MAPPING THE HYPERBOLIC PLANE

INTO A CIRCLE

A Thesis

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CHAPTER I

INTRODUCTION

1.1. Introduction. Much has been written on the subject of Non-Euclidean Geometry, both from an elementary and an advanced viewpoint. However; even in the texts written for one just beginning to study this topic, there is considerable difficulty in visualizing the behavior of lines and angles in the hyperbolic plane.

1.2. Statement of the problem. It was the purpose of this study (1) to construct a mapping of the hyperbolic plane into a circle within the hyperbolic plane; and (2) to show some of the applications of this mapping.

1.3. Importance of the study. The circle obtained under this mapping differs from the circles associated with the study of the hyperbolic plane, by means of the models of Klein and Poincaré. Since this circle is within the hyperbolic plane, it is not a "model" of hyperbolic geometry. Rather, it is a subset of the hyperbolic plane, within which, many of the properties of this plane become easier to visualize. This last statement is extremely important, since it is possible to develop a good deal of hyperbolic geometry by considering what happens in this circle. ([4] , p. 73).

1.4. Undefined terms. The following three terms have been taken as undefined in this thesis:

1) points, denoted by capital letters A,B,C,...,

2) lines, denoted by small letters a,b,c,...,

3) a hyperbolic plane, denoted by the Greek letter π.

In this thesis the discussion has been limited to the study of points and lines within the hyperbolic plane. However, it is possible to study the hyperbolic space in a similar manner by mapping the space into a sphere. ([4], p.111).

1.5. <u>Axioms</u>. This study has used the set of axioms devised by David Hilbert for the Euclidean Plane. ([8], p.9). There are two axioms that need to be altered; (1) the Postulate of Parallels, and (2) the Postulate of Continuity.

The Postulate of Parallels should be changed to the following: given a line a and a point A not lying on a, then there exists, in the plane determined by a and A, more than one line which contains A but not any points of a.

The Postulate of Continuity should be changed to the following; known as the <u>Postulate of Dedekind</u>. If all points of a straight line fall into two classes, such that every point of the first class lies to the left of every point of the second class, then there exists one and only one point which produces this division of all points into two classes, this severing of the straight line into two portions.

1.6. <u>Distance</u>. This study has assumed the existence of a hyperbolic metric; that is, a one-to-one correspondence between hyperbolic line segments and the real numbers. ([7], p.278).

1.7. <u>Definition of terms</u>. Consider two straight lines in plane

π, and sequences of points on them $A_1, B_1, C_1, \ldots$ and $A_2, B_2, C_2, \ldots$ These sequences of points are called Congruent Sequence of Points if and only if $A_1B_1 = A_2B_2$, $A_1C_1 = A_2C_2$, $B_1C_1 = B_2C_2$, etc.

A mapping of plane π onto plane π is called a Symmetry if and only if there exists a line $b \in \pi$, such that if $A \in \pi$, and $A' \in \pi$ is the image of A under this mapping, then b is the perpendicular bisector of AA'. Then it is said that the symmetry maps the point A into the point A'. The notation for this is; $S: A \rightarrow A'$. Reflection may be used as a synonym for symmetry.

A mapping of plane π onto plane π is called a Rotation if and only if there exists a point O and an acute angle α, both in plane π, such that if a point $A \in \pi$ has for its image, under this mapping, a point $A' \in \pi$, then OA = OA' and angle AOA' = α. Then it is said that a rotation of the plane, about O, through angle α, has mapped A into A'. In symbols; $R: A \rightarrow A'$. In this paper, the angle α has been measured in a clockwise direction.

Properties of both the symmetry and the rotation mappings have been considered at the beginning of Chapter III.

Given any two lines a and b both in plane π, and point A on a and B on b, then B is the Perpendicular Projection of A onto b if and only if AB is perpendicular to b. If a and b intersect, the point of intersection is its own perpendicular projection.

1.8. Organization of the remainder of the thesis. The remainder of the thesis has been divided into five main parts.

1) Chapter II gives a short history of hyperbolic geometry.

2) Chapter III develops the transformation that is needed for the mapping.

3) The mapping is given in Chapter IV.

4) Chapter V gives some of the applications of this mapping to the theory of hyperbolic plane geometry.

5) A summary of the thesis has been presented in Chapter VI.

CHAPTER II

HISTORY OF HYPERBOLIC GEOMETRY

2.1. Introduction. Non-Euclidean geometry is a young science; the date of birth may be taken as being 1830, when Lobatchevsky's first publication appeared. ([4] , p.7).

A chapter dealing with the history of this new field of mathematics usually traces the attempts to prove Euclid's parallel postulate. However; seldom is anything said about the events after the discovery of non-Euclidean geometry. This chapter has approached the idea from a slightly different viewpoint. Very little has been said about the attempts to prove Euclid's parallel postulate. Rather, a look at some of the events taking place after the discovery of non-Euclidean geometry has been given in this chapter.

2.2. Attempts to prove Euclid's parallel postulate. Until about 1800 the mathematical world held that a geometrical theorem was provable if its converse was provable. Since the converse of Euclid's parallel postulate was itself a theorem, it was understandable why mathematicians struggled for 2000 years to prove the axiom of parallels. ([6] , p.22).

In 1763, G.S. Kluegal, a student of Kestner, wrote a dissertation in which he brought together and criticized all significant attempts to prove the parallel postulate contributed by mathematicians over the 2000 years between the publication of the "Elements" and his own time. He found, correctly, that all 28 "proofs" were false. ([6] , p.23). Most of these proofs were based on assuming some other property, and then proving the parallel postulate from it. However; in every case the alternate property assumed was eventually shown to be equivalent to the parallel postulate. For example, assuming the sum of the angles of a triangle to be equal to two right angles is equivalent to assuming the parallel postulate.

2.3. Discovery of Non-Euclidean geometry. Evidence indicated Carl Friedrich Gauss (1777-1855) was the first to visualize a consistent geometry in which the parallel postulate was replaced by a contradictory statement. However; Gauss did not publish his results, and thus two developments have resulted. First, even though Gauss was first, he is generally not given credit for discovering hyperbolic geometry. Second, the work Gauss did on the subject has been left to us in his notes, thus giving us, quite possibly, an incomplete account of his findings.

From Gauss's notes it appears that he did not come to very many final results in the field of non-Euclidean geometry. In the more important areas of research, Gauss used the methods of differential geometry instead of synthetic methods. This may be one of the reasons why Gauss did not publish his findings. ([4], p. 54).

The first to publish a complete work on non-Euclidean geometry was Nikolia Ivanovich Lobatchevsky (1793-1856), a professor at the University of Kazan. Lobatchevsky published his results in the 1829-30 numbers of the Kazan journal, but his findings did not reach other countries.

In 1832, Johann Bolyai (1802-1860) published his results in an appendix to one of his father's books. Thus Lobatchevsky was the first to publish on the subject. Because he was the first to publish, this geometry is sometimes referred to as Lobatchevskian geometry.

Since the works of both Lobatchevsky and Bolyai were ignored in their own time, it is interesting to note how the two men reacted to this indifference and lack of recognition. Bolyai completely withdrew from scientific activity. Lobatchevsky, however, attempted to justify his findings. He wrote paper after paper showing the value and truth of his geometry, even indicating its applications in integral calculus.

It was not until after the death of Gauss, Lobatchevsky, and Bolyai that their work became known and accepted. The publication of Gauss's notes, containing his works and a praise of both Lobatchevsky and Bolyai, undoubtedly helped bring on this acceptance.

In 1854, Friedrich Riemann (1826-1866) introduced the other classical non-Euclidean geometry in which there are no parallels.

Despite the work these men did in developing non-Euclidean geometry, the impossibility of proving Euclid's parallel postulate was not shown until 1868. It was at this time that the Italian mathematician Eugenio Beltrami (1835-1900), exhibited a particular model within Euclidean geometry in which the postulates of hyperbolic geometry were satisfied, thereby proving that these were at least relatively consistent. ([9], p.209).

Using Beltrami's work as a starting point, Felix Klein (1849-1925) was able to give the basic idea of the precise proof of the consistency of Lobatchevskian geometry, when, in 1871, he constructed an arithmetic model of Lobatchevskian geometry. Klein also gave the name of hyperbolic to Lobatchevskian geometry, and elliptic to Riemann's geometry.

In 1903, David Hilbert (1862-1943) showed the consistency of Lobatchevskian geometry in a manner similar to that of Klein.

In connection with the study of hyperbolic geometry, there have been two famous models constructed showing the relative consistency of hyperbolic geometry. One, developed by Klein, has already been mentioned. The other is due to Henri Poincaré (1854-1912).

Finally it has been shown that within the hyperbolic space, there is a surface called a horosphere, and that this surface is governed by Euclidean geometry. In other words, if Lobatchevskian geometry is consistent, so is Euclidean. Combining this with the results of the Klein and Poincaré models, it can be said that if either geometry is consistent, so is the other. ([1], p.253).

CHAPTER III

THE NECESSARY TRANSFORMATION

3.1. Introduction. The purpose of this chapter is to define the transformation used in Chapter IV to map plane π into a circle within plane π. This transformation has been introduced in the following manner. First, the properties of the symmetry mapping are listed. Second, the properties of the rotation mapping have been listed and verified. Third, a slightly modified form of the final transformation is given, followed by a list of its properties. Fourth, each of the items on this list is verified. And fifth, the main transformation is given, and a list of properties given for it.

3.2. Properties of symmetry. The symmetry mapping does not change the size and shape of a figure; collinear points are mapped into collinear points, and lengths and angles are preserved. ([8], p.209).

3.3. Properties of rotation. Just as in the case of symmetry, a rotation maps collinear points into collinear points, and also preserves lengths and angles. This is now given as a theorem.

3.4. Theorem 3.1. A rotation of plane π, about a point O, through an angle α, maps collinear points into collinear points, and preserves lengths and angles.

Proof: Let O be any point in plane π. Consider a line $a \in \pi$, such that O is not on a (if O is on a, the proof is obvious). Next, consider three distinct points, X, Y, and Z on a. Perform the rotation about O, through angle α, to obtain the image points X', Y', and Z', Figure 1.

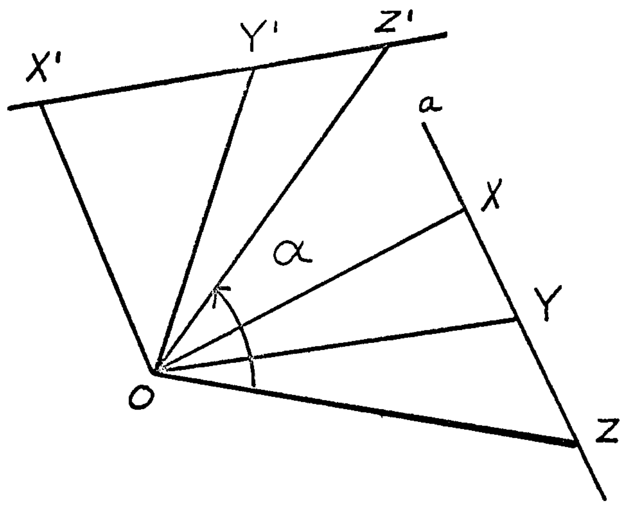


Figure 1

Properties of the Rotation Mapping

Angle YOX' is equal to angle α plus angle Y'OX'. But angle YOX' is also equal to angle α plus angle YOX. Therefore, angle YOX is equal to angle Y'OX', and thus, the size of angles is preserved under a rotation.

To prove that the rotation maps collinear points into collinear points, it is sufficient to show that angle X'Y'Z' equals a straight angle. Since, from the definition of rotation, OX = OX', OY = OY', and $\measuredangle$ YOX = $\measuredangle$ Y'OX', it follows, by SAS, that triangle YOX is congruent to triangle Y'OX'. Also, since by SAS, triangle YOZ is congruent to triangle Y'OZ', the following equalities would hold. First, angle OYX equals angle OY'X'. Second, angle OYZ equals angle OY'Z'. Since the sum of the angles OYX and OYZ is equal to a straight angle, then the sum of the angles OY'X' and OY'Z' is also equal to a straight angle. Thus collinearity has been preserved.

Finally, from the congruence of triangles OXY and OX'Y', it is true that XY is equal to X'Y'. Therefore the rotation has preserved the length of a line segment. Q.E.D.

It is worthwhile to note that the proof for symmetry, while not given here, follows almost the same pattern (i.e., it uses the SAS congruence axiom).

3.5. Transformation h. Consider a fixed acute angle α , a fixed point $O \in \pi$, and an arbitrary point $A \in \pi$. Draw OA, and construct an angle equal to α using OA as its initial side; and call its terminal side b. A point A_o on b is called the image of A under transformation h if and only if A_o is the perpendicular projection of point A onto b, Fig. 2. When using this transformation, the notation $h: A \rightarrow A_o$ has been used.

3.6. Properties of transformation h. The following properties are possessed by transformation h.

1) it leaves the point O unchanged,
2) it maps collinear points into collinear points,
3) it maps angles with vertex O into equal angles with vertex O,
4) it maps circles with center O into circles with center O,
5) it maps right angles with one side passing through O into similar right angles.

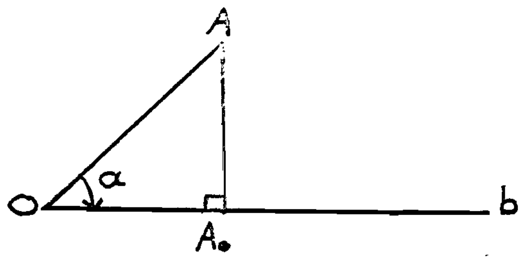


Figure 2

Transformation h

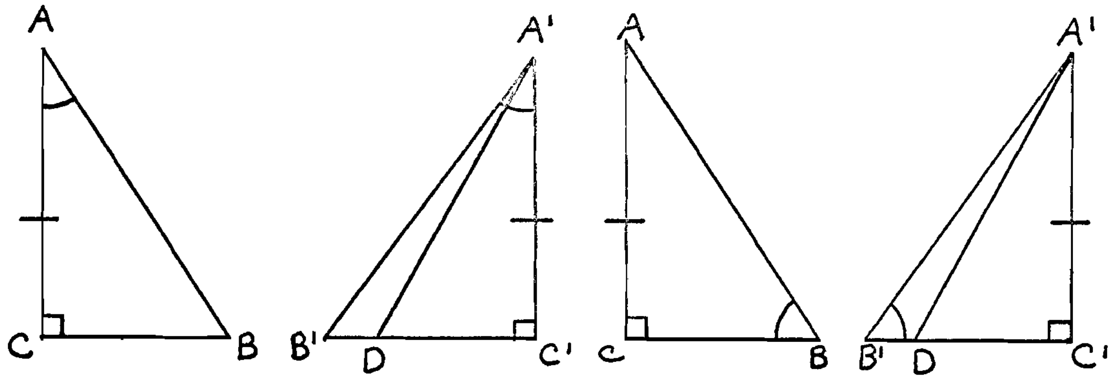
3.7. Proof of property 1. Property 1 states that transformation h leaves the point O unchanged.

Proof: The proof is obvious, since it follows directly from the definition of transformation h.

The proof of property 2 is not so obvious, and requires the use of the following two lemmas.

3.8. The LA lemma. In plane π, two right triangles are congruent if a leg and an acute angle of one are equal to the corresponding leg and acute angle of the other.

Proof: Let triangles ABC and A'B'C' be any two right triangles in plane π, such that $\measuredangle C = \measuredangle C' = 90°$, and $AC = A'C'$, as shown in Figures 3a and 3b respectively. Two cases must be considered; (1) when $\measuredangle A = \measuredangle A'$, and (2) when $\measuredangle B = \measuredangle B'$. In both cases it is sufficient to show $BC = B'C'$, since the SAS congruence axiom could then be used.



a) b)

Figure 3

LA Congruence Lemma

Suppose that $BC \neq B'C'$, and that $B'C'$ is the longer of the two. On $B'C'$, mark point D such that $BC = DC'$. Next draw DA'. By SAS the triangles ABC and A'DC' are congruent.

In case 1 this implies that $\sphericalangle C'A'D = \sphericalangle BAC = \sphericalangle C'A'B'$, which is a contradiction of the fourth axiom on the existence of congruent angles. Therefore $BC = B'C'$.

In case 2 this implies $\sphericalangle A'DC' = \sphericalangle ABC = \sphericalangle A'B'C'$ which contradicts the fact that the exterior angle of a triangle is larger than an interior and opposite angle. Therefore $BC = B'C'$. Q.E.D.

3.9. Hjelmslev's Lemma. The centers of the segments joining the corresponding points of two congruent sequences of points lie on a straight line.

Proof: Consider two straight lines a_1 and a_2, both in plane π, and the congruent sequences of points $A_1,B_1,C_1,\ldots,$ and $A_2,B_2,C_2,\ldots$ on

a_1 and a_2 respectively. Connect A_1 and A_2, and designate the midpoint of this segment as A_0, Fig. 4. Using A_0 as the center, rotate plane π through an angle of 180°. As a result, segment A_1A_0 is mapped onto A_0A_2, and line a_1 is mapped onto a line a_3 passing through A_2. B_1, C_1, ... are mapped into points of a_3 designated by $B_3, C_3, \ldots$, such that $A_1B_1 = A_2B_3$, $B_1C_1 = B_3C_3$, etc.

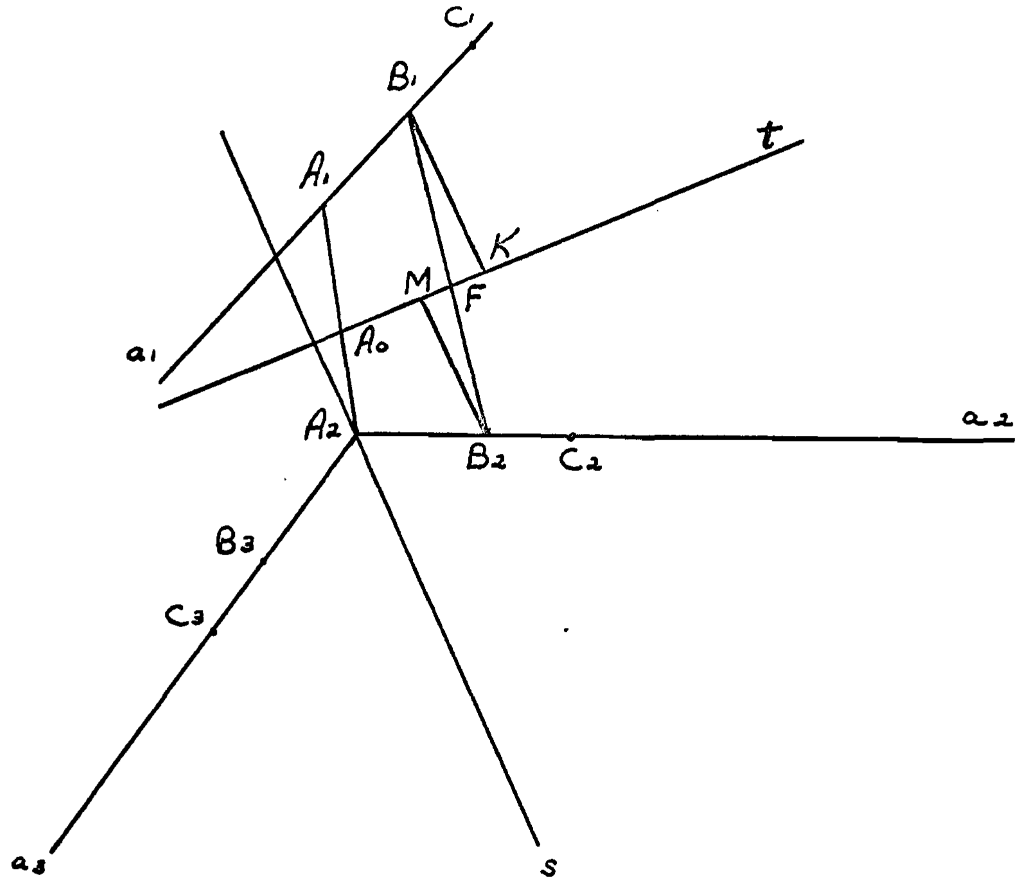


Figure 4

Hjelmslev's Lemma

Next, map $A_2, B_3, C_3, \ldots$ by symmetry with respect to s, the bisector of angle $B_2A_2B_3$. From the definition of symmetry, A_2 is mapped into itself, B_3 is mapped into B_2, C_3 into C_2, etc.

The effect of both the rotation and the symmetry has been the mapping of A_1 into A_2, B_1 into B_2, C_1 into C_2, etc.

Next, construct a perpendicular from A_0 to s, possibly extended, and call the line containing this perpendicular t. When the plane is rotated about an angle of 180°, line t is mapped onto itself, since it passes through the center of the rotation. Draw the perpendicular B_1K from B_1 to line t. By the above mentioned rotation and symmetry, B_1K is mapped onto a segment B_2M, where M is a point on t, and B_2M is perpendicular to t.

Next, draw the segment B_1B_2, and designate by F, the point of intersection of this segment and t. The two right triangles B_1KF and B_2MF are congruent by the LA lemma, and therefore, F is the midpoint of B_1B_2. In other words the midpoint of this segment lies on t. The same argument could be applied to C_1C_2, that is, its midpoint lies on t. Q.E.D.

3.10. Proof of property 2. Property two states that transformation h maps collinear points into collinear points.

Proof: For this property, two cases must be considered; (1) when the points are on a line passing through O, and (2) when the points are on a line that does not pass through O.

Case 1: This follows directly from the definition, and the fourth axiom on the existence of congruent angles.

Case 2: Consider the points A, B, and C which lie on some line $\in \pi$, such that line a does not pass through O. This has been illustrated in Figure 5.

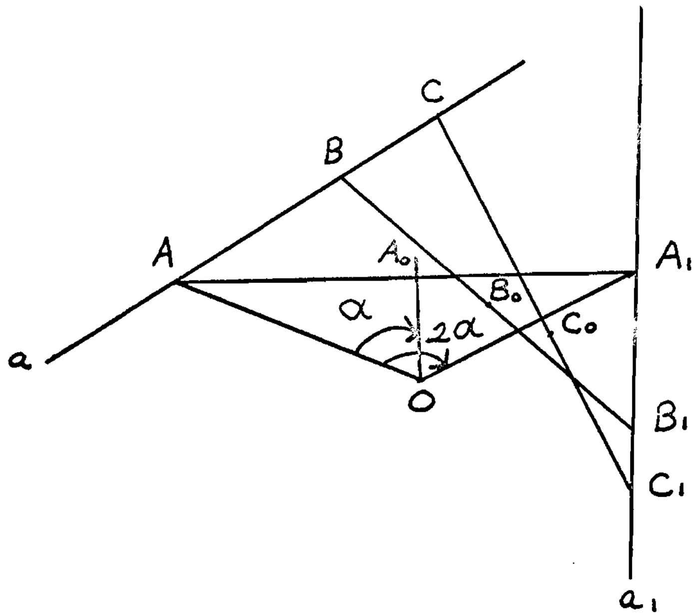


Figure 5

Illustration of Property 2

Rotate the plane about point O, through an angle of 2 α, thus mapping the points A, B, and C into A_1, B_1, and C_1. Since the sequences A_1, B_1, C_1 and A, B, C are congruent, the midpoints of the segments AA_1, BB_1, and CC_1 are collinear, by Hjelmslev's lemma. But the center A_o of the segment AA_1 is the Image of A under transformation h; since triangle AOA_1 is isosceles, and angle $AOA_o = \alpha$, OA_o is perpendicular to AA_1. The same applies to the points B_o, C_o, etc. Q.E.D. Therefore, transformation h maps collinear points into collinear points.

3.11. <u>Proof of property 3</u>. Property 3 states that transformation h maps angles with vertex O into equal angles with vertex O.

<u>Proof</u>: Consider an arbitrary angle AOB in plane π. Perform transformation h on the segments OA and OB. From properties 1 and 2, it is known that these segments are mapped into the segments OA' and OB', thus forming angle A'OB' as shown in Figure 6. To prove property 3, it is sufficient to show that angle AOB is equal to angle A'OB'.

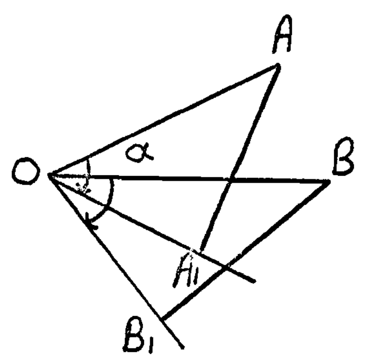


Figure 6

Illustration of Property 3

The equality of these two angles can be shown by writing angle AOB' in two different forms. First, angle AOB' is equal to angle α plus angle A'OB'. Second, angle AOB' is equal to angle AOB plus angle α. Therefore, angle AOB is equal to angle A'OB', and the proof is complete.

Before property 4 has been verified, a lemma has been stated and proven.

3.12. The HA lemma. In plane π, two right triangles are congruent if the hypotenuse and an acute angle of one are equal to the hypotenuse and corresponding acute angle of the other.

Proof: Let triangles ABC and A'B'C' be any two right triangles in plane π, such that $\measuredangle C = \measuredangle C'$ = a right angle, AB = A'B', and $\measuredangle A = \measuredangle A'$, Figure 7.

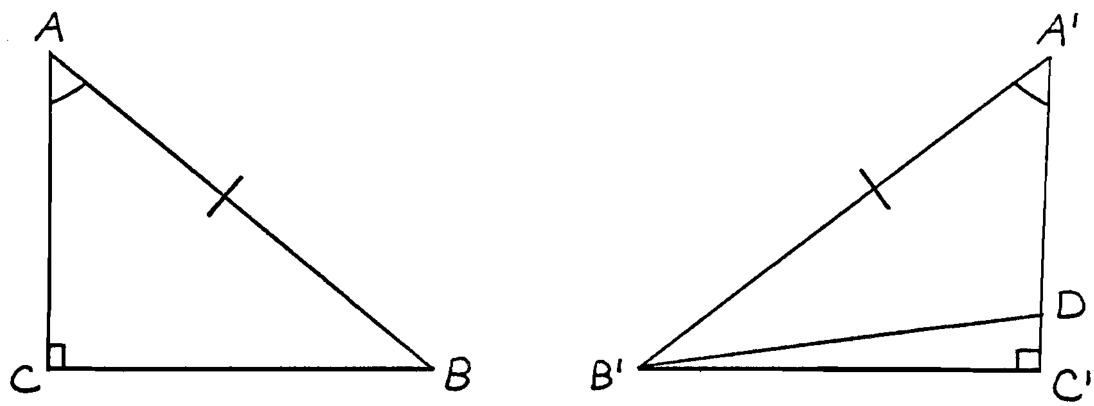


Figure 7

Diagram for the HA Lemma

To show these two triangles congruent, it is sufficient to show that AC = A'C', since then the SAS postulate could be used.

Suppose that AC $\neq$ A'C', and that A'C' is the longer of the two. Then on A'C', mark off point D such that AC = A'D. Next, draw DB'. By SAS, the triangles ABC and A'B'D are congruent. This implies that angle A'DB' = 90°, which is impossible since an exterior angle is larger than a remote interior angle. Thus AC = A'C', and the proof is complete.

3.13. Proof of property 4. Property 4 states that transformation h maps circles with center O into circles with center O.

Proof: It is sufficient to show that if a point at a distance r from O is mapped into a point at a distance r' from O, then the same is true for all points at a distance r from O.

Consider any two points on a circle with center O in plane π. Designate them A and B. Perform transformation h on these two points, obtaining the points A' and B', as illustrated in Figure 8.

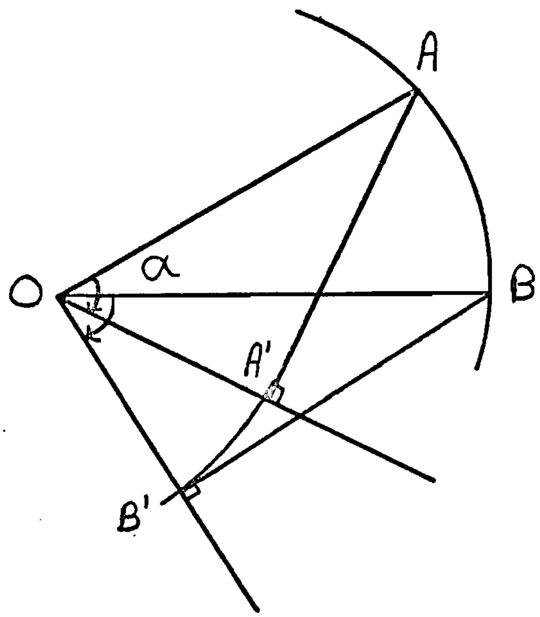


Figure 8

Illustration of Property 4

Consider the two right triangles AOA' and BOB'. It is given that AO = BO. Also, $\measuredangle$ AOA' = $\measuredangle$ BOB' since both of these are equal to α. Therefore by the HA lemma, the two triangles are congruent. Thus, A'O = B'O and the proof is complete.

3.14. <u>Proof of property 5</u>. Property 5 states that transformation h maps right angles with one side passing through O into similar right angles.

<u>Proof</u>: If angle CAO is a right angle with one side passing through the center of the transformation, then the line AC is tangent to the circle with center O and radius OA, as in Figure 9. Since property 4 has shown that, under transformation h, this circle is mapped into another circle with center O, it is sufficient to show that the tangent to the circle with radius OA is mapped onto the tangent to the circle with radius OA'.

Consider a circle with center at O, and radius OA, and a line a tangent to this circle at A, Figure 9. Perform transformation h, mapping this circle into another circle with radius OA'. Line a is mapped onto line a' passing through A' (property 2). If a' is not tangent to

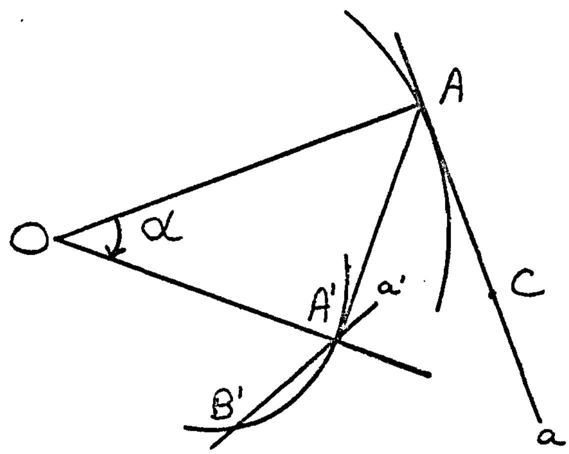


Figure 9

Illustration of Property 5

this circle with radius OA', then there exists some point, say B', different from A', that is the point of intersection of a' and the circle. Combining properties 2 and 4 gives this result: That there is some point E of the original circle such that $h:E \rightarrow B'$; and there is some point D, on line a, such that $h:D \rightarrow B'$. Therefore, $\measuredangle DOB' = \alpha = \measuredangle EOB'$. Now, two cases must be considered. The two cases are (1) when B', D, and E are collinear, and (2) when B', D, and E are not collinear. These have been illustrated in Figures 10 and 11, respectively.

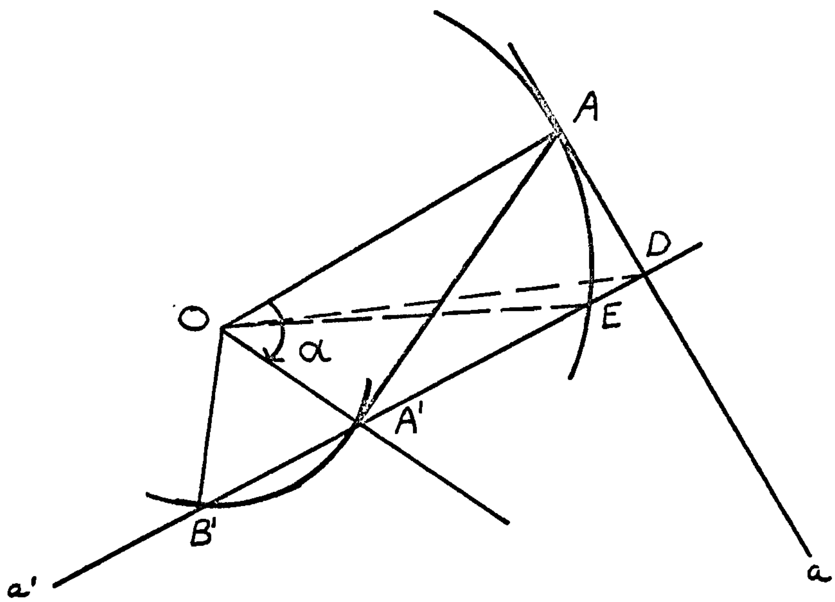


Figure 10

Case I For Property 5

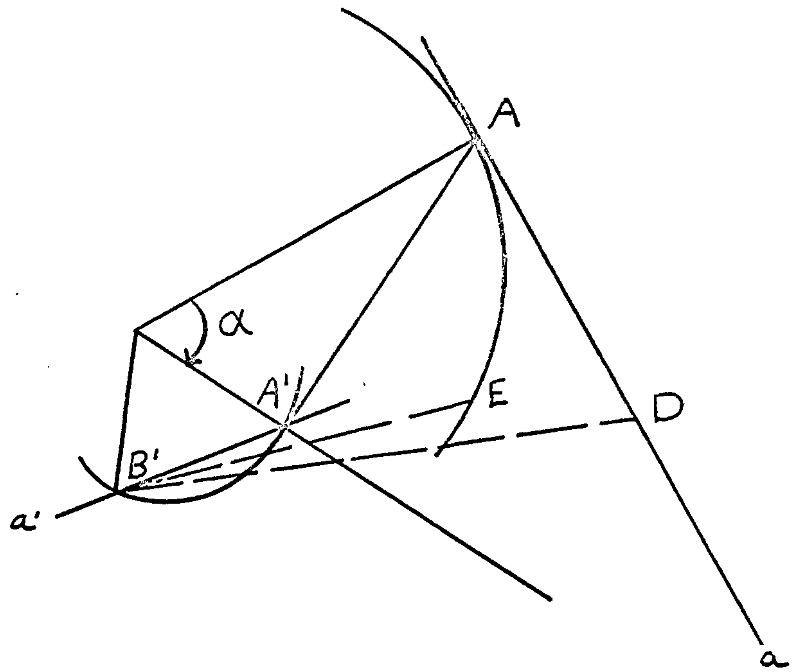


Figure 11

Case II For Property 5

In case 1, two lines, OE and OD, form with OB' an angle of size α. Now since O, E, and D are not collinear, this would be a contradiction of the fourth axiom of existence of congruent angles. Note that if O, E, and D were collinear, this would imply $\measuredangle$ EOB' $= \alpha =$ a straight angle, which is clearly a contradiction.

In case 2, two lines, EB' and DB', form with B'O an angle equal to a right angle. As above E, B', and D could not be collinear. Thus a contradiction of the fourth axiom of existence of congruent angles is obtained. Q.E.D.

3.15. Transformation j. Consider the image of a point in plane π, say A, under the following mappings. First, perform transformation h on point A, thus obtaining point A_0; that is, h: $A \rightarrow A_0$. Next rotate the plane about O, through an angle of $-\alpha$. This will replace A_0 by A'; that is, $R: A_0 \rightarrow A'$. This has been illustrated in Figure 12. Transformation j is a combination of these two. In other words, a point Q is mapped by transformation j into the point Q' (in symbols; j: $Q \rightarrow Q'$) if and only if Q is mapped by h into Q_0, and then Q_0 is mapped by R into Q'.

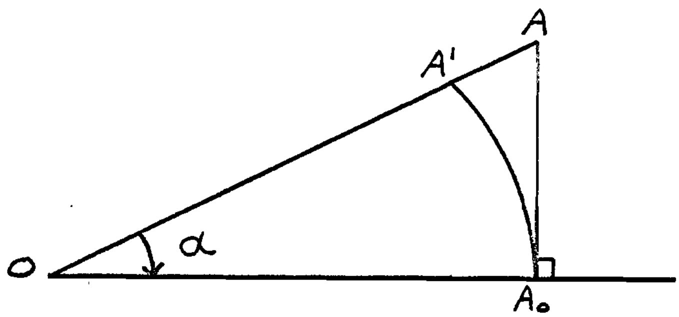


Figure 12

Transformation j

Transformation j possesses the same five properties that were listed for transformation h, and for convenience, have been listed again before the beginning of Chapter IV.

3.16. Properties of transformation j. It leaves the point O unchanged; maps segments onto segments; angles with vertex O into equal angles with the same vertex; circles with center O into circles with center O; right angles with one side passing through O into similar right angles. ([4], p.68).

CHAPTER IV

MAPPING THE PLANE INTO A CIRCLE

4.1. Introduction. The purpose of this chapter is to present the mapping of the hyperbolic plane into the interior of a circle within the hyperbolic plane. This mapping is accomplished by applying transformation j, of the previous chapter, to the points of the hyperbolic plane.

In Chapter I the axiom of parallels for the hyperbolic plane was given to be: given a line a and a point A not lying on a, then there exists, in the plane determined by a and A, more than one line which contains A but not any point of a.

On the basis of this postulate, the following theorem can be derived.

4.2. Theorem A. If l is any line and P is any point not on l, then there are always two lines through P which do not intersect l, which make equal acute angles with the perpendicular from P to l, as in Figure 13, which are such that every line through P lying within the angle containing that perpendicular intersects l, while every other line through P does not. ([8], p.67).

The acute angle referred to in theorem A has been designated as β . Throughout the remainder of this thesis, when β is used, it will be used in this sense. The following is an immediated consequence of Theorem A.

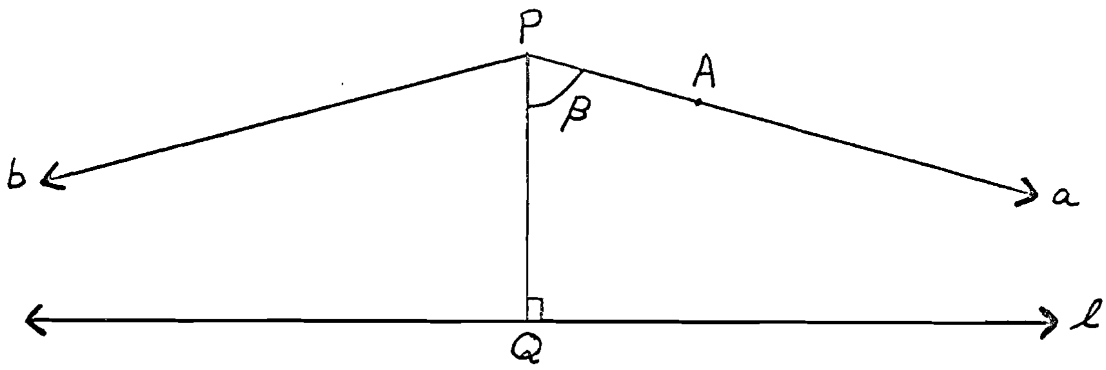


Figure 13

Illustration of Theorem A

4.3. Corollary A. In plane π, there exists an acute angle β (angle APQ in Fig. 13) such that a perpendicular to one of its sides ($l \perp PQ$ in Fig. 13) does not intersect the other side of the angle.

Two lemmas related to Corollary A are now proven.

4.4. Lemma A_1. Given angle ABC in plane π, such that angle ABC = β, and a point H on the half-line BC such that the perpendicular erected at H does not intersect the half-line AB, and a point R on BC is such that BR > BH, then the perpendicular erected at R does not intersect the half-line AB. This is illustrated in Figure 14.

Proof: Suppose the perpendicular at R intersects the half-line AB, say at the point T, as in Figure 14. Then triangle BTR is formed, and since the perpendicular at H passes through BR, it must, by Pasch's axiom, pass through either BT or TR. It cannot cut BT since it was given as a perpendicular that did not intersect the half-line AB. Neither can it cut TR, for if it did, say in Q, then triangle HQR would have two right angles, also a contradiction. Therefore, the perpendicular at R does not intersect the half-line AB. Q.E.D.

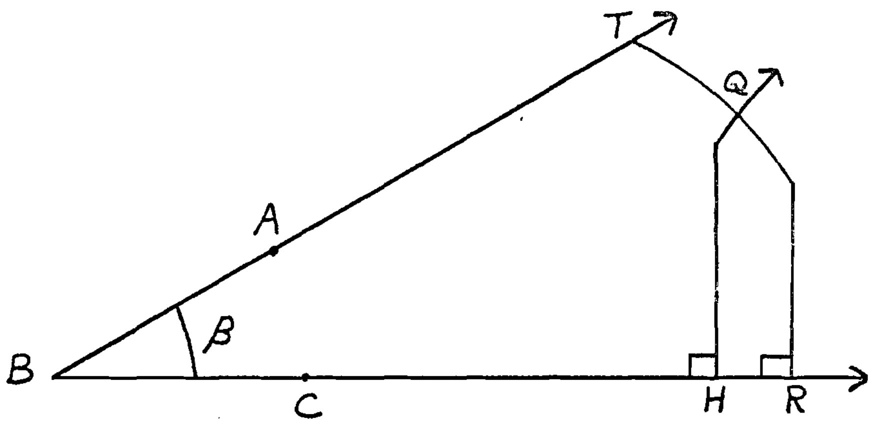


Figure 14

Illustration of Lemma A_1

4.5. <u>Lemma A_2</u>. Given angle ABC in plane π, such that angle ABC = β, then there exists a point H on one side of the angle, say BC, such that the perpendicular erected at H is the first perpendicular that fails to intersect the other side of the angle AB. This is shown in Figure 15.

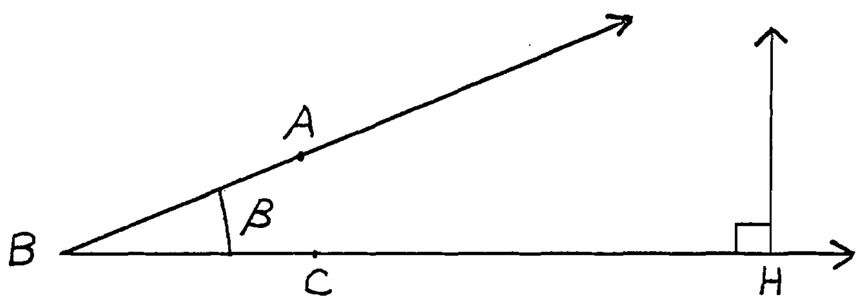


Figure 15

Illustration of Lemma A_2

Proof: If at points of BC, perpendiculars are erected, some of these will intersect AB while by Corollary A, there exists at least one that does not intersect AB, as in Figure 16. Thus the points of half-line BC are divided into two sets: those at which the perpendiculars intersect AB, and those at which the perpendiculars do not, each point of the first set preceding each point of the second by Lemma A_1. Under these conditions, the Postulate of Dedekind asserts that there exists a point such that the perpendicular at this point brings about this division. Designate by H the point that divides the sets. Since the perpendicular at H itself either cuts AB or does not cut it, it must either be the last of the cutting perpendiculars or the first of the non-cutting ones. Suppose it was the last of the cutting ones, intersecting AB in point F, Figure 16.

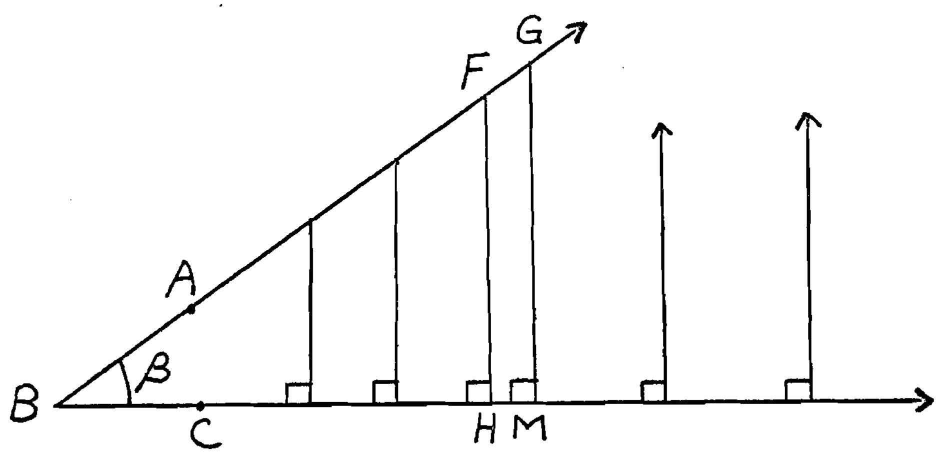


Figure 16

Proof of Lemma A_2

Then measure off FG on AB such that BG is greater than BF, and drop a perpendicular from G to BC at M (To a given infinite straight line, from a given point which is not on it, to draw a perpendicular straight line). Then MG is a cutting perpendicular, and by an argument similar to the one used in the proof of Lemma A_1, MB is greater than HB. In other words M lies to the right of H, and a contradiction is reached since HF was given to be the last cutting perpendicular. Therefore the perpendicular at H is the first of the non-cutting perpendiculars. Q.E.D.

4.6. Theorem B. If OC and OA represent any two distinct half-lines in plane π such that angle AOC equals β, then the perpendicular projections of the points on the half-line OA map onto a finite segment OH, not including H, of OC, Figure 17, where H is the point at which the perpendicular first fails to intersect the half-line OA.

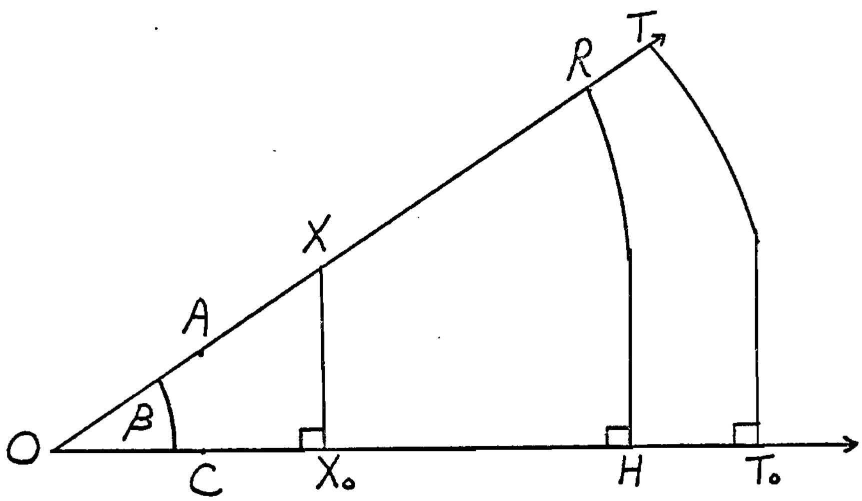


Figure 17

Illustration of Theorem B

Proof: Two things must be shown; (1) every point of OH, not including H, is the perpendicular projection of some point on the half-line OA, and (2) no point to the right of H is the perpendicular projection of any point on the half-line OA.

Now the point O is the perpendicular projection of itself, by definition of perpendicular projection. Next, consider any point X_o of OH such that $0 < OX_o < OH$. At X_o construct a perpendicular. This perpendicular intersects the half-line OA at X since H is the first point where this fails to happen. But X_o is the perpendicular projection of X on OC. Now if H was the perpendicular projection of some point, say R, of half-line OA, as in Figure 17, then RH would be perpendicular to OC which is clearly a contradiction. Therefore, every point of OH, not including H, is the perpendicular projection of some point of the half-line OA.

Suppose a point of half-line OA, say T, had T_o on OC as its perpendicular projection, and that $OT_o > OH$. This would imply that the perpendicular erected at T_o would intersect the half-line OA. But this contradicts Lemma A_1. Therefore, no point to the right of H is the perpendicular projection of any point on the half-line OA. Q.E.D.

Hereafter in this thesis, whenever H is used to represent a point, it is used as the point H of Lemma A_2, that is, as the point at which the perpendicular first fails to intersect the other side of the angle.

By an argument similar to the one just given, it is clear that the same would apply if OA and OC were extended to form lines. Since vertical angles are equal, the points on the half-line obtained by extending OA would have their perpendicular projections mapped onto a finite segment of the half-line obtained by extending OC. This is illustrated in Figure 18.

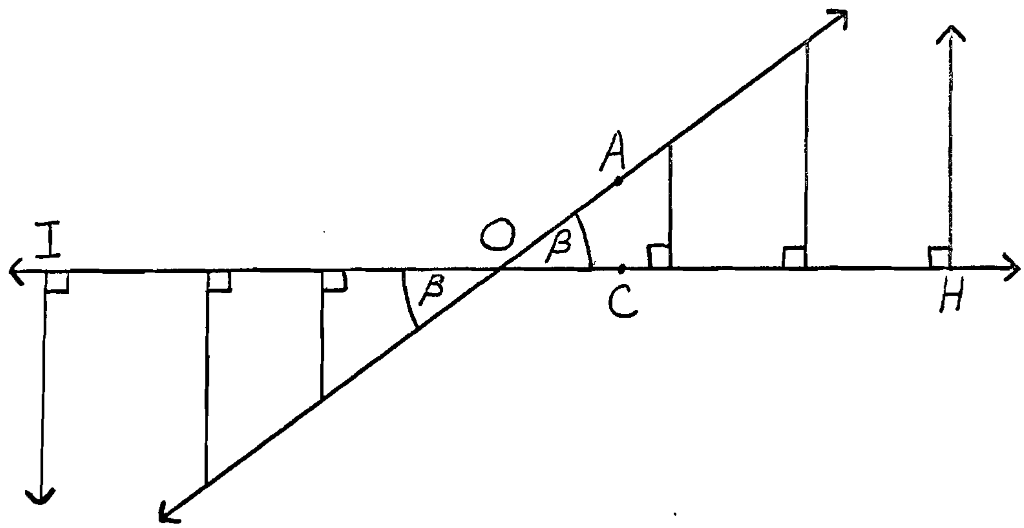


Figure 18

Theorem B Extended to Lines

If the point on OC extended that corresponds to H is designated by I, it is true that OH = OI. This is now given as a lemma.

4.7. G's Lemma. For the acute angle β in plane π, the distance from the vertex of this angle to the point on one side, from which the perpendicular erected first fails to cut the other side, is unique.

Proof: Consider, in plane π, two angles CAB and C'A'B', such that $\measuredangle$ CAB $= \measuredangle$ C'A'B' $= \beta$, Figure 19. Let H and H' represent the points on AB and A'B' respectively where the perpendiculars erected at these are the first that do not intersect the other sides of the angles. Lemma A_2 guarantees the existence of the points H and H'.

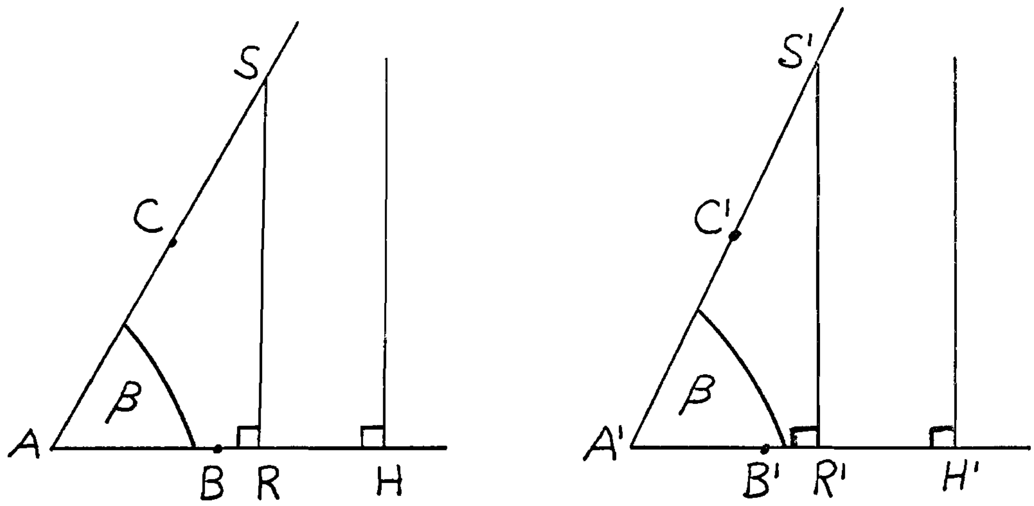


Figure 19

Illustration of G's Lemma

Suppose that AH $\neq$ A'H', and that A'H' is the longer of the two. Then, on A'H', mark off A'R', such that AH $=$ A'R'. The perpendicular erected at R' intersects A'C' at some point, say S', since the first non-intersecting perpendicular was given to be at point H'. On AC mark off AS $=$ A'S', and drop a perpendicular SR to AB at R. By SAS, triangle SAR and triangle S'A'R' are congruent, from whence AR $=$ A'R'.

But A'R' $=$ AH, thus implying that AR $=$ AH. But this is impossible since the perpendicular at H does not intersect AC. Q.E.D.

4.8. <u>Applying transformation j to the hyperbolic plane</u>. First, apply transformation j to the points of some half-line b, in plane π, using O as the center and β as the angle. This is illustrated in Figure 20.

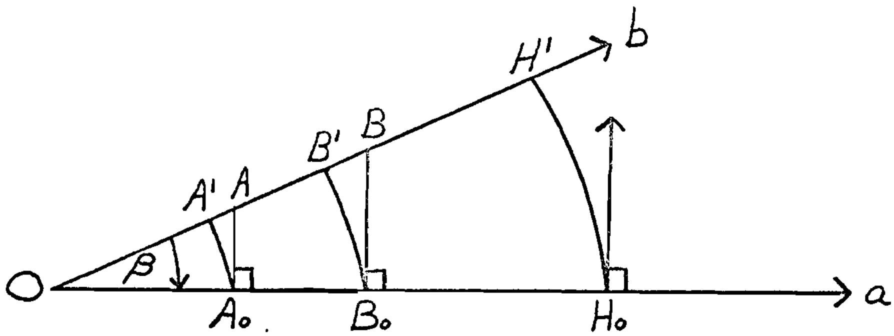


Figure 20

Transformation j Applied to a Half-line

Recall that transformation j replaces each point X, on half-line b, by its perpendicular projection X_o on some half-line a, where the angle between a and b is some acute angle (in this case β), and then rotates the plane about O, through an angle that is the negative of the given acute angle (in this case $-\beta$), so that X_o is mapped into X' on b. Thus, the end result is: $j:A \rightarrow A'$, $j:B \rightarrow B'$, etc.

If H_o is used to denote the point at which the perpendicular first fails to intersect b, then transformation j would not assign any point of b to H'. This is designated by saying that H' represents an "infinitely distant" point of b.

By a similar argument, transformation j could be extended to the points on the lines formed by extending a and b. This has been illustrated in Figure 21, where I_o corresponds to H_o. Then $OI_o = OH_o$ by G's Lemma, and where I' would not be the image of any point on line b under transformation j.

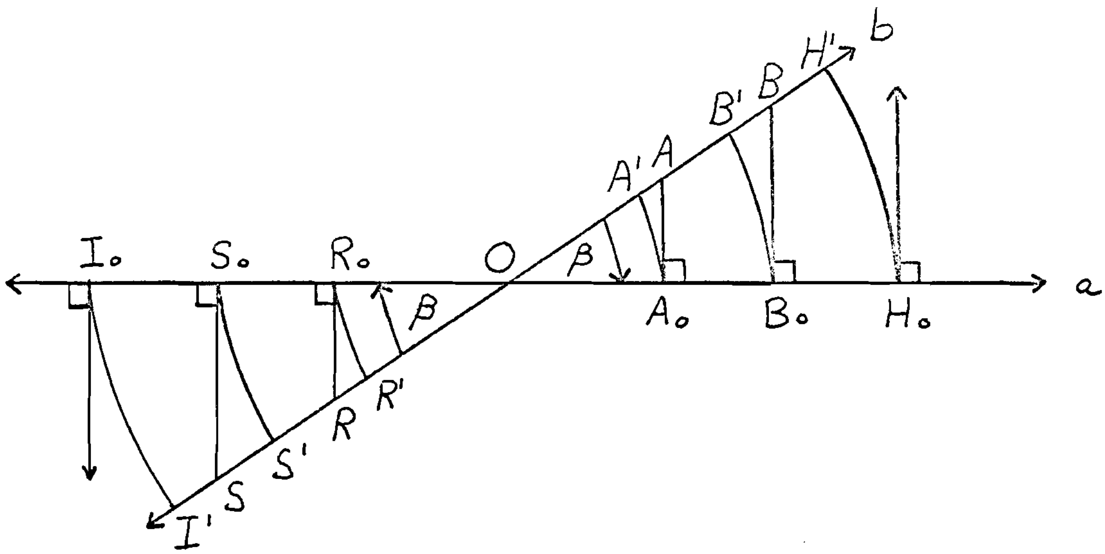


Figure 21

Transformation j Applied to a Line

Now, if transformation j is performed on all the lines passing through O, using angle β, a mapping of plane π into a circle with center O and radius OH' is obtained.

4.9. Notation Used With the Circle. The notation used so far has associated, under transformation j, the point A with A', point B with B', etc. Therefore; in an effort to be consistent, the following notation is used in the next chapter.

If A, B, C,..., and a, b, c,... represent points and lines in plane π, then A', B', C', ..., and a', b', c',... designate the corresponding points and lines within the circle. This is shown in Figure 22. ([4], p.71).

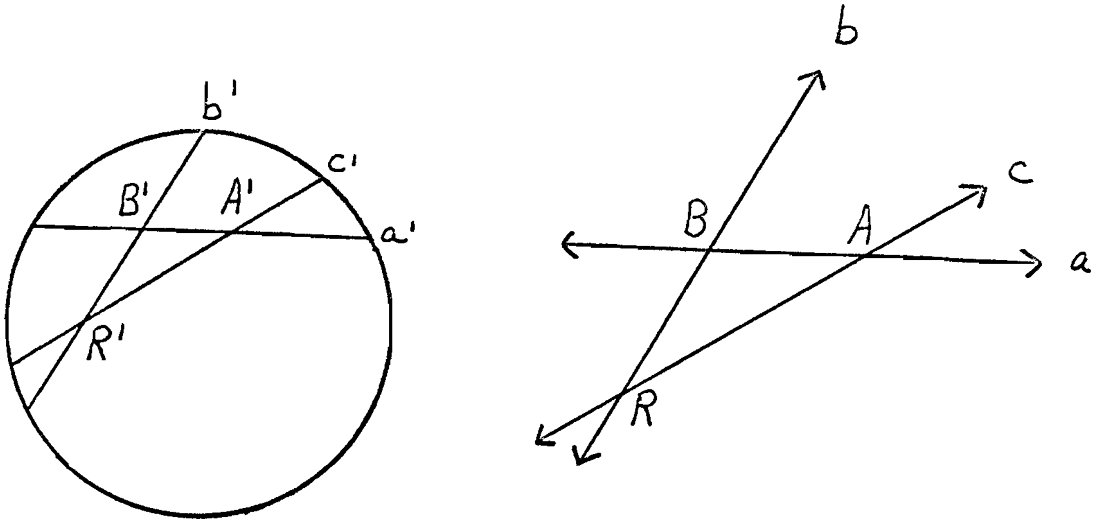


Figure 22

Notation Within the Circle

CHAPTER V

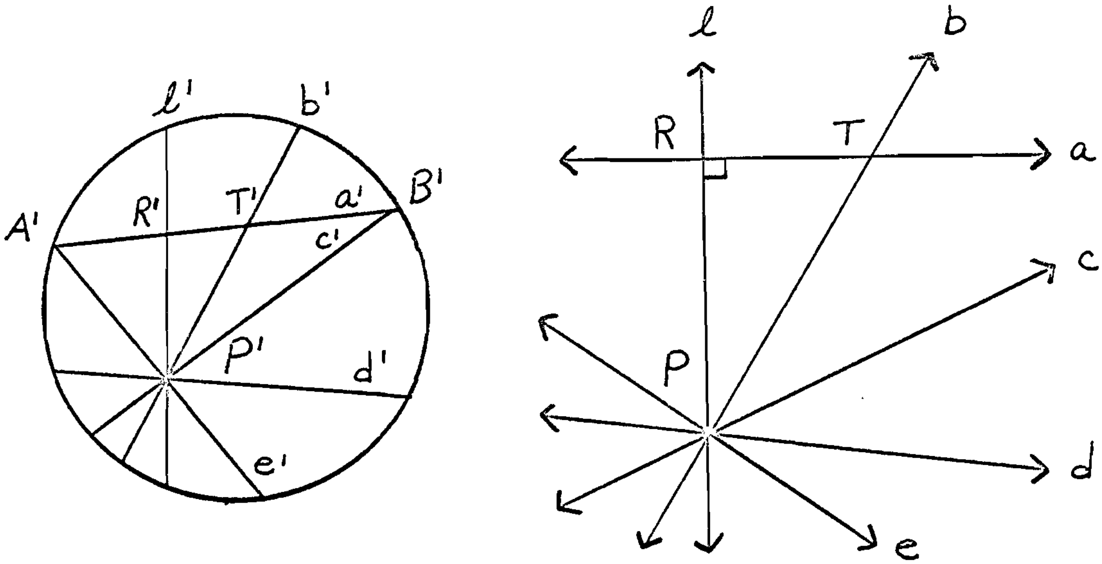
APPLICATIONS OF THE MAPPING

5.1. Introduction. In Chapter IV, a mapping of plane π into the interior of a circle was shown to exist. This mapping was accomplished by choosing a point O, of plane π, arbitrarily, and then applying transformation j to the points of π, where the angle used was equal to β. The radius of the circle was OH'. Straight lines of the plane are represented in this circle by chords. Two types of angles have their measures preserved under the mapping: (1) those with vertex at O, and (2) right angles one side of which contain O.

It is possible, using the circle as the instrument of research, to develop some of the theory of hyperbolic geometry. This chapter in no way exhausts the use of the circle as a means of developing hyperbolic geometry. Rather, it was the chapter's purpose to apply the properties of this circle to some of the more "familiar" characteristics of the plane, such as the notion of parallelism, and the angle of parallelism.

This mapping shows that many of the intuitive results obtainable by use of models of the hyperbolic plane in Euclidean Space can be obtained without reference to another geometry; i.e., within the hyperbolic plane.

5.2. Notion of parallelism. Consider the points and chords of the circle as shown in Figure 23a. In the plane these chords represent the lines l,b,a,c,d, and e, while the points in the circle represent the points R, T, and P of the plane, as shown in Figure 23b.



a) b)

Figure 23

Notion of Parallelism

Take l as the perpendicular from P to a (recall that l' would not be perpendicular to a' unless either a' or l' contained O). Now rotate l about P in a clockwise direction. It is concluded from the map, Figure 23a, that l would intersect a until it reached the position c, since on the map c' is the first line through P' in that direction that does not intersect a' (recall that the points B' and A' do not represent any points of the plane under the mapping). If instead l is rotated in the other direction about P, it would intersect a until it reached the position e, since in the circle e' is the first that fails to intersect a' in that direction. Also, from the map, it is clear that any chord, such as d', lying within the angle formed by c' and e' does not intersect a'. Thus in the plane any line, such as d, lying within the angle formed by e and c fails to intersect a.

Thus lines c and e separate the lines that intersect a from those that do not intersect a. Lines c and e are therefore defined to be the parallels to a through P, one in each direction. That is, c and e are the Parallels to a through P, in each direction, if and only if c and e are the two lines that separate the lines through P into intersecting and non-intersecting.

It is known from Theorem A of Chapter IV that the lines c and e form equal acute angles with l, the perpendicular from P to a. Therefore, c and e are symmetrical to each other with respect to l.

On the map, straight lines that are parallel to each other are represented by chords meeting on the perimeter of the circle, and conversely, chords meeting on the circle represent lines that are parallel to each other, Figure 24.

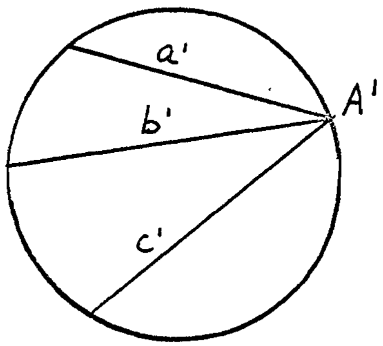


Figure 24

Illustration of Parallels

Now, if in the plane, lines b and c are both parallel to a in the same direction, this would imply that in the circle a' and b' intersect on the perimeter, say at A', and also that a' and c' intersect at A' also, Figure 24.

The following two theorems are an immediate consequence.

5.3. Theorem T. If three straight lines a, b, and c, in plane π, are such that both a and b are parallel to c in the same direction, then a and b are parallel to each other.

Proof: Using the notation of Figure 24, if a is parallel to c, then a' and c' intersect at A', where A' is some point on the perimeter of the circle. Also, if b is parallel to c then b' and c' intersect at A'. But this gives the result that a' and b' intersect at A'. Therefore, a is parallel to b. Q.E.D.

5.4. Theorem R. If in plane π, two lines a and b are such that a is parallel to b, then b is parallel to a.

Proof: Using the notation of Figure 24, if a is parallel to b, then a' and b' intersect at a point on the perimeter of the circle A'. But this implies that b' and a' intersect at A'. Therefore b is parallel to a. Q.E.D.

5.5. Divergent lines. If a and b represent any two lines in plane π, a and b are Divergent if and only if a and b do not intersect and are not parallel.

A theorem and its converse are now proven concerning divergent lines.

5.6. Theorem D. If three distinct lines, in plane π, a, b, and c are such that a and b are both perpendicular to c, then a and b are divergent lines.

Proof: Consider distinct lines a, b, and c such that a and b are both perpendicular to c, as in Figure 25.

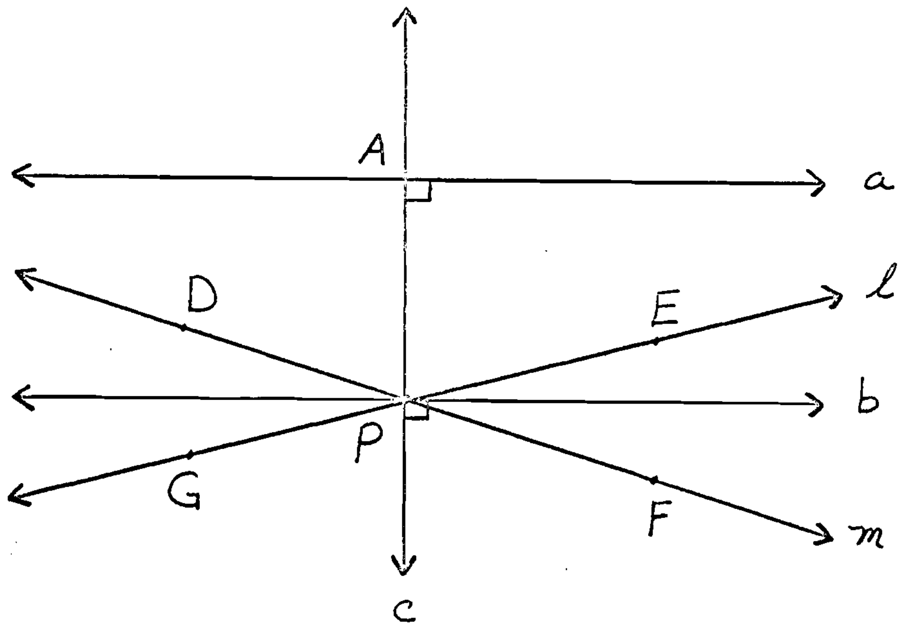


Figure 25

Illustration of Theorem D

Since AP is perpendicular to a, there exists lines l and m, through P, parallel to a, such that l and m form equal acute angles with PA by Theorem A. Thus l and m are distinct from b, so b is not parallel to a.

Since $\measuredangle$ APD $= \measuredangle$ FPG, b cannot fall within $\measuredangle$ FPG, since this would imply an acute angle is greater than a right angle which is impossible. Therefore b lies within $\measuredangle$ EPF, which implies b is non-intersecting with respect to a. Since b does not intersect a, and b is not parallel to a, then the lines a and b are divergent. Q.E.D.

5.7. Theorem D'. If a and b represent any two divergent lines in plane π, then there exists a line c, in π, such that c is perpendicular to both a and b.

Proof: Consider the divergent lines a and b, as in Figure 26. To prove that there exists a common perpendicular, it is sufficient to show that there exists, through a point D, between a and b (i.e., $AD + DB = AB$, if A, B, and D are collinear), a pair of lines e and f such that e and f are parallel to both a and b. For then the perpendiculars from D to a and b at K and L respectively would form a straight line, since $\angle KDL =$ a straight angle.

Therefore, consider the mapping of plane π into a circle. Divergent lines a and b would be represented by the chords a' and b' that have no point in common, as in Figure 27. Draw R'N', R'M', and M'S'.

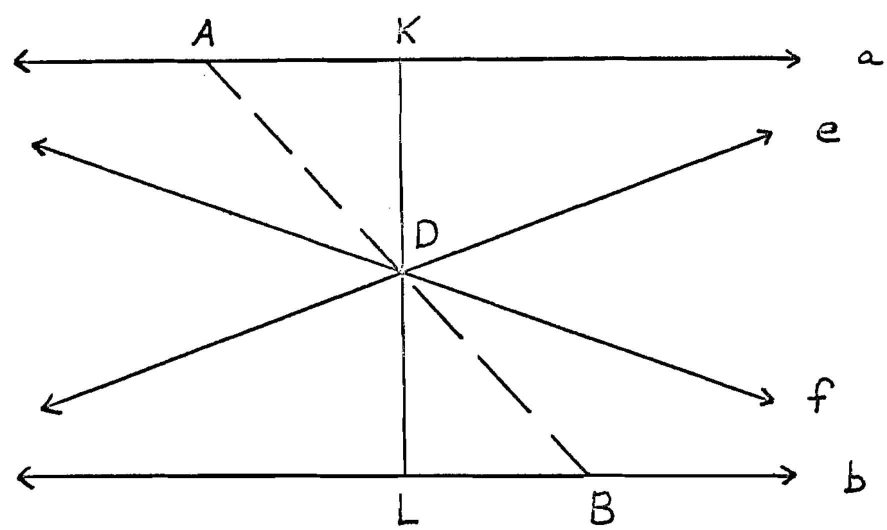


Figure 26

Illustration A of Theorem D'

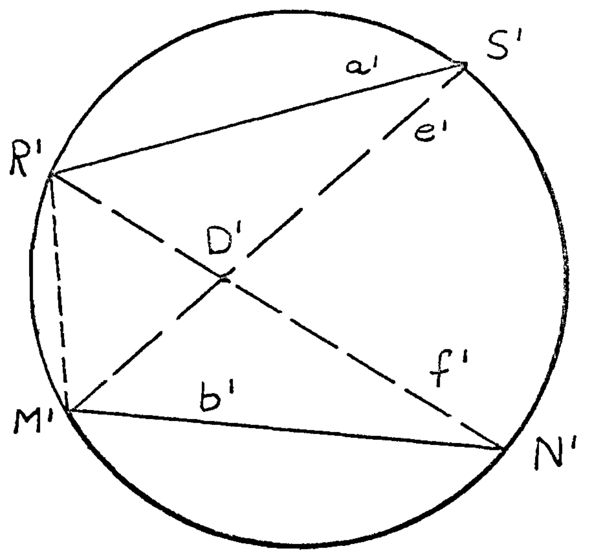


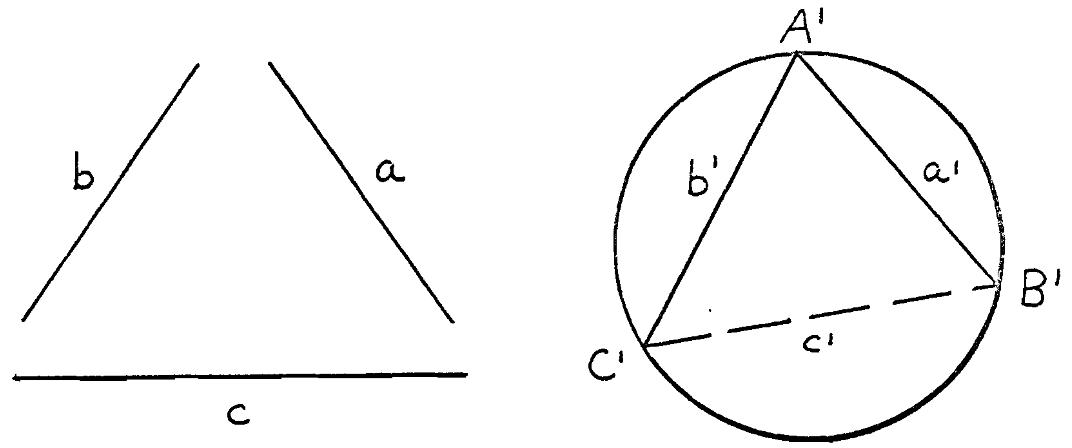
Figure 27

Illustration B of Theorem D'

Since M'S' enters triangle R'M'N' at M', it follows from Pasch's axiom that M'S' must intersect R'N' at a point distinct from either R' or N'. Denote this point by D'. Also denote M'S' as e', and R'N' as f'. In plane π, point D is the point required, for it is clear that through D, both e and f are parallel to the lines a and b. Therefore, there exists a common perpendicular to two divergent lines. Q.E.D.

5.8. Theorem U. If a and b are any two lines in plane π such that a and b are parallel, then there exists a line c, also in π, such that c is parallel to a in one direction, and c is parallel to b in the other direction.

Proof: Consider any two lines a and b such that a is parallel to b, as in Figure 28a. Then map plane π into the circle, obtaining the chords a' and b'. Since a and b are parallel, a' and b' meet on the perimeter of the circle, say at A', as in Figure 28b.



a) b)

Figure 28

Illustration of Theorem U

Designate a' and b' as A'B' and A'C' respectively. Draw chord B'C', calling it c'. Since c' and b' intersect on the perimeter at C', then c is parallel to b. Also, since a' and c' meet on the perimeter at B', it follows that c is parallel to a. Therefore c is parallel to both a and b. Since B' and C' do not represent the same point, it follows that c is parallel to a in one direction and parallel to b in the other direction.

5.9. <u>Improper Triangle</u>. An arrangement of points and lines in plane π is called an <u>Improper Triangle</u> if and only if it is mapped onto a triangle, when π is mapped into a circle, such that the triangle has one, two, or all three of its vertices on the perimeter of the circle.

Thus the three mutually parallel lines of Theorem U form an improper triangle.

5.10. Angle of Parallelism. In Chapter III, Theorem A stated that, in plane π, there existed an acute angle β such that the perpendicular to one of its sides does not intersect the other side of the angle. H was used to denote the first point at which the constructed perpendicular failed to cut the other side. This is illustrated in Figure 29.

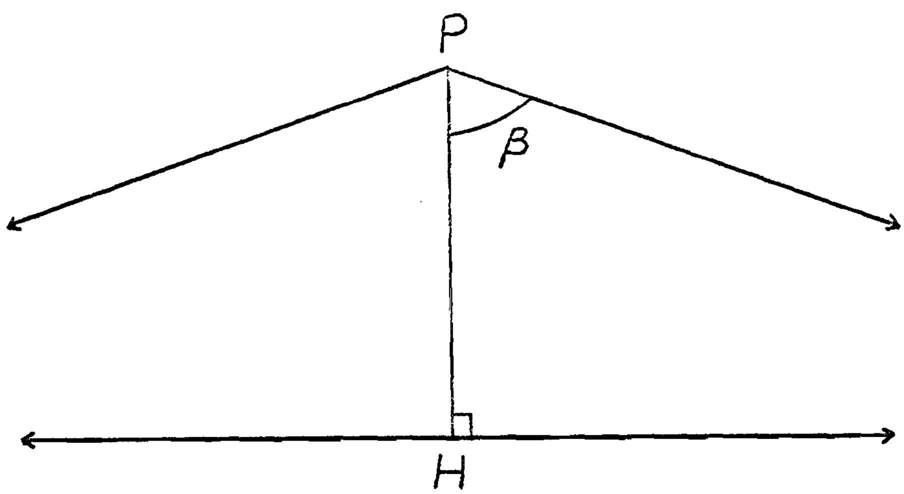


Figure 29

Angle of Parallelism

Denote the length of PH by q. Angle β is then called the angle of parallelism for q. That is, an angle α, in plane π, is called the Angle of Parallelism for some distance q if and only if q is the distance, on one side of α, to the point where the perpendicular at this point first fails to intersect the other side of α.

It can be proven, although the proof is not presented in this thesis, that the angle of parallelism for any given distance is constant. That is, the angle of parallelism is a function of q. The notation to be used for this is: $\alpha = \pi(q)$.

By utilizing the mapping of Chapter IV, it is possible to arrive at an "elementary" result concerning the relation between α and q.

5.11 <u>Theorem L</u>. If, in plane π, α is the angle of parallelism for length q, then α approaches 90° as q approaches 0, and 0° as q approaches infinity.

<u>Proof</u>: Map plane π into a circle choosing the center of transformation j to be the vertex of α. Thus both sides of α are mapped onto radii of the circle, Figure 30.

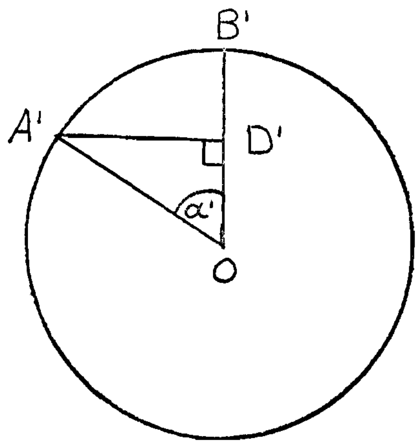


Figure 30

Illustration A of Theorem L

Denote the sides of α' by OB' and OA'. By Property 2 of transformation j, $\measuredangle \alpha = \measuredangle \alpha'$, since its vertex is at O. Drop a perpendicular from A' to OB' at D'. Since one side of the right angle A'D'O passes through O, Property 4 of transformation j states that $\measuredangle$ A'D'O is the image of a similar right angle in plane π'.

Thus, since OA' and A'D' represent parallel lines in the plane such that A'D' represents the first non-intersecting perpendicular to the line represented by OD', it follows that α is the angle of parallelism for the segment OD, as in Figure 31.

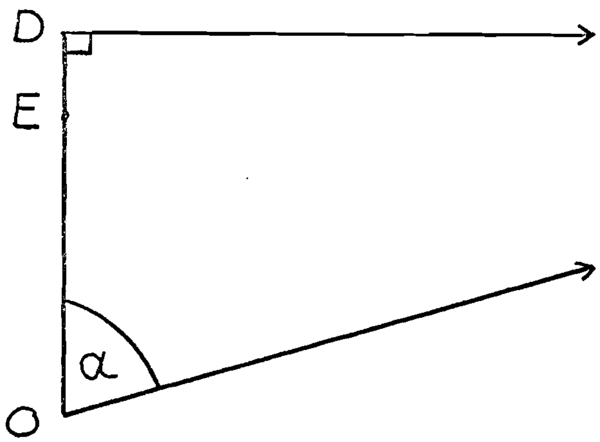


Figure 31

Illustration B of Theorem L

Take a point E on OD such that OE < OD. On the map this corresponds to a point E' on OD' such that OE' < OD', Figure 32.

At E' construct a perpendicular cutting the circle at F'. F' is distinct from A', since if they were the same, triangle A'D'E' would contain two right angles which is impossible.

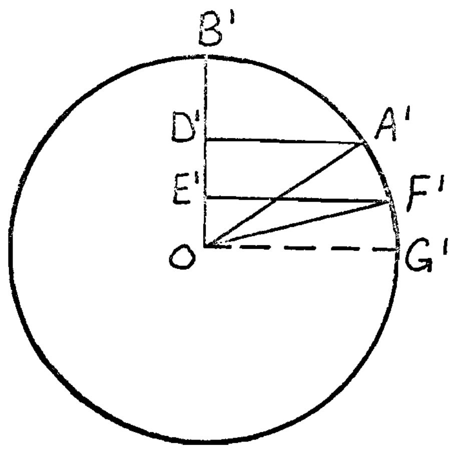


Figure 32

Illustration C of Theorem L

Draw OF'. Angle F'OD' is larger than angle A'OD' since angle F'OD' equals angle A'OD' plus angle A'OF'. Now if E' and F' are considered as arbitrary points, then as E' approaches O, F' approaches the point G' where G' is the point on the perimeter of the circle where the perpendicular at O cuts the circle. Thus the segment OG' is the limiting position of E'F'. From this it is clear that angle F'OE' approaches 90° as OE' approaches O.

By an argument similar to the one used to show angle F'OE' is larger than angle A'OD', it is shown that angle F'OE' gets smaller as OE' becomes larger. As E' approaches B', F' also approaches B'. Thus OB' is the limiting position of OF' as E' approaches B'.

Since angle B'OB' = 0°, it follows that angle F'OE' approaches 0° as OE' approaches OB'.

Therefore, since α' varies between 0° and 90°, then α varies between 0° and 90° as the length of q varies. Q.E.D.

CHAPTER VI

CONCLUSION

6.1. Summary. The purpose of this thesis has been to present a method of mapping the hyperbolic plane into the interior of a circle in the hyperbolic plane, and to show some of the applications of this mapping. As was mentioned at the beginning of Chapter V, no attempt was made to exhaust the uses of this circle.

In Chapter I, the list of axioms, defined, and undefined terms, that were to be used in this study, were given.

Chapter II gave a short history of hyperbolic geometry, concentrating on the events taking place after Lobatchevsky published his original work in the Kazan journal.

In Chapter III, the transformation used in mapping the hyperbolic plane into a circle was given. This transformation was called "transformation j". The central problem in setting up transformation j was the question of preserving the collinearity of points on a line that did not pass through the center of the transformation. This was done by means of Hjelmslev's lemma.

In Chapter IV, transformation j was applied to the points of the hyperbolic plane. The result of this was that the plane was, so to speak, compressed into a circle with a finite radius such that the points on the perimeter of the circle were not the images of any points in the plane.

In Chapter V, some of the applications of this mapping were given. Since lines in the plane were mapped onto chords, and certain angles had their measures preserved, it was possible to prove some of the "basic" theorems of hyperbolic plane geometry.

6.2. Suggested research. There are two main areas for further research suggested by this thesis.

First, recall that a particular acute angle β was used in transformation j in mapping the plane into a circle. If an arbitrary acute angle is used, can the plane still be mapped into a circle? If so, what effect would it have on the radius of the circle?

Secondly, in Chapter I, it was mentioned that the hyperbolic space could be mapped into the interior of a sphere with a finite radius. How would this be done? Would there be fewer, the same, or more properties preserved under such a mapping?

Besides the two above questions, a third natural area of research would be to see how much of the theory of hyperbolic plane geometry could be developed using the properties of the circle only. For example, Figure 28b hints at the possibility that, in the hyperbolic plane, there exists a triangle having a maximum area. That is, no other triangle in the plane would have an area exceeding the area of a certain triangle. Does the triangle in the hyperbolic plane that is mapped onto triangle A'B'C' possess the property that its area is greater than or equal to the area of any other triangle in the plane? If so, could this be proven by the use of the mapping only?

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