left(\frac{a+p}{r}\right)^2 = -1$. Thus, if a is in D and not in F, real numbers p and r can be found such that $\left(\frac{a+p}{r}\right)^2 = -1$. If D is commutative, pick a in D and not in F and let $i = \left(\frac{a+p}{r}\right)$ where p and r are chosen in F so that $i^2 = -1$. Therefore D contains a field isomorphic to the field of complex numbers; call it F(i). Since D is commutative and algebraic over F it is algebraic over F(i) and by lemma 3.1. $D = F(i)$. Therefore, if D is commutative it is either F or F(i).

Assume now that D is non-commutative. The center of D must be F for if a is in the center and not in F, then for some p and r in F $\left(\frac{a+p}{r}\right)^2 = -1$. Therefore, the center contains a field isomorphic to the complex numbers. But by lemma 3.1. if the complex numbers (or an isomorph) are in the center of D, then $D = C$ forcing D to be commutative. Hence F is the center of D.

Now let a be in D and not in F. For some p and r in F, $i = \frac{a+p}{r}$ satisfies $i^2 = -1$. Since i is not in F, i is not in the center of F. Therefore, there is an element b in D so that $c = bi - ib \neq 0$. Now compute $i \cdot c + c \cdot i$; $ic + ci = i(bi - ib) + (bi - ib)i = ibi - i^2b + bi^2 - ibi = 0$ since $i^2 = -1$. Thus, $ic = -ci$. Also, $ic^2 = -c(i \cdot c) = -c(-c \cdot i) = c^2 i$ and c^2 commutes with i. Now c satisfies some quadratic equation over F, $c^2 + tc + m = 0$. Since c^2 and m commute with i, tc must commute with i. That is, $tci = itc = tic = -tci$. Therefore $2tci = 0$ and since $2ci \neq 0$, then $t = 0$ and $c^2 = -m$. Since c is not in F (for $ci = -ic \neq ic$) it can be said that m is positive and so $m = v^2$ where v is in F. Therefore $c^2 = -v^2$; let $j = \frac{c}{v}$.

Then j satisfies:

(1.) $j^2 = \frac{c^2}{v^2} = -1$

(2.) $ji + ij = \frac{c}{v}i + i\frac{c}{v} = \frac{ci + ic}{v} = 0$.

Let $k = ij$. The i, j and k behave like those for the quaternions, so that $T = \{p_0 + p_1 i + p_2 j + p_3 k \mid p_0, p_1, p_2$ and p_3 are in $F\}$ forms a subdivision ring of D isomorphic to the quaternions over the real numbers. It must now be shown that $T = D$.

If g in D satisfies $g^2 = -1$, let $N(g) = \{x \mid x$ is in D, $x \cdot g = g \cdot x\}$. N(g) is a subdivision ring of D; moreover g, and so all $p_0 + p_1 g$, where p_0 and p_1 are in F, are in the center of N(g). By lemma 3.1. it follows that $N(g) = \{p_0 + p_1 g \mid p_0$ and p_1 are in $F\}$ and if $x \cdot g = g \cdot x$, then $x = p_0 + p_1 g$ for some p_0 and p_1 in F.

Suppose that u is any element in D and not in F. For some p and q in F, $w = \frac{u-p}{q}$ satisfies $w^2 = -1$. Now $wi + iw$ commutes with both i and w since $i(wi + iw) = iwi + i^2w = iwi + wi^2 =$

$(iw + wi)i$, where $i^2 = -1$. Similarly $w(wi + iw) = (wi + iw)w$. From the preceding paragraph it can be stated that $wi + iw = p_0' + p_1'i = p_0 + p_1 w$. If w is not in T, then $p_1 = 0$ (since otherwise w could be solved in terms of i). Therefore $wi + iw = p_0$ for some p_0 in F. Similarly $wj + jw = q_0$ for some q_0 in F and $wk + kw = r_0$ for some r_0 in F. Let $z = w + \frac{p_0}{2}i + \frac{q_0}{2}j + \frac{r_0}{2}k$. Then $zi + iz = wi + iw + \frac{p_0}{2}\cdot(i^2 + i^2) + \frac{q_0}{2}\cdot(ji + ij) + \frac{r_0}{2}\cdot(ki + ik) = p_0 - p_0 = 0$. Similarly $zj + jz = 0$ and $zk + kz = 0$. Now, $0 = zk + kz = zij + ijz = (zi + iz)j + i(jz - zj) = i(jz - zj)$, since $zi + iz = 0$. However, $i \neq 0$ and since this is in a division ring, it follows that $jz - zj = 0$. But, $jz + zj \quad 0$. Thus, $2jz = 0$ and $2j \neq 0$, implies that $z = 0$. Now, $z = w + \frac{p_0}{2}i + \frac{q_0}{2}j + \frac{r_0}{2}k = 0$ and w is in T, contradicting w not in T. Since $w = \frac{u-p}{q}$, $u = qw + p$, then u is in T. Therefore, any element in D is in T and since T is contained in D, then $D = T$. As T is isomorphic to the quaternions over the field of real numbers, D is also isomorphic to the division ring of real quaternions. This completes the proof.[8]

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[8] Ibid., pp. 327-329

CHAPTER IV

THE COMMUTATIVE QUATERNIONS

4.1. Introduction. Quaternions are not generally commutative under the operation of multiplication. In this chapter the conditions for two quaternions to be commutative will be determined and some of the properties of the sets of quaternions which are commutative will be examined.

Attention will first be directed to finding the conditions for two quaternions to be commutative. That is, for quaternions $Q_1 = a + bi + cj + dk$ and $Q_2 = x + yi + zj + wk$ when is $Q_1 \cdot Q_2 = Q_2 \cdot Q_1$?

4.2. Conditions for commutativity. By definition 3.4. $Q_1 \cdot Q_2 = (ax - by - cz - dw) + (ay + bx + cw - dz)i + (az - bw + cx + dy)j + (aw + bz - cy + dx)k$ and $Q_2 \cdot Q_1 = (ax - by - cz - dw) + (bx + ay + dz - cw)i + (cx - dy + az + bw)j + (dx + cy - bz + aw)k$. By definition 3.1. $Q_1 \cdot Q_2 = Q_2 \cdot Q_1$ if and only if:

(1.) $ax - by - cz - dw = ax - by - cz - dw$ and

(2.) $ay + bx + cw - dz = bx + ay + dz - cw$ and

(3.) $az - bw + cx + dy = cx - dy + az + bw$ and

(4.) $aw + bz - cy + dx = dx + cy - bz + aw$.

It suffices to show that $Q_1 \cdot Q_2 = Q_2 \cdot Q_1$ when statements (2.), (3.) and (4.) are satisfied simultaneously. That is, in simplified form

(2.) $cw = dz$,

(3.) $dy = bw$, and

(4.) $bz = cy$.

must be satisfied simultaneously. Obviously, if b, c and d are all zero and/or y, z and w are all zero the above equations are satisfied. This implies that $Q_1 \cdot Q_2 = Q_2 \cdot Q_1$ when one or both of Q_1 and Q_2 are real valued quaternions. Also, all three equations are satisfied when the quaternions Q_1 and Q_2 are complex valued. That is, if Q_1 and Q_2 are elements of a subset of quaternions isomorphic to the complex numbers. (If Q_1 and Q_2 are both of the form $a + bi + oj + ok$, or both of the form $a + oi + bj + ok$, or both of the form $a + oi + oj + bk$.) Now suppose that Q_1 and Q_2 are not both real valued or of the complex forms. Then equations (2.), (3.) and (4.) are satisfied if $\frac{b}{y} = \frac{c}{z} = \frac{d}{w}$. The following theorem can now be stated.

<u>Theorem</u>. 4.1. Quaternions $Q_1 = a + bi + cj + dk$ and $Q_2 = x + yi + zj + wk$ are commutative if:

(1.) Q_1 and/or Q_2 are real valued quaternions

(2.) Q_1 and Q_2 are complex valued quaternions

(3.) $\frac{b}{y} = \frac{c}{z} = \frac{d}{w}$.

Case 1. of theorem 4.1 implies that any real valued quaternion will commute multiplicatively with any other quaternion. Case 2. implies that a complex valued quaternion will commute only with another complex valued quaternion of the same form. Case 3. implies that any two quaternions which have their corresponding coefficients of their respective basis units in constant ratio will commute.

As the sets of quaternions which satisfy case 1. or case 2. are the sets of quaternions isomorphic to the real numbers and the complex numbers respectively, the remainder of this chapter will be devoted to the sets of quaternions which satisfy case 3. of theorem 4.1.

In case 3. the coefficients of the basis unit 1 of Q_1 and Q_2 play no part in determining the commutativity of Q_1 and Q_2. In other words, given any quaternion $Q = a + bi + cj + dk$ all other quaternions which commute with Q would be of the form $e + n.bi + n.cj + n.dk$ where e and n vary over the set of real numbers and the real numbers b, c and d are fixed by the choice of Q. Thus, for each choice of Q there exists an infinite set which contains Q and all quaternions which commute with Q.

4.3. Quaternions which commute. Consider now the set of all quaternions which commute with some quaternion $Q = a + bi + cj + dk$. This set of course includes Q, all real valued quaternions and all quaternions of the form $e + nbi + ncj + ndk$. Denote this subset of the quaternions as CQ. The operations of addition and multiplication on CQ are the operations defined on all quaternions in chapter III.

Theorem. 4.2. The set of quaternions CQ form a field.

Proof: First, CQ is an additive abelian group. Let $Q_1 = (a + nbi + ncj + ndk)$ and $Q_2 = (e + mbi + mcj + mdk)$. Then, $Q_1 + Q_2 = \left[(a+e) + (n+m)bi + (n+m)cj + (n+m)dk\right]$ which is an element of CQ. Thus, addition is closed in CQ. Associativity and commutativity for addition follows as in theorem 3.1. The additive identity $o + oi + oj + ok$ is an element of CQ. For any element $(a + nbi + ncj + ndk)$, its additive inverse $(-a - nbi - ncj - ndk)$ is also in CQ. Thus, CQ is an additive abelian group.

Now, multiplication is closed in CQ. Let $Q_1 = (a + nbi + ncj + ndk)$ and $Q_2 = (e + mbi + mcj + mdk)$ where Q_1 and Q_2 are any elements of CQ. Then, $Q_1 \cdot Q_2 = \left[(ae - mnb^2 - mnc^2 - mnd^2) + (ne + ma)bi + (ne + ma)cj + (ne + ma)dk\right]$ and $Q_1 \cdot Q_2$ is in CQ. Multiplication is associative and also is distributive with respect to addition on both the left and right as a result of theorem 3.2. and closure in CQ of multiplication and addition. The multiplicative identity $1 + oi + oj + ok$ is in CQ. From theorem 3.3., the multiplicative inverse for every non-zero quaternion in CQ is again in CQ. Since this set is the set of quaternions which are commutative under multiplication, CQ is a field.

According to the theorem of Frobenius, if CQ is algebraic over the field of real numbers, then CQ is isomorphic to either the field of real numbers, the field of complex numbers or the division ring of quaternions over the real numbers. To show that this is the case, the following theorem must be proved.

<u>Theorem</u>. 4.3. The field CQ is algebraic over the field of real numbers.

Proof: To prove this theorem two conditions must be satisfied:

(1.) the real numbers must be contained in the center of CQ;

(2.) every element of CQ must satisfy a non-trivial polynomial with coefficients in the real numbers.

Condition (1.) is easily satisfied as CQ contains all real valued quaternions which are isomorphic to the field of real numbers and of course commute with every other element in CQ. To show that condition (2.) is satisfied, let Q be any element of CQ. Consider the polynomial $p(x) = (x - Q) \cdot (x - \overline{Q})$ so that $p(x) = x^2 - (\overline{Q} + Q)x + Q \cdot \overline{Q}$. The coefficients of this polynomial are real numbers and $p(Q) = 0$. Thus, CQ is algebraic over the field of real numbers and must be isomorphic to either the field of real numbers, the field of complex numbers or the division ring of quaternions over the real numbers.

Theorem. 4.4. CQ is isomorphic to the field of complex numbers.

Proof: The mapping of CQ onto the field of complex numbers is given by:

$$a + bi + cj + dk \longleftrightarrow a + (1 \cdot \sqrt{b^2 + c^2 + d^2})\, i .$$

Then,

$$e + n \cdot bi + n \cdot cj + n \cdot dk \longleftrightarrow e + (n \cdot \sqrt{b^2 + c^2 + d^2})\, i$$

and

$$f + r \cdot bi + r \cdot cj + r \cdot dk \longleftrightarrow f + (r \cdot \sqrt{b^2 + c^2 + d^2})\, i$$

so that

$$(e+f) + (n+r)bi + (n+r)cj + (n+r)dk \longleftrightarrow (e+f) + \left[(n+r)\cdot\sqrt{b^2+c^2+d^2}\right]i = \left[e + (n\cdot\sqrt{b^2+c^2+d^2})\,i\right] + \left[f + (r\cdot\sqrt{b^2+c^2+d^2})\,i\right]$$

and

$$\left[ef - nr(b^2+c^2+d^2) + (nf+re)bi + (nf+re)cj + (nf+re)dk\right] \longleftrightarrow \left[ef - nr(b^2+c^2+d^2)\right] + \left[(nf+re)\sqrt{b^2+c^2+d^2}\right]i = \left[e + (n\sqrt{b^2+c^2+d^2})\,i\right] \cdot \left[f + (r\sqrt{b^2+c^2+d^2})\,i\right].$$

Thus, operations are preserved.

Now, if

$$e + nbi + ncj + ndk \longleftrightarrow e + (n\cdot\sqrt{b^2+c^2+d^2})\,i$$

and

$$f + rbi + rcj + rdk \longleftrightarrow e + (n\cdot\sqrt{b^2+c^2+d^2})\,i$$

so that

$$\left[(e+f) + (n+r)bi + (n+r)cj + (n+r)dk\right] \longleftrightarrow (e+f) + \left[(n+r)\sqrt{b^2+c^2+d^2}\right]i = \left[e + (n\cdot\sqrt{b^2+c^2+d^2})\cdot i\right] + \left[e + (n\cdot\sqrt{b^2+c^2+d^2})\cdot i\right] = \left\{(e+e) + \left[(n+n)\sqrt{b^2+c^2+d^2}\right]i\right\},$$

then the complex numbers $(e+f) + \left[(n+r)\cdot\sqrt{b^2+c^2+d^2}\right]i$ and $(e+e) + \left[(n+n)\cdot\sqrt{b^2+c^2+d^2}\right]i$ are equal if and only if $f = e$ and $r = n$, establishing the one-to-one correspondence. Thus, the isomorphism is established.

CHAPTER V

REPRESENTATIONS OF QUATERNIONS

5.1. Introduction. Various ways of representing the set of quaternions over the field or real numbers will be given in this chapter. To accomplish this, an isomorphism between each proposed representation and the quaternions which were defined in chapter III will be shown.

5.2. Ordered quadruples. The first representation will be a representation of the quaternions by the set A consisting of all ordered quadruples (a_1, a_2, a_3, a_4) where a_1, a_2, a_3 and a_4 are real numbers. The usual manner for defining equality of n-tuples will be assumed to hold for the elements of set A.

Definition. 5.1. i) addition of ordered quadruples.

$$(a_1,a_2,a_3,a_4) + (b_1,b_2,b_3,b_4) = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ a_4 + b_4)$$

ii) multiplication of ordered quadruples.

$$(a_1,a_2,a_3,a_4) \cdot (b_1,b_2,b_3,b_4) = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4,\ a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3,\ a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2,\ a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1).$$

With these definitions on the set A of ordered quadruples there exists a natural isomorphism with the set of quaternions over the real number system.

Theorem. 5.1. The quaternions over the field of real numbers are isomorphic to the set A of ordered quadruples.

Proof: The mapping of the quaternions onto set A is given by:

$$a + bi + cj + dk \longleftrightarrow (a, b, c, d).$$

If

$$e + fi + gj + hk \longleftrightarrow (e, f, g, h)$$

then

$$\left[(a+e) + (b+f)i + (c+g)j + (d+h)k\right] \longleftrightarrow (a+e, b+f, c+g, d+h) = (a,b,c,d) + (e,f,g,h)$$

and

$$\left[(ae-bf-cg-dh) + (af+be+ch-dg)i + (ag-bh+ce+df)j + (ah+bg-cf+ed)k\right] \longleftrightarrow (ae-bf-cg-dh,\ af+be+ch-dg,\ ag-bh+ce+df,\ ah+bg-cf+de) = (a,b,c,d) \cdot (e,f,g,h)$$

establishing the one-to-one correspondence and preservation of operations.

5.3. Matrices. The following representations of quaternions involve matrix algebra requiring the following definitions.

Definition. 5.2. An mxn matrix over a field F is a rectangular array of elements of F consisting of m rows and n columns of the form:

$$\begin{pmatrix} a_{11}a_{12}\cdots a_{1n} \\ a_{21}a_{22}\cdots a_{2n} \\ \vdots \quad\quad \vdots \\ a_{m1} \quad\quad a_{mn} \end{pmatrix} = {}_m(a_{ij})_n.$$ [1]

The usual manner for defining equality of matrices will be assumed.

Definition. 5.3. Matrix addition.

The sum of the matrices over F, $A = {}_m(a_{ij})_n$ and $B = {}_m(b_{ij})_n$ is the matrix $C = A+B = {}_m(a_{ij}+b_{ij})_n$ in F.[2]

Definition. 5.4. Multiplication of a matrix by an element of F.

The product of ${}_m(a_{ij})_n$ by an element f of F is the matrix ${}_m(f\cdot a_{ij})_n$.[3]

Definition. 5.5. Matrix multiplication.

The product ${}_m(a_{ij})_n \cdot {}_n(b_{ij})_p$ is the matrix ${}_m(c_{ij})_p$, where $c_{ij} = \sum_{k=1}^{n} \left(a_{ik} \cdot b_{kj} \right)$, $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,p)$.[4]

Definition. 5.6. A square matrix is a matrix with n rows and n columns.

With the use of the preceding definitions the following theorem is stated and proved which yields the second

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[1]Charles P. Benner and others, Topics in Modern Algebra, New York, 1962, p. 31.

[2]Ibid., p. 31.

[3]Ibid., p. 32.

[4]Ibid., p. 34.

representation of the quaternions over the field of real numbers.

Theorem. 5.2. The quaternion over the field of real numbers are isomorphic to a system of 4x4 matrices over the field of real numbers.

Proof: The mapping of the quaternions onto the set of 4x4 matrices is given by:

$$a+bi+cj+dk \longleftrightarrow \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix}.$$

If

$$e+fi+gj+hk \longleftrightarrow \begin{pmatrix} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{pmatrix}$$

then

$$\left[(a+e)+(b+f)i + (c+g)j+(d+h)k\right] \longleftrightarrow \begin{pmatrix} a+e & b+f & c+g & d+h \\ -(b+f) & a+e & -(d+h) & c+g \\ -(c+g) & d+h & a+e & -(b+f) \\ -(d+h) & -(c+g) & b+f & a+e \end{pmatrix} =$$

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} + \begin{pmatrix} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{pmatrix}$$

and

$$\left[(ae-bf-cg-dh)+(af+be+ch-dg)i + (ag-bh+ce+df)j+(ah+bg-cf+ed)k\right] \longleftrightarrow$$

$$\begin{pmatrix} ae-bf-cg-dh & af+be+ch-dg \\ -(af+be+ch-dg) & ae-bf-cg-dh \\ -(ag-bh+ce+df) & ah+bg-cf+ed \\ -(ah+bg-cf+ed) & -(ag-bh+ce+df) \end{pmatrix}$$

$$\begin{pmatrix} ag-bh+ce+df & ah+bg-cf+ed \\ -(ah+bg-cf+ed & ag-bh+ce+df \\ ae-bf-cg-dh & -(af+be+ch-dg) \\ af+be+ch-dg & ae-bf-cg-dh \end{pmatrix} =$$

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \cdot \begin{pmatrix} e & f & g & h \\ -f & e & -h & g \\ -g & h & e & -f \\ -h & -g & f & e \end{pmatrix}$$

establishing the one-to-one correspondence and preservation of operations.

The last representation of the quaternions over the field of real numbers to be taken up in this chapter is given by the following theorem.

Theorem. 5.3. The quaternions over the field of real numbers are isomorphic to a system of 2x2 matrices over the field of complex numbers.

Proof: The mapping of the quaternions onto the set of 2x2 matrices is given by:

$$a+bi+cj+dk \longleftrightarrow \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}.$$

If

$$e+fi+gj+hk \longleftrightarrow \begin{pmatrix} e+fi & g+hi \\ -g+hi & e-fi \end{pmatrix}$$

then

$$\left[(a+e)+(b+f)i + (c+g)j + (h+d)k\right] \longleftrightarrow \begin{pmatrix} (a+e)+(b+f)i & (c+g)+(d+h)i \\ -(c+g)+(d+h)i & (a+e)-(b+f)i \end{pmatrix} =$$

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} + \begin{pmatrix} e+fi & g+hi \\ -g+hi & e-fi \end{pmatrix}$$

and

$$\left[(ae-bf-cg-dh)+(af+be+ch-dg)i + (ag-bh+ce+df)j + (ah+bg-cf+ed)k\right] \longleftrightarrow$$

$$\begin{pmatrix} (ae-bf-cg-dh)+(af+be+ch-dg)i & (ag-bh+ce+df)+(ah+bg-cf+ed)i \\ -(ag-bh+ce+df)+(ah+bg-cf+ed)i & (ae-bf-cg-dh)-(af+be+ch-dg)i \end{pmatrix} =$$

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \cdot \begin{pmatrix} e+fi & g+hi \\ -g+hi & e-fi \end{pmatrix}$$

establishing the one-to-one correspondence and preservation of operations.

CHAPTER VI

CONCLUSION

6.1. Summary. The quaternions were devised by Hamilton as a mathematical tool for the application to the solution of physics problems in three dimensional space. However, even though Hamilton's expectations for the use of quaternions were never fulfilled completely, from the quaternions emerged the more applicable subject of vector analysis.

The quaternions had an effect on the study of structures of number systems which was similar to the effects that the discovery of the non-Euclidean geometries had on the study of geometry. With the development of the quaternions it was shown that a consistent algebra existed which contradicted what was thought to be an immutable postulate of algebra, the commutative principle for multiplication.

In conclusion, the effects of Frobenius' theorem in the development of this study should not be overlooked. It is hoped that the reader has gained a fuller appreciation for the power of this theorem with the isomorphism that was presented between the commutative quaternions and the field of complex numbers.

6.2. A suggestion for further study. In chapter two it was stated that the algebra of quaternions was the number system of the basis units 1, i, j and k over any field. The question as to what properties are contained by the quaternions taken over a finite field or infinite fields other than the reals might merit further investigation.

BIBLIOGRAPHY

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Bell, E.T. Men of Mathematics. New York: Simon and Schuster, Inc., 1937.

Benner, Charles P., and others. Topics in Modern Algebra. New York: Harper and Row Publishers, 1962.

Birkhoff, Garrett and MacLane, Saunders. A Survey of Modern Algebra. New York: The Macmillan Co., 1941.

DuVal, Patrick. Homographies, Quaternions and Rotations. Oxford: The Clarendon Press, 1964.

Hardy, A.S. Elements of Quaternions. Boston: Ginn, Heath and Co., 1881.

Hathaway, Arthur S. A Primer of Quaternions. New York: Macmillan and Co., 1896.

Herstein, I.N. Topics in Algebra. New York: Blaisdell Publishing Co., 1964.

Kelland, P. and Tait, P.G. Introduction to Quaternions. London: Macmillan and Co., 1882.

Lanczos, C. "William Rowan Hamilton - An Appreciation," American Scientist, 55:129-143, June, 1967.

MacDuffee, Cyrus C. An Introduction to Abstract Algebra. New York: John Wiley and Sons Inc., 1940.

Miller, Kenneth S. Elements of Modern Abstract Algebra. New York: Harper and Row Publishers, 1958.

Tait, P.G. An Elementary Treatise on Quaternions. Oxford: The Clarendon Press, 1867.