METHODS OF REPRESENTING A CURVE

HAVING COMPLEX POINTS

515

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CHAPTER I

INTRODUCTION

The problem. Very early in his training, the student of mathematics is confronted with the problem of making a graph to represent a relationship between two variables. As there is a one-to-one correspondence between the real numbers and the points on a line, one axis is usually used for one variable and another for the other. Then points on one line map to points of the other according to the relation defined between the two variables. Thus, if the variables are x and y and the relation is

$$y = x^2,$$

the point 2 on x maps to 4 on y, -3 to 9, $\sqrt{15}$ to 15, and so on. This kind of model is not satisfactory for conveying some information about the behavior of the relation. For if x is allowed to range over the whole line, y will range over the half line consisting of the points corresponding to

$$y \geq 0.$$

Moreover, any point on the second line is the map of two points of the first line.

Now if the two lines are set at right angles, a plane is determined, called a Cartesian plane after Rene Descartes, and the correspondence of variables in a relation is represented by points of the plane identified by ordered number pairs. When all points satisfying the relation have been located, the result is the graph of the relation.

Thus the graph is a subset of R X R where R is the set of real numbers. For most purposes, this graph is ideal. It satisfies most of the things for which a graph is desired. It will exhibit zeros of a function, show simultaneous solutions to two equations, exhibit critical points, represent the rate of increase or decrease in a function, and it can be used to exhibit such things as the trigonometric functions.

This thesis is concerned with ways of exhibiting some of the above mentioned things for which a graph is used when the replacement set for the variables in the relation to be graphed is the set of complex numbers.

The usual representation of a complex number is not a point on a line, but rather a point on a plane, where one axis is real and the other imaginary. If R represents the set of all real numbers and I the set of real coefficients of all imaginary numbers, then the set of ordered number pairs R X I represents the set of all complex numbers. Given a complex variable z and another complex variable w, it is desired to make a graphical representation of a relation between these two variables. As in the first paragraph, where a mapping was considered between two lines, here a mapping may be considered between two planes. If the relation is

$$w = z^2,$$

the point (2,0) on the z-plane maps to (4,0) on the w-plane, (0,3) maps to (-9,0), (3,2) maps to (5,12), and so on. As z varies over all of its plane, w ranges over all of its plane. Just as was true for real variables, this is not entirely satisfactory for getting an over-all view of the relation.

Proceeding in a manner similar to what was done before, let the two planes be placed at right angles to one another, with each of the axes vertical to each of the other axes. Here is where the problem arises. There are now four axes, each to be perpendicular to the other three. The requirement for this, in a straight forward approach, is a four dimensional space, and this does not appear to be available.

No method has been found which does as adequate a job of showing the relation between two complex variables as simply as the Cartesian graph for real variables. While a number of graphical methods have been suggested, the degree of success each has attained in a complete representation is usually dependent on the amount of elaboration in a given method, which, in turn, results in a decrease in simplicity, with a corresponding decrease of intuitive understanding.

Forsyth groups the important methods in three categories [4, p. 5]. In the first, a four-dimensional space is used with the four axes each perpendicular to the others. The second method uses a line, curved or straight, the whole line or sometimes a segment of it, as representing the two variables simultaneously. The third procedure is for each variable to be associated with a point in a plane, or in two different planes, such that the two points represent the two variables simultaneously.

Organization of the thesis. No single method of representation of points is used exclusively in this thesis. Rather, several of the basic applications of graphs are considered, and ways are shown of exhibiting those characteristics of graphs relating to that particular application. Following a brief history of the problem in Chapter II, methods of showing the roots of a polynomial equation in one variable are considered in Chapter III. Graphically, this is generally done by setting the polynomial function equal to a second variable, graphing the resulting curve, and noting the points where the second variable is zero. Chapter IV is concerned with the simultaneous solution of two equations in two variables, accomplished graphically by identifying the points of intersection of the curves representing the two equations. The third application, Chapter V, is some of the topics considered in work with the circle, including tangents to the circle, the radical axis to two circles, and the definitions of the circular (trigonometric) functions, and their relation to the hyperbolic functions. Chapter VI investigates the derivative of a function, to see what relevance this has to the graph of the function. A summary of the developments of the thesis is given in Chapter VII.

Definitions. There are a few terms whose definitions are needed in order to understand their use in this thesis. They are given here.

Complex number. A complex number is any number of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

Absolute value of a complex number. The absolute value of a complex number $a + bi$ is $\sqrt{a^2 + b^2}$.

Ideal point. An ideal point is a point at infinity, added to a line or plane so that it is not necessary to state exceptions to certain theorems.

<u>Asymptote</u>. An asymptote to a curve is a line which intersects the curve at an ideal point.

<u>Imaginary number</u>. An imaginary number is a complex number whose real part is zero ($a = 0$).

<u>Supplementary</u>. A supplementary is the total of those points of a graph for which one or both of the co-ordinates is complex.

CHAPTER II

HISTORY OF GRAPHS USING COMPLEX NUMBERS

It took mathematicians centuries to recognize the existence of negative numbers, partially because of their inability to accept a line as having negative length. It took longer to admit to imaginary and complex numbers, the very name imaginary being a witness to this. However, since in the solution of quadratic equations, solutions involving the square roots of negative numbers were often obtained, mathematicians were forced to accept such results as numbers, and they began to provide ways to represent such entities. The history given here is divided into two parts, the first being concerned with the representation of a single complex number, and the second the graph of a relation between two complex variables.

Representing complex numbers. The first known attempt to make a representation of complex numbers was John Wallis in his Algebra, published in 1685 [2, p. 13]. He gives some ingenious arguments for the existence of complex numbers, and gives many examples of drawings suggested by quadratic equations whose roots are complex. He did not have a general method of representing complex values of a given variable.

There seems to have been no further work with the geometrical representation of complex numbers for over sixty years until Heinrich Kuhn, challenged to the problem by Euler's invitation for him to cube $-1 \pm \sqrt{-3}$, published a book on the subject in 1753 [2, p. 16].

Actually, his methods were more primitive than those of Wallis, and he presented no progress towards an ultimate solution.

Caspar Wessel, an obscure Norwegian surveyor, had a paper published by the Royal Danish Academy in 1799 that might have given an acceptable general geometrical method much earlier had it become widely known. Instead, it was nearly one hundred years before the work was really discovered. He used a vector approach in which each vector, radiating from a common origin, had real and imaginary components. While his work was awaiting its time of discovery, several other men achieved nearly identical results. Jean Robert Argand, with some advice from Legendre, used imaginary numbers as the mean proportional of a positive and a negative number, with the real numbers represented on horizontally directed lines, and the mean proportionals directed vertically. His work was published in 1806 [2, p. 26]. C.V. Mourey and the Rev. John Warren, both writing in 1828, also conceived of imaginary numbers as vectors, and placed them in vertical directions with real numbers as vectors in horizontal positions [2, p. 27].

It was Gauss, in 1831, who contributed the method of representation of complex numbers in use today [2, p. 28]. He is responsible for considering the points 1, i, -1, and -i as points on a plane in four different unit directions from the origin, rather than the vector quantities that had been previously used. This is the classical representation that has become the standard means of graphing complex numbers, and while other methods have since been pursued, none has met with the standard of completeness and simplicity as this standard model.

The only thing lacking in the Gauss representation is the infinite domain. For the correspondence between the complex numbers and the plane to be complete, the entire infinite domain of the plane consists of exactly one point, a concept that is not immediately intuitive [1, p. 71].

Of those attempts to further refine methods of representation, the most important is an application of the principle of duality by representing a complex point by a real line in the plane.

The graph of the relation of two variables. Jean Victor Poncelet first published his classic work, Traite des proprietes projectives des figures, in 1822. He was able to show, if the equation of a conic is of the form

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

any point P on the x-axis will have ordinates of length

$$y = \frac{b}{a}\sqrt{\pm(a^2 - x^2)}.$$

If P is in the interior of the conic, the ordinate length is a real number, but if P is in the exterior of the conic, y will be imaginary. The segment joining the two points of the conic in the latter case is called an ideal chord. The locus of all of the points with imaginary ordinates is called the supplementary to the conic, and has the equation

$$\frac{x^2}{a^2} \mp \frac{y^2}{b^2} = 1 \qquad [2, pp. 68-69].$$

In 1839, the English mathematician Gregory started a more complicated line of thought by using a surface in space to represent a curve [2, p. 70]. He located points of the graph of the equation

$$f(z,w) = 0,\ z = x + iy,\ w = u + iv$$

by two perpendicular complex planes. If a point (x,y) is located in the z-plane by the Gauss scheme, the w-plane is then established passing through the origin of the z-plane and perpendicular to the line passing through the origin and the point (x,y). The origin of each plane is the same point. The real axis u of the w-plane is normal to the z-plane, while the imaginary axis v lies in the z-plane. The point (u,v) corresponding to (x,y) is then located in the w-plane, a line is drawn joining (x,y) to (u,v), and then the origin is projected normal to this line. This gives the point (z,w).

A somewhat simpler approach was taken by Walton in 1852 [2, p. 71]. He represented the point (z,w), where

$$z = a + bi \text{ and } w = c + di$$

in a three-space using axes X, Y, and Z. The point was represented by the ordered number triple (X,Y,Z) where

$$X = a,\ Y = c,\ \text{and } Z = b + d.$$

In 1878, Appell used an approach similar to that of Walton, except instead of $z = b + d$, he used $z = \sqrt{b^2 + d^2}$ [2, p. 72]. This has a great detraction in that a complex point and its conjugate are indistinguishable. In 1933, however, Jahnke and Emde, in their bilingual book, Tables of Functions with Formulas and Curves, in which are found many tables for use in higher mathematics not normally found in books of tables, make use of Appell's method and have made rather elaborate graphs of a number of functions which they call the relief of the function [9, p. IX].

In 1869, Sophus Lie devised a method of representing the points of a curve by the use of the real points in a three-space. The point $(a + bi, c + di)$ was represented by the real point (x,y,z) where

$$x = a, \; y = b, \; z = c,$$

to which is attached a weight d. Thus he had a correspondence of complex non-weighted points of the plane with weighted real points of space [2, p. 104].

It has already been mentioned that a line was sometimes used to represent a complex number. Weierstrass, in 1892, and Van Uven, in 1911, used real lines to show the relation between two complex variables. They graphed the point $(a + ib, c + id)$ by a real line lying in a three-space. The point $a + ib$ was graphed as $(a,b,0)$, while $c + id$ was placed at (c,d,k), where k was some arbitrary constant. The line joining these two points represented the complex point [2, pp. 80-81].

Henschell, in 1892, and Vivanti, in 1895, also proposed to represent points of a curve by lines in space, their methods becoming quite involved, using spheres and stereographic projections [2, p. 82]. In 1948, Laird used Plucker's line coordinates to set up a one-to-one correspondence between the complex points and real lines in space [10, p. 40].

This survey of some of the developers of representative procedures for graphing complex points and curves using complex variables is certainly not complete. Many other mathematicians have worked on it. Some of them, like the Abbe Buee [2, p. 24] and Bjerknes [2, p. 74], presented methods inferior to what had been already developed. Others, such as Laguerre [2, p. 85], Paulus [2, p. 76], and Marie [2, p. 78], presented some tremendous strides in developing and improving previous work. Riemann, by introducing the idea of Riemann surfaces, was able to provide a method of giving a single value to w in functions such as

$$w = z^{1/n},$$

n a natural number, where previously multiple values existed in the w-plane [1, p. 275]. Klein used the complex tangents to the curve rather than the curve itself, incorporating Riemann surfaces in his work [2, p. 83]. The Von Staudt theory defines a complex point as "an elliptic involution on a line, together with a sense of description for that line" [2, p. 230].

It is hoped that this summary gives some idea of the importance that has been attached to the problem of representing complex numbers and graphing relations involving complex variables by some of the great mathematicians of history. No method has yet been given that includes the completeness, simplicity, and intuitiveness of the Cartesian representation of real curves. The methods of representation used in this thesis will try to maintain, as much as possible, these attributes, using points of Gaussian planes to represent complex numbers whenever this is feasible.

CHAPTER III

ROOTS OF A POLYNOMIAL EQUATION

One of the most important uses of graphs of polynomial functions is that of representing the roots of a polynomial equation. If the function

$$F(x) = \sum_{n=0}^{k} a_n x^n,$$

where the a_n are real, is graphed in the usual manner using Cartesian co-ordinates, all real roots of the equation

$$F(x) = 0$$

become readily available as the abscissas of the points of intersection of the curve and the x-axis. Now it is well-known that a polynomial equation of the form $F(x) = 0$ will have k roots. If the graph of $y = F(x)$ does not indicate all k of the roots, then the ones remaining must be complex. This chapter is concerned with methods of indicating these complex roots by the use of graphs.

Methods of representation. There are basically two ways in which complex roots may be demonstrated. The first method makes use of the fact that the real curve $y = F(x)$, while not indicating the roots directly, has certain characteristics determined by them. The other procedure is a direct representation showing real roots as the points of intersection of the curve $y = F(x)$ with the x-axis, while complex roots are indicated by the intersection of the curve with the complex x-plane. In the latter case, the variables z and w are used instead of x and y, respectively.

The dual plane. The first method considered is a way of finding the roots of a quadratic equation in x, making use of a single plane serving a dual purpose. The plane is first considered as real, but when the roots of the quadratic are complex, then, after the procedures described below are completed, the same plane is taken as a complex plane [3, p. 130].

In order to find the roots of a quadratic equation of the form $F(x) = 0$, graph $y = F(x)$ on the real plane in the usual manner, such as in Figure 1. When the parabola does not intersect the x-axis, the roots of the equation are complex. To identify the roots on the graph, first draw the axis of the parabola, line AB in Figure 1(a). From the point C, where AB cuts the x-axis, draw the two tangents to the curve, determining points T_1 and T_2 as the points of tangency. Draw line T_1T_2 cutting the axis AB in a point. Call this point D. Then mark off the length T_1D on AB on either side of C, locating points P and Q. If the plane is now considered as a complex plane, the co-ordinates of P and Q are the desired roots.

Proof: It is first necessary to find the equations of the tangent lines CT_1 and CT_2 of Figure 1(a). The parabola formed from the general quadratic,

$$y = ax^2 + bx + c \tag{1}$$

will have an axis passing vertically through its vertex. The vertex is the point where $x = -b/2a$, which provides that the point C will have co-ordinates $(-b/2a,0)$. Thus the equation of the set of lines passing through C is

$$y = m(x + b/2a) \tag{2}$$

where m is the slope of a line. In equations (1) and (2), substitution to eliminate y gives

$$ax^2 + (b - m)x + (c - mb/2a) = 0. \tag{3}$$

The solution of (3) for x will give the abscissas of the points of intersection of (1) and (2). If a line of (2) is to be tangent to the curve, the roots of (3) must be real and equal. Thus

$$(b - m)^2 - 4ac + 2mb = 0.$$

Solving this for m, the slope of the tangent line is

$$2ax + b = \pm\sqrt{4ac - b^2}.$$

Solving this for x,

$$x = \frac{-b \pm \sqrt{4ac - b^2}}{2a}.$$

This indicates the points T_1 and T_2 have abscissas $\sqrt{4ac - b^2}/2a$ units on either side of the axis $x = -b/2a$, that is, the line AB. This distance corresponds to the real number which is the coefficient of i given by the quadratic formula. Thus, when the distance T_1D is marked off on AB on either side of C, the resulting points, P and Q, when considered as points of a complex plane, correspond to the roots of the quadratic equation. This justifies the dual use of the plane.

<u>A more general graphical interpretation</u>. For the purposes of this section, consider the polynomial equation

$$F(x) = \sum_{n=0}^{k} a_n x^n = 0, \; k \leq 2, \; a_k = 1,$$

which has at least one pair of complex roots, $a \pm bi$. There is a procedure which will indicate the complex roots, $a \pm bi$, on the usual real graph. The equation can be written in the form

$$F(x) = (x^2 - 2ax + a^2 + b^2)f(x) = 0,$$

where $f(x)$ is a polynomial function of degree $n - 2$. Here the family of curves $y = mf(x)$ is introduced, m being a real parameter. Then roots of the polynomial equation $F(x) = 0$ can be obtained graphically by means of the theorem which follows.

Theorem: If $F(x) = (x^2 - 2ax + a^2 + b^2)\ f(x)$, where $b \neq 0$, and $f(x)$ is a polynomial function with real coefficients, then there is a curve of the family $y = mf(x)$ that is tangent to $y = F(x)$ at a point (h,k) on the real plane, with $a = h$ and $b = \sqrt{m}$ [7, p. 238].

Proof: Two of the roots of the equation $F(x) = 0$ are $a \pm bi$. $F(x) = 0$ and each member of $mf(x) = 0$ will have $n - 2$ roots in common. Equating the two functions,

$$(x^2 - 2ax + a^2 + b^2)f(x) = mf(x).$$

Then if $f(x) \neq 0$,

$$x^2 - 2ax + a^2 + b^2 - m = 0.$$

By the quadratic formula,

$$x = a \pm \sqrt{m - b^2},$$

the values of x being the abscissas of the points of intersection of the two curves. By proper choice of the parameter m, the values of x can be made to be real, which puts the intersections on the real plane. The midpoint of the line segment joining the two points of intersection has an abscissa a. A vertical line through this midpoint intersects the curve $y = F(x)$ in a point also having abscissa a. Now if m is allowed to approach b^2 as a limit, the secants joining the points of intersection of $y = F(x)$ and $y = mf(x)$ will approach the tangent to

$y = F(x)$, at the point where $x = a$, as a limit. Therefore, when $m = b^2$, the two curves will be tangent, the point of tangency being the point (h,k) of the theorem, and $a = h$ and $b = \sqrt{m}$.

The theorem thus having been established, all that remains in finding the complex roots $a \pm bi$ of the polynomial equation $F(x) = 0$ is to determine the value of m making $y = F(x)$ and $y = mf(x)$ tangent, and to find the abscissa of the point of tangency. In order to see how this theorem applies to actual problems, applications are here made to solve quadratic and cubic equations for complex roots.

Let the quadratic equation

$$x^2 + a_1x + a_o = 0$$

have complex roots $a \pm bi$. Then

$$F(x) = x^2 - 2ax + a^2 + b^2,$$

and

$$mf(x) = m.$$

When $y = F(x)$ is graphed, it is a parabola with a vertical axis. The family $y = mf(x)$ is the set of horizontal lines $y = m$, with one of these lines tangent to the parabola at its vertex. If the parabola has its vertex at (h,k), then $a = h$, $b = \sqrt{m} = \sqrt{k}$, and the roots of the equation are $x = h \pm i\sqrt{k}$.

As an example of this, consider the equation

$$x^2 + 2x + 5 = 0.$$

The graph of Figure 1(b), page 18, is the parabola

$$y = x^2 + 2x + 5,$$

from which the co-ordinates of the vertex, $(-1,4)$, are obtained. Hence,

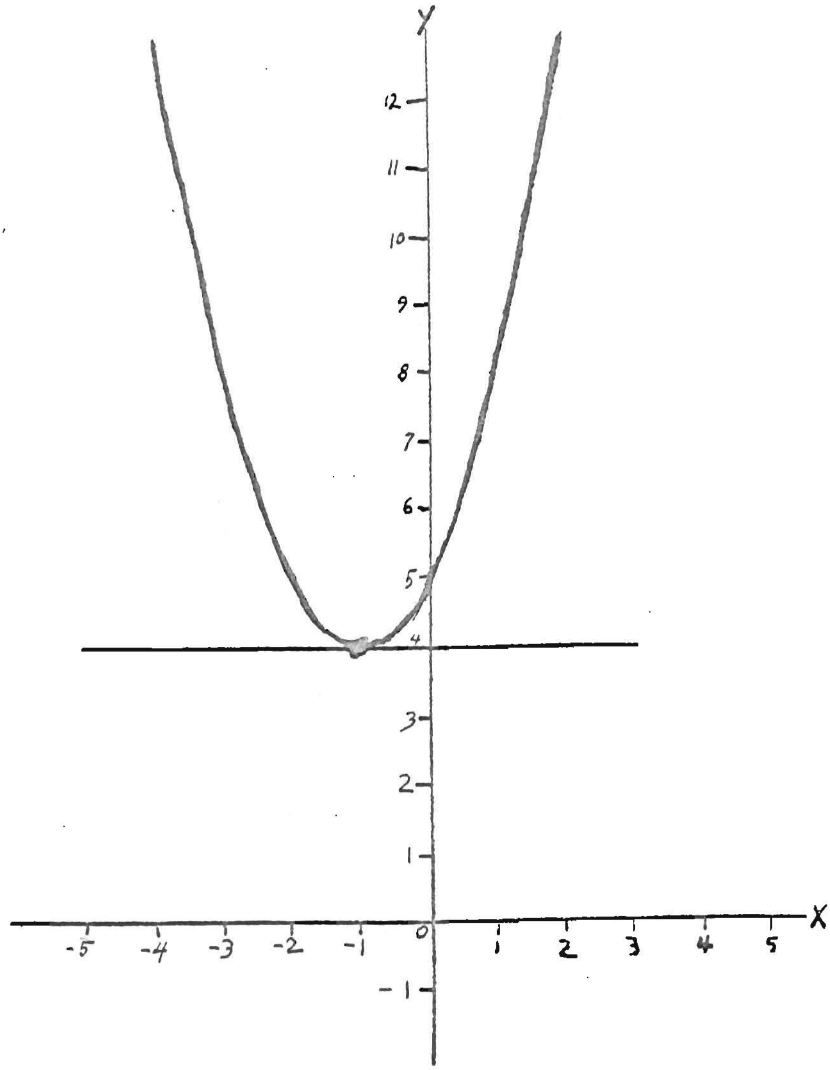

FIGURE 1(b)

$y = x^2 + 2x + 5$

THE $y = m$ TANGENT LINE SOLUTION OF $x^2 + 2x + 5 = 0$

$m = 4$, and the roots of the equation are $x = -1 \pm 2i$.

Let the cubic equation

$$x^3 + a_2x^2 + a_1x + a_o = 0$$

have one real root, r, and two complex roots, $a \pm bi$. From this

$$F(x) = (x^2 - 2ax + a^2 + b^2)(x - r).$$

Then

$$mf(x) = m(x - r).$$

The family $y = m(x - r)$ is a set of straight lines passing through $(r,0)$, and having slope m. One of these lines will be tangent to the curve $y = F(x)$. The complex roots of $F(x) = 0$ will be $a \pm bi$, where a is the abscissa of the point of tangency, and b is the square root of the slope of the tangent line.

For example, consider the graph of

$$y = x^3 + 2x^2 - (15/4)x - 17/2,$$

shown in Figure 2. Then

$$F(x) = (x^2 + 4x + 17/4)(x - 2),$$

and

$$mf(x) = m(x - 2).$$

The curve $y = F(x)$ intersects the x-axis at $x = 2$. Any secant line is then drawn through $(2,0)$, intersecting $y = F(x)$ in two other points, S_1 and S_2. The possible secant lines have the equation $y = m(x - 2)$. The midpoint of chord S_1S_2 has an abscissa of -2. The line tangent to the curve $y = F(x)$ at $x = -2$, is $y = 1/4\ (x - 2)$. Then b has the value $\sqrt{1/4}$, and the required complex roots of $F(x) = 0$ are $x = -2 \pm 1/2\ i$.

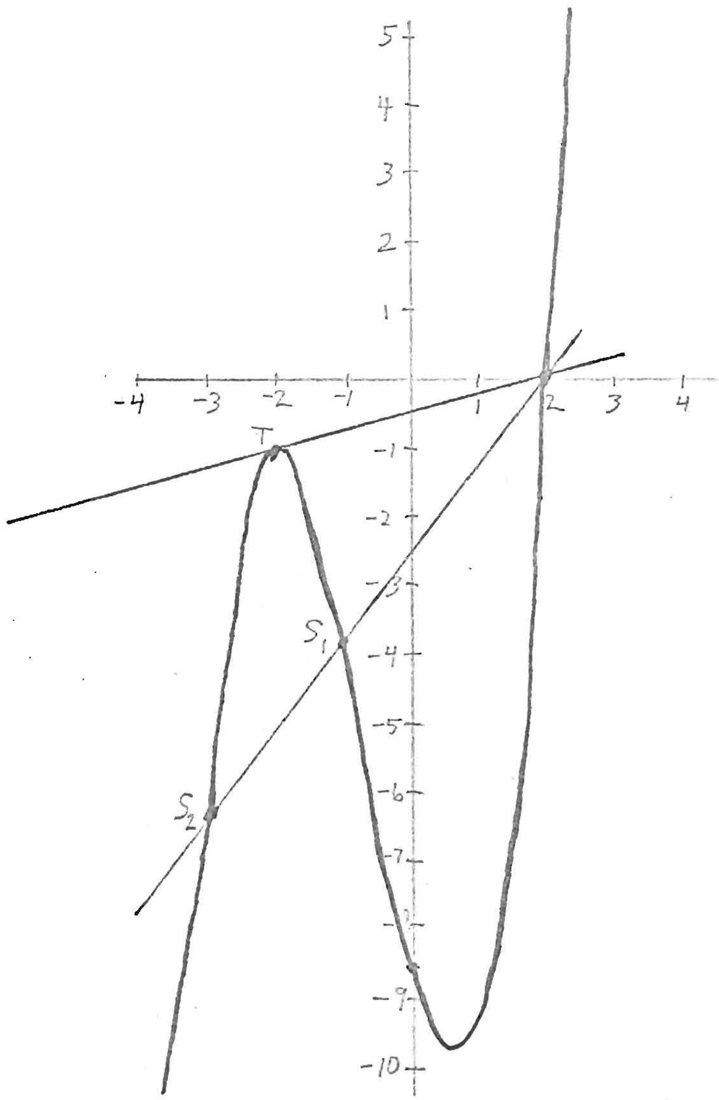

FIGURE 2

$$y = x^3 + 2x^2 - (15/4)x - (17/2)$$

Direct representation. The other method of indicating roots of an equation is more direct. For the purposes of this representation, $w = f(z)$ is the functional form used, where $z = x + iy$ and $w = u + iv$. Given any equation $f(z) = 0$, in which coefficients are real, numbers satisfying the equation may be identified directly from the graph of the function $w = f(z)$. If the independent variable is allowed to vary over the complex numbers, both variables will at times be complex, and each requires a plane for locating points, making it necessary to use four dimensions to obtain a complete graph. The points where the graph and the (x,y) plane intersect represent the roots of the equation.

In solving $f(z) = 0$ by graphing $w = f(z)$, only the three-dimensional cross-section, in which $v = 0$, of the four-space is needed, as the points where the graph crosses the (x,y) plane will always have a zero co-ordinate for w. This makes the graph for which w is real while z is complex ideal for representing the solution to this kind of problem [5, p. 410].

Consider the linear equation $w = az + b$. The independent variable z is allowed as a domain the entire field of complex numbers. Then w will have as a range $u + iv = (ax + b) + i(ay)$. If w is to be real, then $y = 0$. Thus, all real values of w are in the (x,u) plane, and the results are the same as in the graph of the function in real co-ordinates.

To represent roots of the quadratic equation

$$az^2 + bz + c = 0,$$

graph the function

$$W = az^2 + bz + c.$$

If $z = x + iy$, then

$$w = (ax^2 - ay^2 + bx + c) + iy(2ax + b).$$

If w is to be real, then $y = 0$ or $2ax + b = 0$. Therefore, the desired graph of $w = az^2 + bz + c$ consists of

$$u = ax^2 + bx + c, \tag{1}$$

the usual real graph of the quadratic equation on the real plane; and

$$u = -ay^2 - (b^2 - 4ac)/4a, \tag{2}$$

a parabola lying on the $x = -b/2a$ plane. The curve will intersect the (x,y) plane in two points in all cases. If the discriminant, $b^2 - 4ac$, is zero, the points are not distinct, but are in all cases where $b^2 - 4ac \neq 0$. If the discriminant is positive or zero, the parabola of the equation (1) intersects the x-axis, giving the same results as in the ordinary graph in the real plane. If the discriminant is negative, the curve of the equation (2) intersects the (x,y) plane, yielding two complex points.

As a specific example, consider the function

$$w = z^2 - 2z + 5.$$

Figure 3 shows that part of the graph for which w is real, that is, the graph whose domain has $z = x$, and $z = 1 + iy$. The curve lies on the planes $y = 0$ and $x = 1$, and intersects the (x,y) plane in points corresponding to the roots, $z = 1 \pm 2i$, of the quadratic equation

$$z^2 - 2z + 5 = 0.$$

There is a mechanical method for solving any quadratic graphically using a single well-made graph of the function $u' = (z')^2$ [11, p. 108].

This graph will consist of points on the (x,u) and (y,u) planes, as shown in Figure 4. To solve $az^2 + bz + c = 0$, obtain a unit coefficient for the z^2 term,

$$z^2 + (b/a)z + c/a = 0.$$

Reduce the function

$$u = z^2 + \frac{b}{a}z + \frac{c}{a} \tag{1}$$

to

$$u' = (z')^2. \tag{2}$$

To accomplish this, substitute

$$z = z' - b/2a$$

in (1). This reduces the function w to

$$u = (z')^2 - \frac{b^2 - 4ac}{4a^2}.$$

Graphically, the substitution translates the axes b/2a units along the x-axis. Let

$$\frac{b^2 - 4ac}{4a^2} = k.$$

Then

$$u = (z')^2 - k$$

Now let

$$u = u' - k.$$

Then

$$u' = (z')^2.$$

The last substitution translates the axes k units along the u-axis. The substitutions given result in axes translations only and have no effect on the form of the curve. On the graph of (2) can be found the roots desired, by following through the translations given.

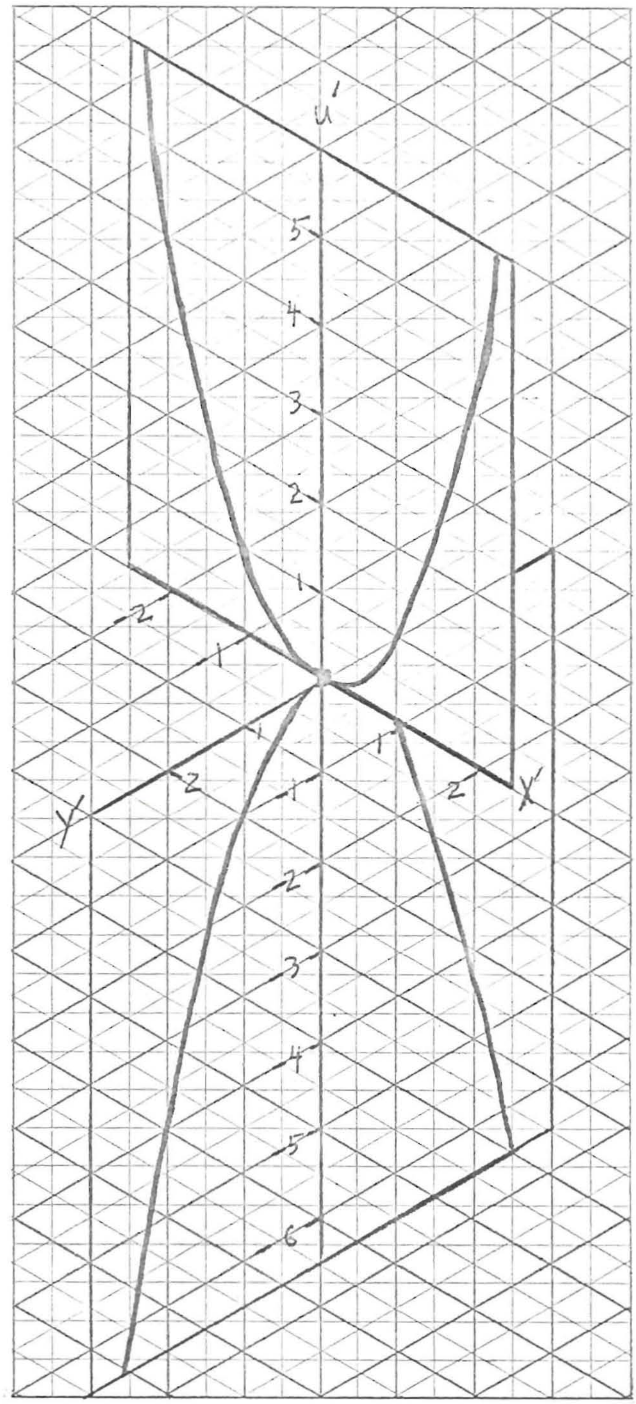

FIGURE 4

$u' = z'^2$

For example, to solve

$$z^2 - 2z + 5 = 0,$$

let

$$z = z' + 1.$$

The equation thus formed is

$$(z')^2 + 4 = 0.$$

Move -4 units on the u-axis on Figure 4, and read the corresponding values of z', $z' = \pm 2i$. Since $z = z' + 1$, the desired roots are

$$z = 1 \pm 2i.$$

In order to find the roots of a cubic equation by the present method, it is necessary to graph the function

$$u = z^3 + bz^2 + cz + d.$$

The curve will appear as the graph of

$$u = (z')^3 + c'z' + d'$$

if a proper translation of the axes is made. The substitution in the first function which accomplishes this translation is

$$z = z' - b/3,$$

and it translates the axes b/3 units along the x-axis. The form of the graph depends only on the resulting coefficient of z'. For this reason, only the cubic of the form

$$u = z^3 + cz + d \tag{1}$$

will need to be considered here.

Since $z = x + iy$,

$$u = x^3 - 3xy^2 + cx + d + iy(3x^2 - y^2 + c). \tag{2}$$

As u is real, the coefficient of i must be zero. The real curve is obtained by letting $y = 0$, from which

$$u = x^3 + cx + d, \tag{3}$$

a curve lying on the (x,u) plane. This plane intersects the (x,y) plane in the line

$$y = 0,\ u = 0. \tag{4}$$

The supplementary is obtained by letting $3x^2 - y^2 + c = 0$, and will lie on the surface

$$y^2 - 3x^2 = c,$$

which intersects the (x,y) plane in the hyperbola

$$y^2 - 3x^2 = c,\ u = 0. \tag{5}$$

The line will be the major axis of the hyperbola if c is negative, and its conjugate axis if c is positive.

The supplementary intersects the (x,u) plane and the real curve at the extrema points of the real curve, if those points are real. The projection of the supplementary onto the (x,y) plane is the hyperbola (5). If the substitution

$$y^2 = 3x^2 + c$$

is made in (2), the result is the cubic

$$u = -8x^3 - 2cx + d, \tag{6}$$

which represents the projection of the supplementary onto the (x,u) plane, for values of x at and beyond the extrema points of (3).

Those points of the graph of (1) which are also points of the hyperbola (5), if any, will represent the complex solution of

$$z^3 + cz + d = 0. \tag{7}$$

The point or points in common with the x-axis will show the real solution.

The graph of

$$u = z^3 + cz + d$$

will take one of three forms, depending on the value of the coefficient of z. Figures 5, 6, and 7 show these three forms. In Figure 5, c is negative, in Figure 6, c is zero, while in Figure 7, c is positive. These graphs are made with $d = 0$. The real part of each graph, lying in the (x,u) plane, is the cubic (3), as it is usually graphed using real variables.

If c is negative, the graph appears as in Figure 5. The supplementary intersects the real graph in the extrema points of the real graph. If the supplementary is projected onto the (x,y) plane, the resulting locus is the hyperbola (5), with the x-axis being the transverse axis of the hyperbola. If the supplementary is projected onto the (x,u) plane, the resulting locus is the cubic (6), for those values of x beyond the extrema points of (3). As c increases in value, the extrema points of the real graph move closer to the origin, and the two branches of the supplementary move closer to the u-axis.

When $c = 0$, the graph appears as in Figure 6. Here, the real graph and both branches of the supplementary intersect at the origin. The real graph (3) is the cubic

$$u = x^3,$$

the projection of the supplementary to the (x,y) plane is the two intersecting lines from (5),

$$y - 3x = 0 \text{ and } y + 3x = 0,$$

and the projection of the supplementary to the (x,u) plane is the

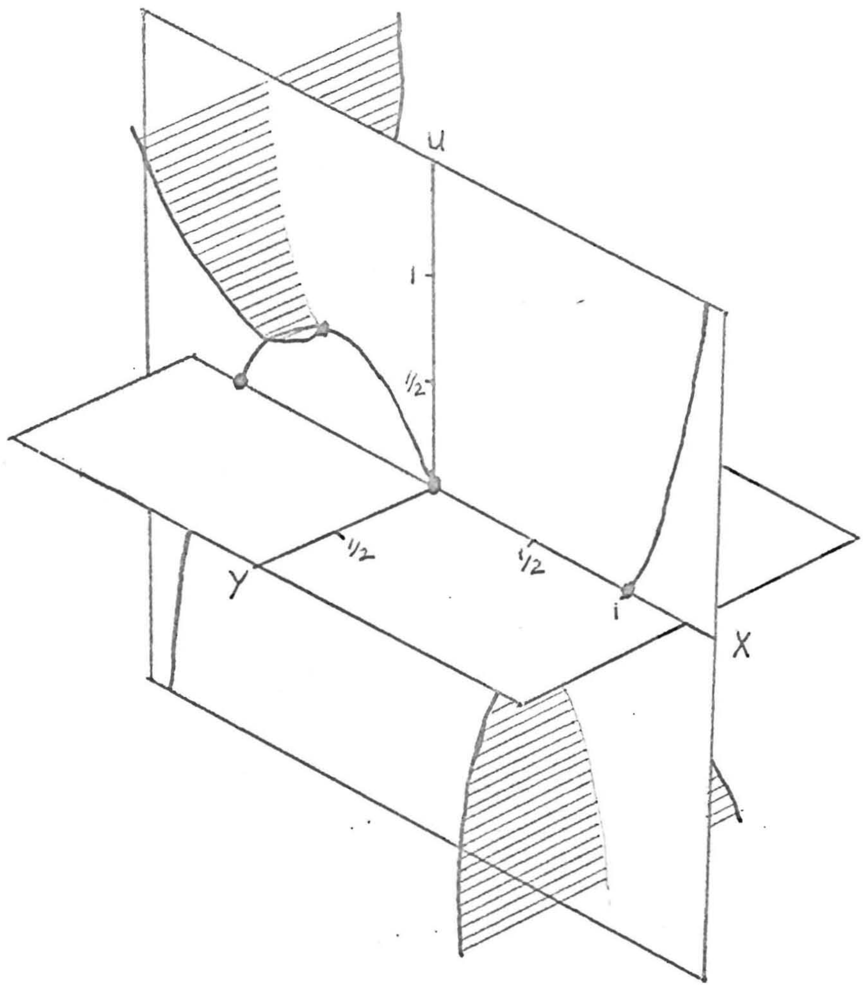

FIGURE 5

$u = z^3 - z$

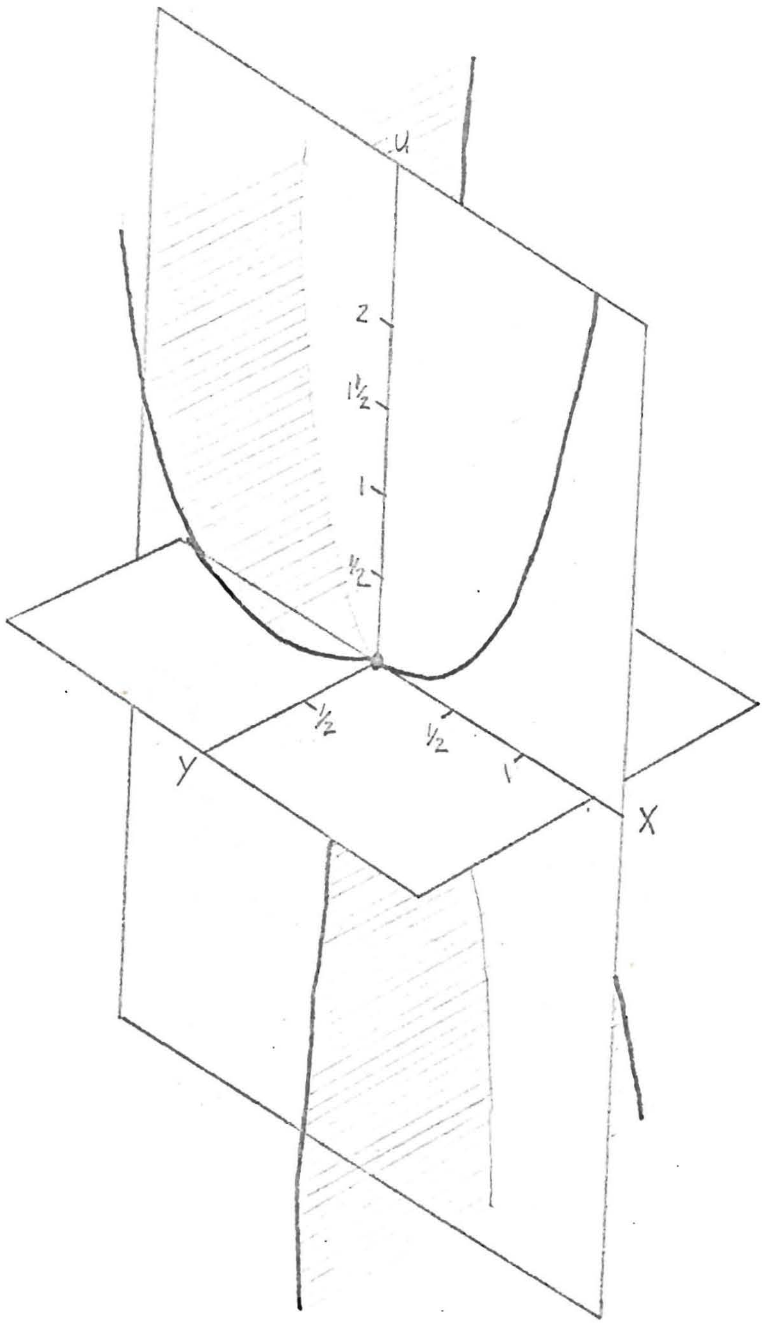

FIGURE 6

$u = z^3$

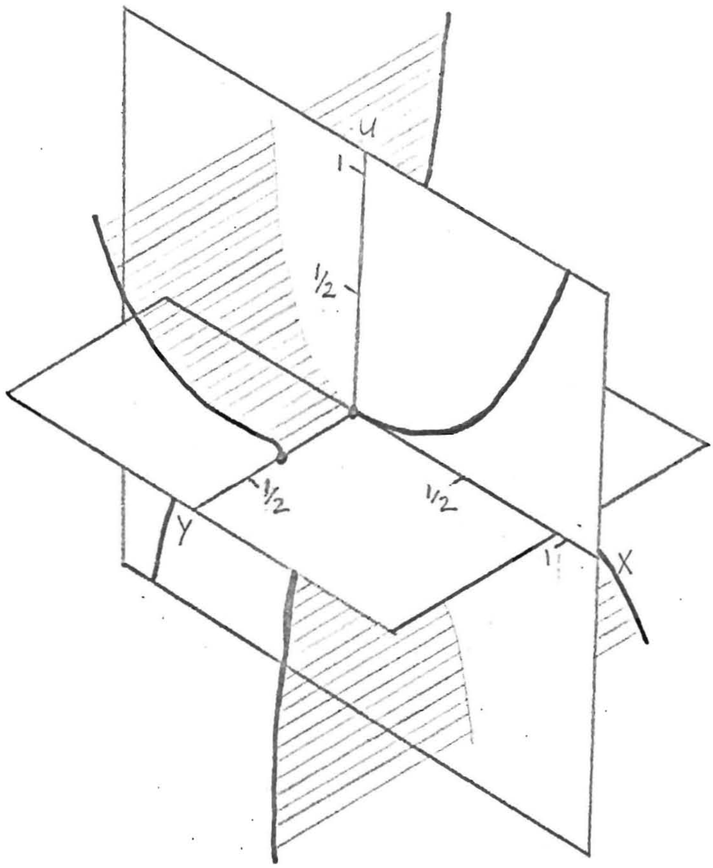

FIGURE 7

$u = z^3 + (1/10)z$

cubic from (6),

$$u = -8x^3.$$

The third form the cubic may take is when c is positive, as shown in Figure 7. Here the projection of the supplementary is still the cubic (3) on the (x,u) plane and the hyperbola (5) on the (x,y) plane, but now the transverse axis of the hyperbola is the y-axis. As c continues to increase, the supplementary approaches two straight lines, as indicated in Figure 8.

If d = 0, the axes will be translated -d units along the u-axis. When the supplementary intersects the (x,y) plane, the complex solutions to the equation (6) are represented by the points of intersection.

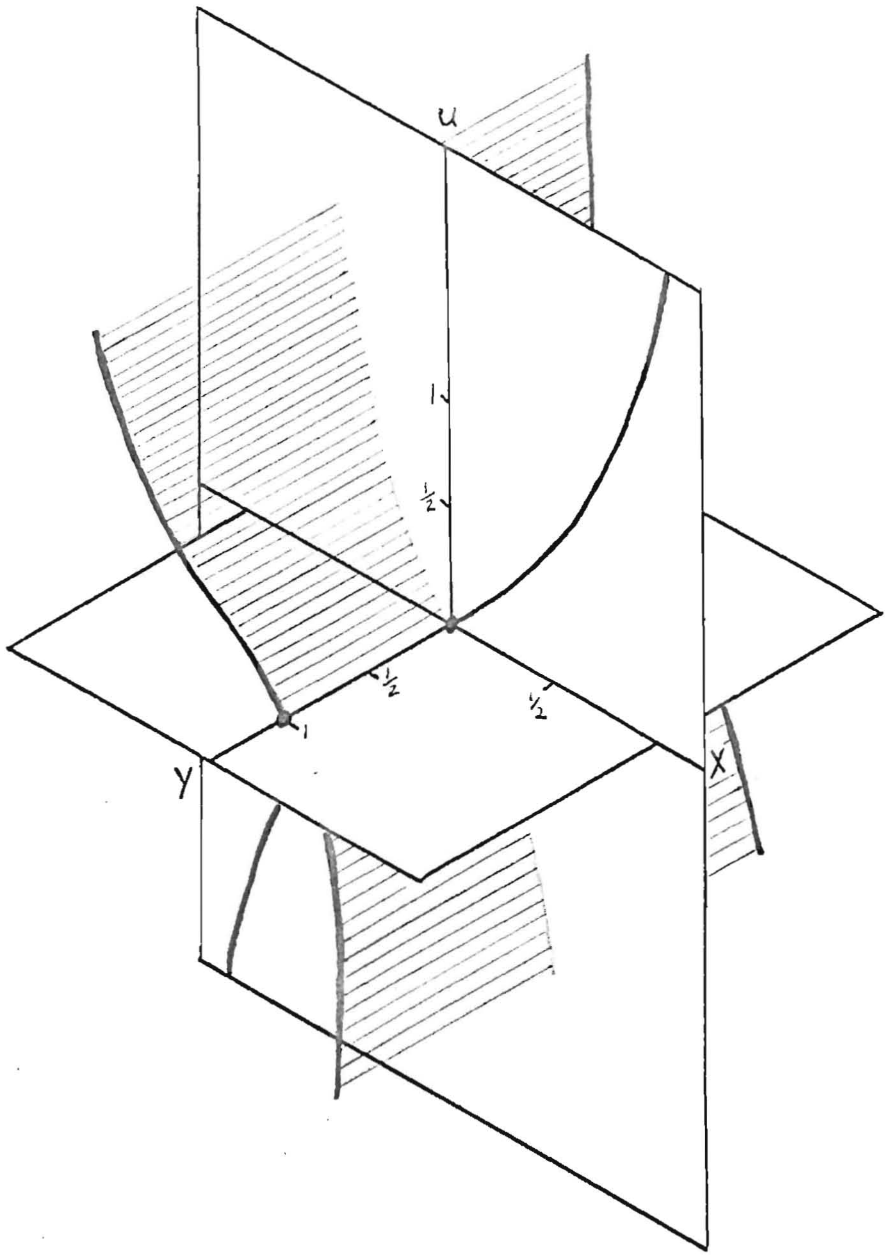

FIGURE 8

$u = z^3 + z$

CHAPTER IV

THE INTERSECTION OF TWO CURVES

One of the first uses of graphs the student of mathematics encounters is that of finding the common solution to two equations. Geometrically, this is represented by the points of intersection of the graphs of the two equations. The co-ordinates of these points correspond to the algebraic solution.

Number of points of intersection. One of the important theorems having to do with the intersections of curves is that attributed to Bezout. This theorem states that, given a curve of degree m and a second curve of degree n, the two curves will intersect in mn points [12, p. 54]. These points need not be distinct, and some or all of them may be points added to the plane, commonly called ideal points of points at infinity.

The set of points at infinity, called the infinite domain, of a complex plane consists of a single ideal point corresponding to the value of z_0 when

$$z_0 = \lim_{|z|\to 0} \frac{1}{z}$$

for z a complex number. This will provide one ideal point on each of the two complex planes of the four-space determined by the axes x, y, u, and v, and the ideal point may be approached by going in any direction on these planes. On other planes of the space, such as the (x,v) plane, a single point at infinity is added to the plane for each line which has a distinct slope. Thus, each set of parallel lines will have a point in common. Each line in the space has one and only one ideal point on it.

Graphical representation. The real graphs of two equations will indicate the common points when the equations are linear, and for many equations of higher degree. However, if in the simultaneous solution of two equations, when at least one of them is of degree two or more, not all of the solutions are necessarily real, and the complex solutions do not appear on the real graph.

For instance, if

$$z = x + iy \text{ and } w = u + iv,$$

the graph of the parabola

$$w = z^2$$

and of the straight line

$$w = z - 1$$

do not intersect in the real plane. Yet algebraically, they have two points of intersection, $\left(\frac{1 + i\sqrt{3}}{2}, \frac{-1 + i\sqrt{3}}{2}\right)$ and $\left(\frac{1 - i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}\right)$. This chapter shows how a graph may be drawn which will demonstrate intersections which show complex roots as actual geometric entities.

Consider, as a first example, the two circles

$$z^2 + w^2 = r^2$$

and

$$(z - a)^2 + w^2 = r^2,$$

where r is real. Algebraically, these equations yield a simultaneous solution consisting of the ordered number pairs (z,w), $\left(\frac{a}{2}, \frac{\sqrt{4r^2 - a^2}}{2}\right)$ and $\left(\frac{a}{2}, -\frac{\sqrt{4r^2 - a^2}}{2}\right)$. The circles intersect in two points, which will be not real if a^2 is greater than $4r^2$.

In representing graphically the intersection of these two circles, if

$$a^2 \leq 4r^2,$$

the intersection points will have $y = v = 0$. If

$$a^2 > 4r^2,$$

the intersection points will have $y = u = 0$. In either situation, $y = 0$. Therefore, a graph which will always exhibit the intersections may be made in a three-space, using x, u, and v for axes, with $y = 0$.

The equation of a circle with center at the origin and real radius r, when expanded, becomes

$$(x^2 + u^2) - (y^2 + v^2) + (2xy + 2uv)i = r^2.$$

From this it follows that

$$(x^2 + u^2) - (y^2 + v^2) = r^2 \tag{1}$$

and

$$xy + uv = 0. \tag{2}$$

If y is zero then, from (2), u is zero or v is zero. If v is then zero, from (1),

$$x^2 + u^2 = r^2,$$

which is the equation of the real circle in the (x,u) plane. If u is zero,

$$x^2 - v^2 = r^2,$$

which represents an equilateral hyperbola, and appears as such in the (x,v) plane, but is actually a part of the locus of the circle.

Figure 9 is the graph of

$$z^2 + w^2 = 4.$$

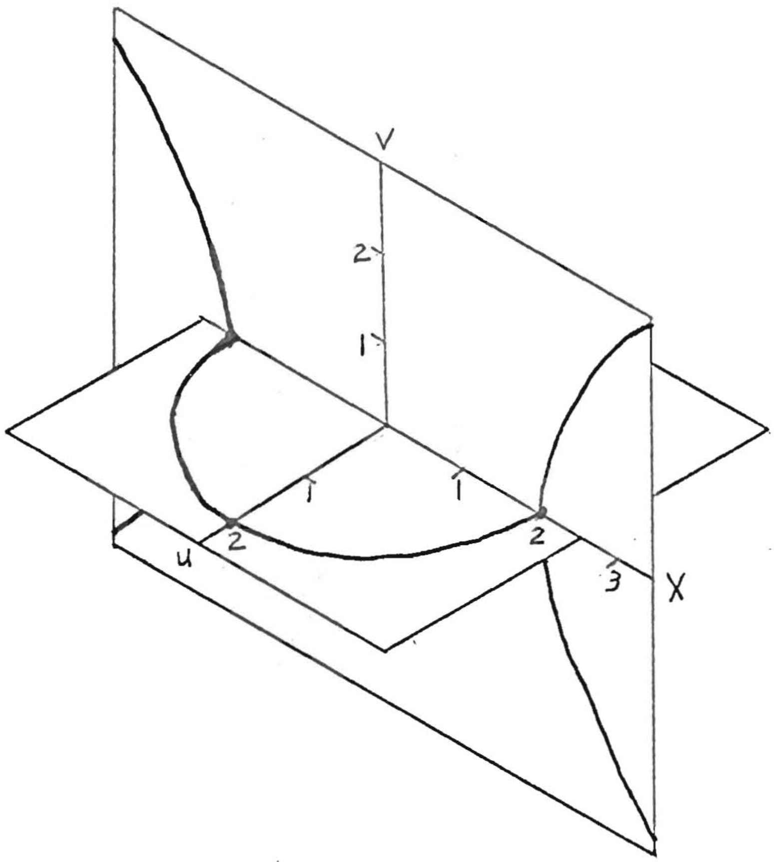

FIGURE 9

$z^2 + w^2 = 4, \ y = 0$

The curve consists of the real circle in the (x,u) plane which, at the points (2,0) and (-2,0), undergoes an abrupt right-angle change onto the (x,v) plane and follows the path of what appears to be a hyperbola. These points satisfy the equation of the circle, so the points of the "hyperbola" are actually points of the circle.

Circles with no real points of intersection. Now if the circle

$$(z - 6)^2 + w^2 = 4$$

is graphed on the same axes, the intersections of the two circles, although in the (x,v) plane, are seen by Figure 10 to be actual occurrences. The imaginary branches of the circles intersect at the points $(3, \sqrt{5}\, i)$ and $(3, -\sqrt{5}\, i)$.

As circles have equations of degree two, reference to Bezout's Theorem indicates there should be four points of intersection. The two points not shown will be points at infinity [2, pp. 69-78], and the graph of the curves indicates, in an intuitive way, these points. If the asymptotes of the hyperbolic-type supplementarys of the circles in the (x,v) plane are drawn, they will, by definition, intersect the curve in an ideal point. Since the asymptotes of the two circles are parallel, they also intersect in an ideal point, giving the other two points of intersection of the circles as points at infinity.

A graph in four variables. While there are many instances of the type just given of the intersections of two curves where the said intersections occur in planes determined by two of the four axes, these remain special cases of a much more general problem. A more elaborate

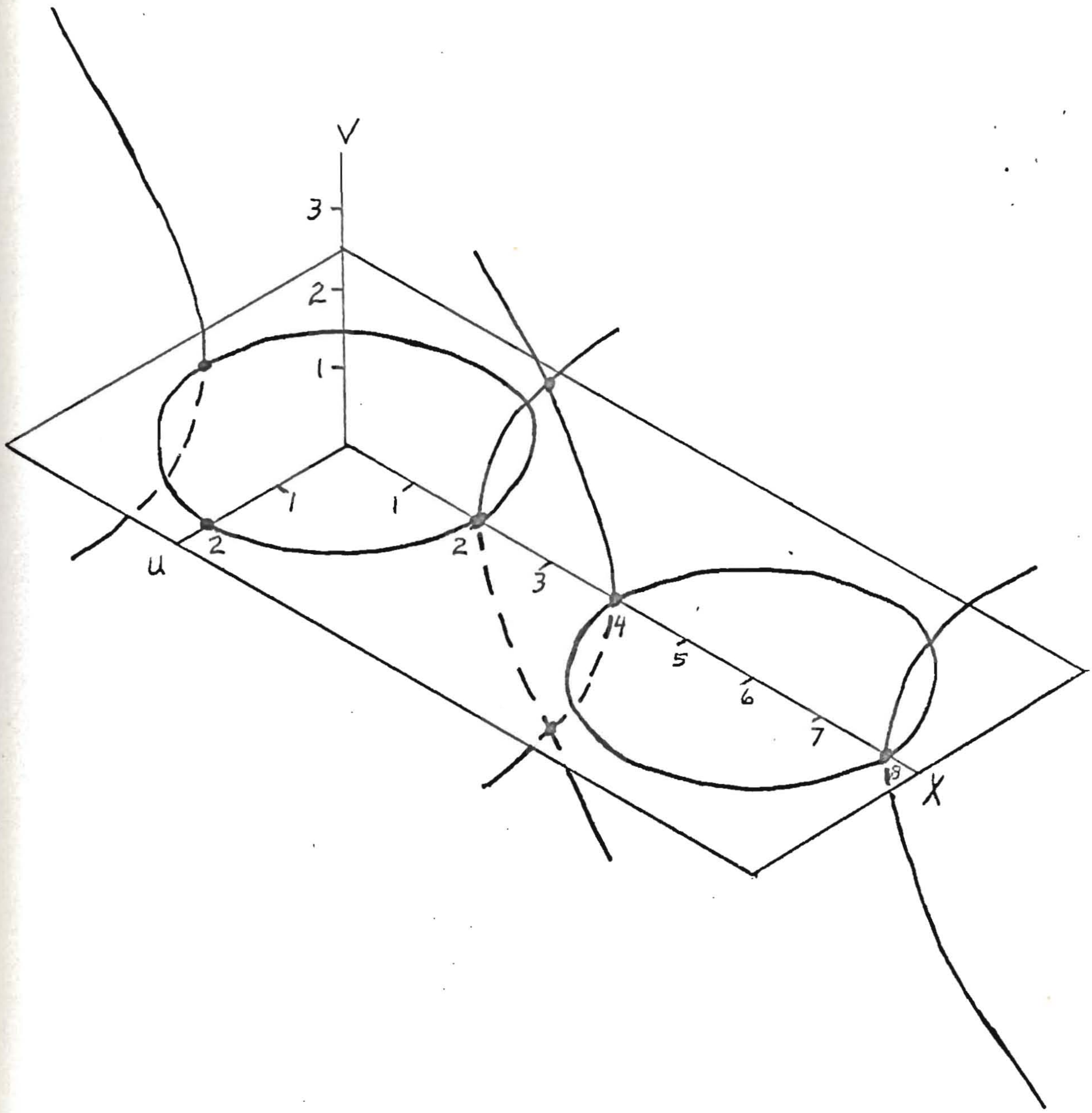

FIGURE 10

THE INTERSECTION OF

$z^2 + w^2 = 4$ AND $(z - 6)^2 + w^2 = 4$

graph must be devised to show intersections of the type represented by the problem given at the opening of this chapter. For the solution of this problem, all four of the variables x, y, u, and v must be represented simultaneously.

This can be accomplished by a partial departure from the representation of points in a strictly Cartesian co-ordinate system. The variables x, y, and u might be represented in a three-axis system with v represented by graphing x, y, and u for various values of v. There are at least two ways this can be done. While both are essentially the same, they are separated here for purposes of application, the first being a more complete representation, with the second probably easier to visualize in the context of the present problem.

Figure 11 depicts the graph of

$$w = z^2$$

for discreet values of v where v attains the values zero, one-half, one, and two. Only values of u greater than zero are shown. The points of the curve lie on the surfaces $v = 2xy$, which cut the (x,y) plane in the family of hyperbolas

$$v = 2xy, \; u = 0,$$

here called a path equation. As x and y vary on a path v, values of u are obtained from

$$u = x^2 - y^2,$$

here called a length equation. These equations are obtained from the expansion of

$$u + iv = (x + iy)^2$$

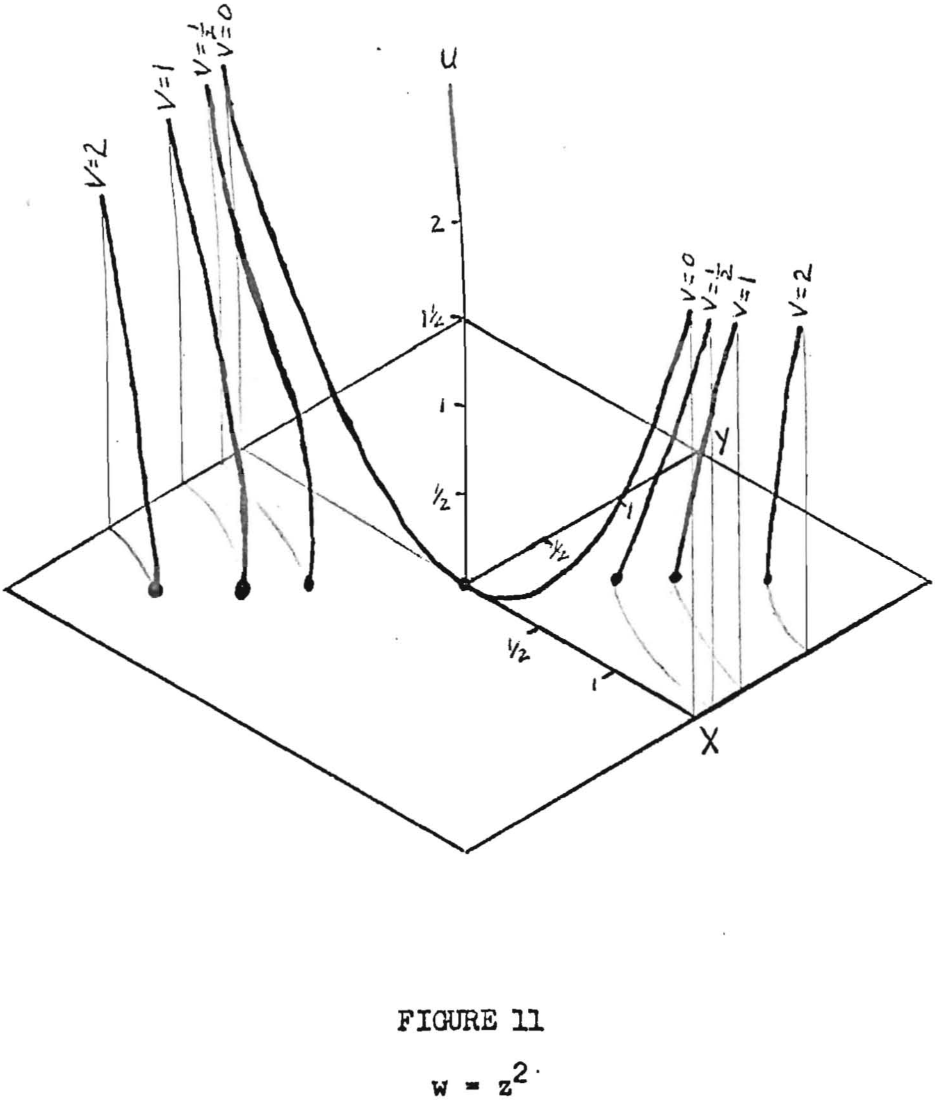

FIGURE 11

$w = z^2$.

yielding

$$u + iv = x^2 - y^2 + 2xyi,$$

producing the path and length equations just given. This graph gives an idea of what happens between, and beyond, those values of v shown by interpolation and extrapolation, and, indeed, indicates a surface with values of v as parameters, producing curves on the surface. By graphing a function from different vantage points, or with one of the other variables as the parameter, a better idea of how the function behaves may sometimes be obtained, although any one graph is complete.

The graph of the linear function

$$w = z - 1$$

is shown in Figure 12 as a plane in three-space with v as a parameter. Now if the graph of Figure 12 were superimposed on that of Figure 11, the intersections of the surfaces at the points where the parameters are equal would give the simultaneous solution. However, the graphs then become rather difficult to visualize. A somewhat different approach might be adopted at this point.

Instead of illustrating intersections by means of a graph attempting to show continuous values of all four variables, a series of graphs, each representing a single value of one variable graphed against the other three variables may be used. Thus the graph of

$$w = z^2$$

is shown in Figure 13 for values of v of 0, 1, and 2. Part (a) shows the curve for $v = 0$, with the $u \geq 0$ part that part of the parabola lying in the (x,u) plane, the plane of reals. The $u \leq 0$ part of the

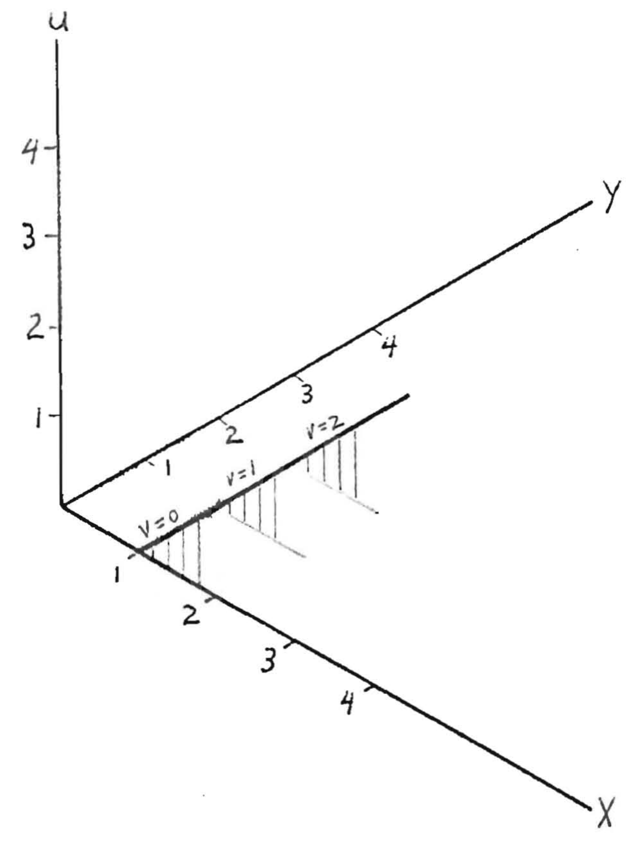

FIGURE 12

$w = z - 1$

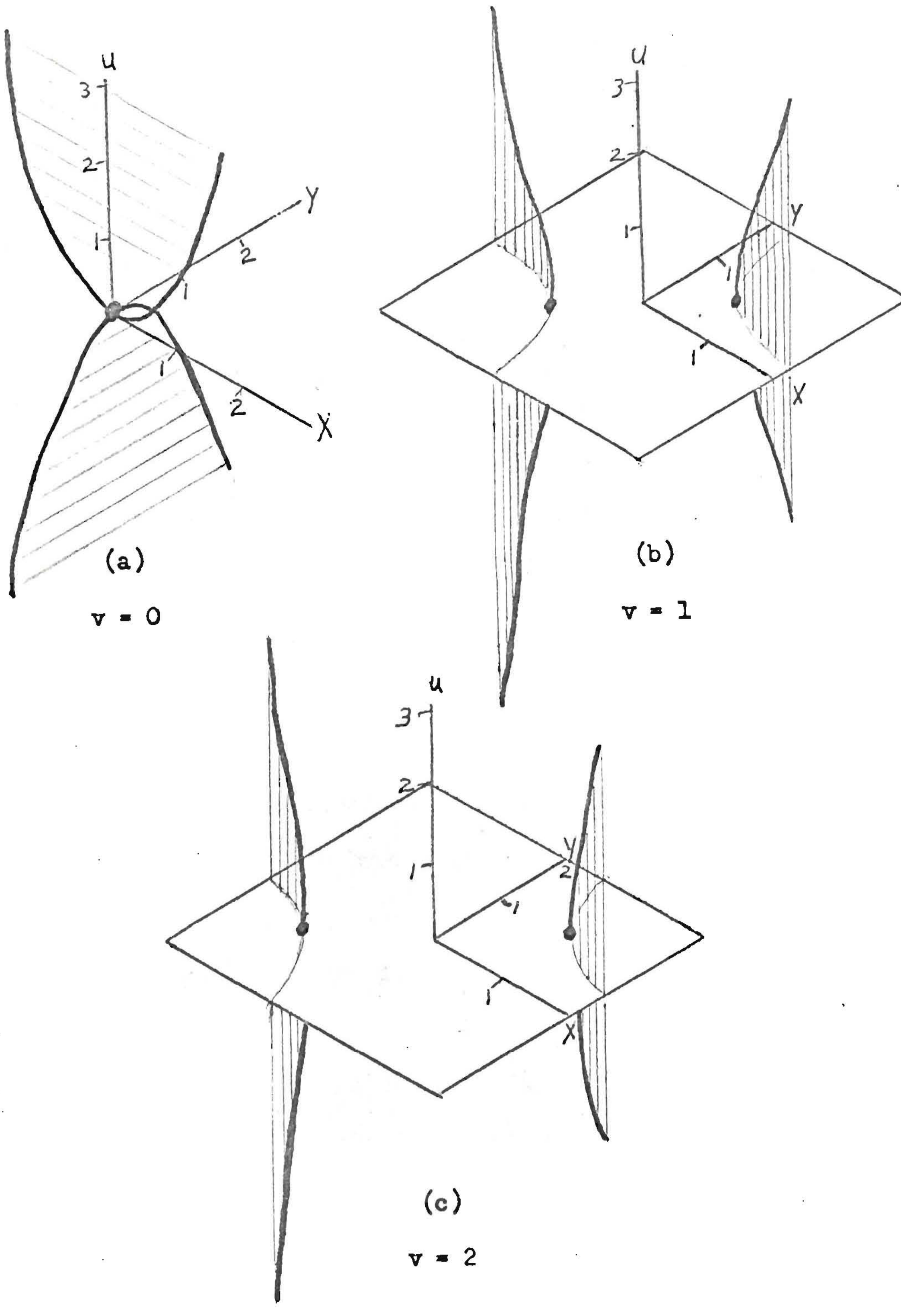

FIGURE 13

$w = z^2$

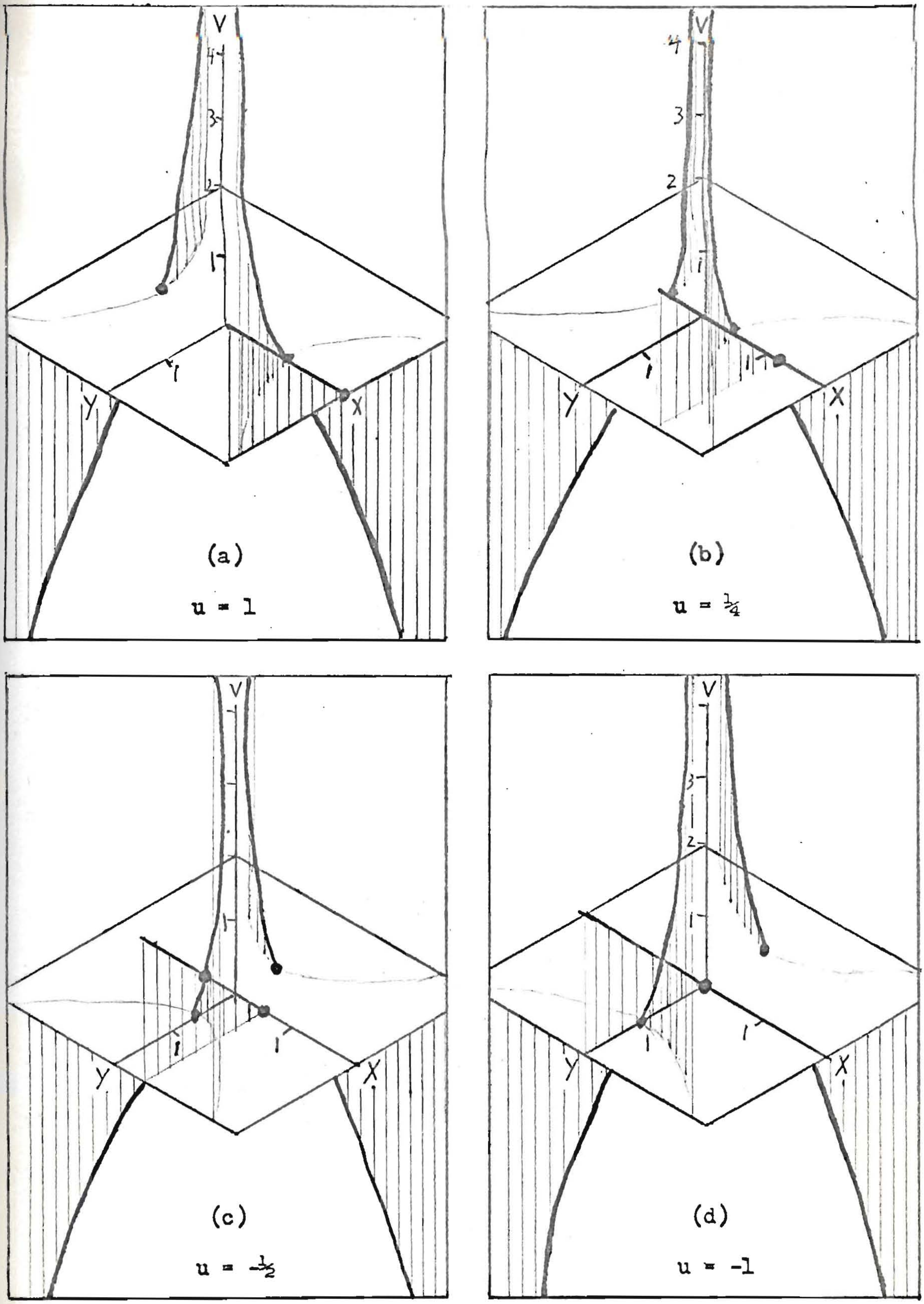

FIGURE 14

THE INTERSECTION OF $w = z^2$ AND $w = z - 1$

for instance, gives

$$(u^2 + x^2) - (v^2 + y^2) = r^2$$

and

$$uv + xy = 0.$$

Here, no apparent path equations in any plane are present, nor do length equations appear. Even more sophisticated procedures must be developed for these. Before turning to this more general type of problem, it is first shown when the procedures developed above may be applied.

<u>Types of functions to be graphed</u>. The function

$$w = f(z) \tag{1}$$

can be written in the form

$$u + iv = \phi(x,y) + i\,\psi(x,y). \tag{2}$$

From the definition of equality of complex numbers, it follows that

$$u = \phi(x,y) \tag{3}$$

and

$$v = \psi(x,y). \tag{4}$$

Thus, for any given value of v, a definite relationship exists between x and y, and from each of these, a value for u. Hence, (4) may serve as a path equation in the (x,y) plane for each value of v, with (3) as a length equation. Therefore any explicit function of the form (1), from which functions of the form (3) and (4) may be obtained, can be graphed in the manner described.

For implicit functions of the form

$$f(w,z) = 0, \tag{1}$$

there are some procedures which can be used that will provide a graph of the function. For this, resolve the function into its component parts,

$$\phi(x,y,u,v) + i\psi(x,y,u,v) = 0,$$

from which

$$\phi(x,y,u,v) = 0 \tag{2}$$

and

$$\psi(x,y,u,v) = 0 \tag{3}$$

are obtained. Define any convenient function

$$y = \mu(x). \tag{4}$$

The type of function defined may be a straight line, circle, hyperbola, or some other curve, in the (x,y) plane, suggested or dictated by the original function. Using this defined function as a path in the (x,y) plane, f then serves as an operator, transforming (4) in the (x,y) plane into some locus

$$h(u,v) = 0 \tag{5}$$

in the (u,v) plane. The function (5) is obtained by substituting (4) into (2) and (3) and eliminating x from these, if possible.

Since values of x and y do not appear in the final result, graphs of both (4) in the (x,y) plane and (5) in the (u,v) plane need to be made with occasional indications as to where certain points of (4) map to (5) [5, p. 418].

Because of the wide variety of functions possible which may not be satisfactorily served by the methods of this chapter, the next step in the progress to more sophisticated problems is the theory of complex functions where maps of regions in the domain plane are studied in some detail. Most textbooks on complex variables or complex functions will treat this matter. Churchill, for instance, gives an appendix showing the transformations of some of the more widely used functions [1, pp. 284-291].

Conclusion. This chapter shows a method for representing graphically the intersection of two curves when the points of intersection are complex as well as when they are real. In order to accomplish this in the general sense, each point of a curve must, in some way, exhibit the relation of the four variables, x, y, u, and v. Hence, some procedures that may be used for making such a representation are given. The graphs of explicit functions are found to be a special case of the more general graphs with implicit functions. The situation where all intersections of two curves occur in the (x,u,v) space with y = 0 is a special, although very useful, case of explicit function. While many functions do not lend themselves nicely to simultaneous solutions in the present context, it has been demonstrated that such a process is often possible, and not particularly difficult for the simpler functions.

CHAPTER V

THE CIRCLE

Introduction. This chapter is concerned primarily with applications of graphs to problems involving circles. The specific problems considered are the tangent to a circle from a point, the representation of circles with imaginary radius, the representation of circles with radius zero, the radical axis to two circles, and the relationship existing between the circular (trigonometric) functions and the hyperbolic functions.

Tangent to a circle. In the study of elementary analytic geometry, there are a number of things of importance concerning the tangent to a curve, and especially to a circle, such as the equation of the tangent line at a point on the circle and the length of the tangent from a point. Usually the tangent to a circle is stated to have been drawn from a point outside the circle. It is of interest here to represent the tangent to a circle from a real point, with no restrictions on where that point is.

Consider the relation

$$z^2 + w^2 = r^2.$$

When z and w are real, this is the equation of a circle of radius r and with center at the origin. It is here still considered as a circle when $z = x + iy$ and $w = u + iv$. Since the tangents desired are from real points, $y = 0$. Without loss of generality, due to the symmetry of the circle, only tangents from the points on the x-axis will be considered.

If the abscissa of a point on the circle is x, the point will have an ordinate $w = \pm \sqrt{r^2 - x^2}$. When $x^2 \leq r^2$, then $w = u$; otherwise $w = iv$. Therefore, the graph in three dimensions of x, u, and v, with $y = 0$, will be used [5, p. 415].

Figure 15 shows the circle

$$z^2 + w^2 = 9, \; y = 0,$$

with the part of the locus that falls in the (x,u) plane represented by

$$x^2 + u^2 = 9$$

and the supplementary in the (x,v) plane represented by

$$x^2 - v^2 = 9.$$

For any point (h,0) on the (x,u) plane such that $h^2 > 9$, the tangent from the point to the circle lies in the (x,u) plane. This is the usual tangent to a circle from a point outside the circle. When $0 < h^2 < 9$, the point lying within the real circle, the tangent passes out of the (x,u) plane into the (x,v) plane, and is tangent to one of the hyperbolic branches of the supplementary, the tangents being drawn to the "right" branch if $h > 0$ and to the "left" branch if $h < 0$. For instance, the line from the point (1.8,0) tangent to the circle is found to be tangent at the point (5,4) in the (x,v) plane, that is, the point (z,w)= (5,4i).

Three special points have need to be considered individually. These are the points (h,0) on the (x,u) plane with $h = \pm r$, where r is the radius of the circle; and the center of the circle. If the point is on the (x,u) plane with $h = \pm r$, then lines lying in both the (x,u) and the (x,v) planes perpendicular to the x-axis at the points designated

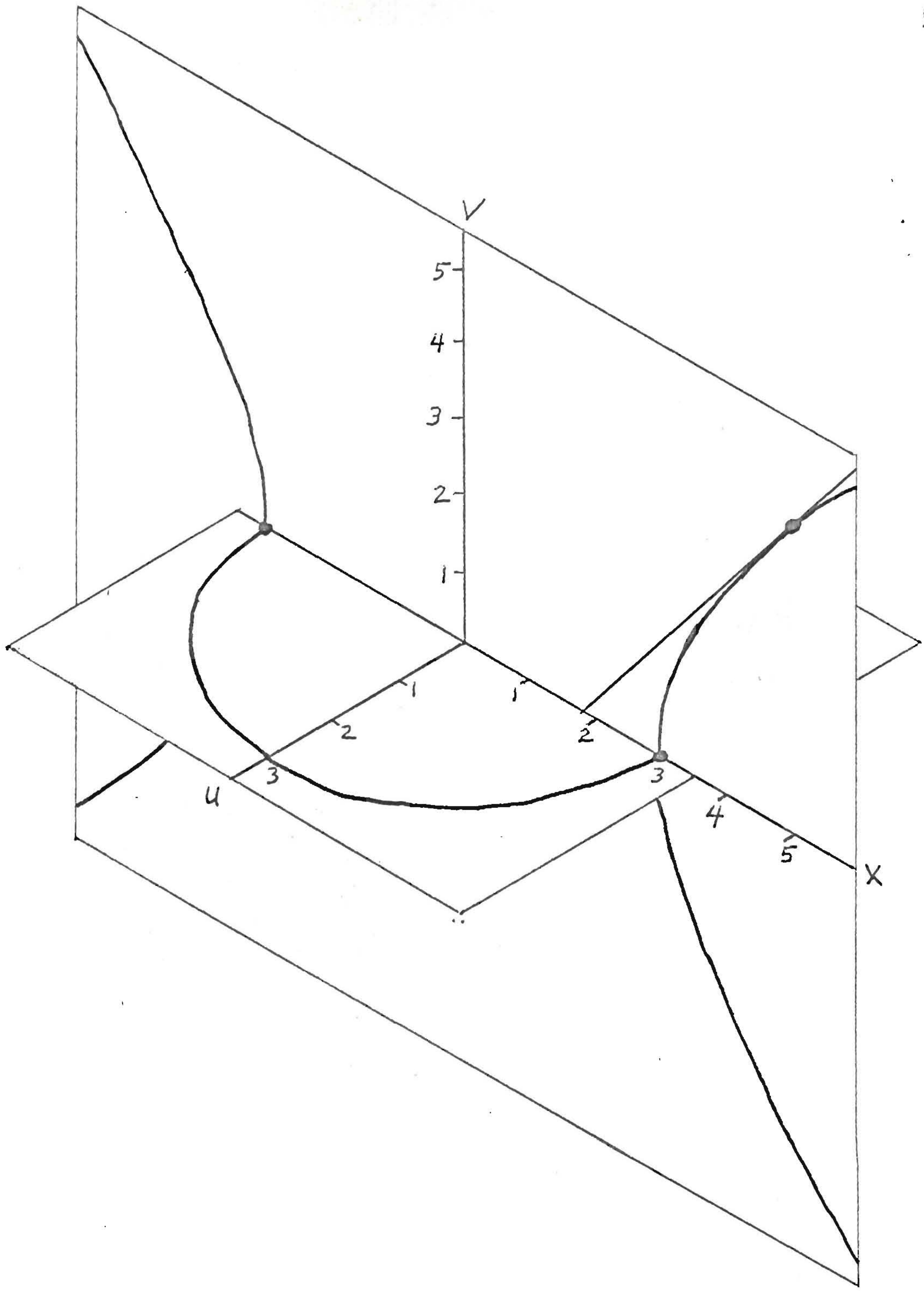

FIGURE 15

TANGENT LINE TO $z^2 + w^2 = 9$ at $(z, w) = (5, 4i)$

will be tangent to the curve. The equations

$$z = \pm r$$

represent two planes whose intersections with the (x,u) and (x,v) planes are the tangents to the real curve and the supplementary, respectively. This leaves the point $(0,0)$, the center of the circle, from which it is desired to draw a tangent to the circle. It will be remembered that the supplementary of the circle, lying in the (x,v) plane, has the equation

$$x^2 - v^2 = r^2$$

which has the form of an equilateral hyperbola whose asymptotes pass through the origin. Therefore these asymptotes may be considered as tangents to the circle, intersecting the circle at the ideal points of the asymptotes.

The equation of a tangent line to a circle of radius r at a point (a,b) on the circle $z^2 + w^2 = r^2$ is

$$az + bw = r^2.$$

This holds true whether or not the point (a,b) is on the real plane. As an illustration, consider the point $(5,4i)$ on the circle. Then the equation of the tangent line is

$$5z + 4wi = 9.$$

When expanded, this gives

$$5x - 4v = 9 \text{ and } 5y + 4u = 0.$$

Since $y = 0$ in the cross section being used here, then $u = 0$ and only the first of these lines lies in the (x,u,v) space. It is, indeed, the tangent to the circle, lying in the (x,v) plane, and passing through the x-axis at $x = 1.8$ as shown in Figure 15.

Power of a point. Given a circle

$$z^2 + w^2 = r^2$$

and a point (h,k), the power of the point is defined to be t where

$$t^2 = h^2 + k^2 - f^2.$$

If the point (h,k) is outside the circle, t is real and is the length of a tangent from the point to the circle. If the point is on the circle, the power of the point is zero. If the point lies within the circle, t is imaginary. Although when t is imaginary, it is not a distance, there are certain observations related to the idea of a distance, that can be made. Suppose this equation is applied to a point on the x-axis such that $0 < x^2 < r^2$. Then $t^2 = x^2 - r^2$. Now since

$$0 < |x^2 - r^2| < r^2,$$

t will always be an imaginary number with coefficient less than the measure of the radius of the circle. Then the absolute value of the power of a point within the circle, but not at the center, is less than the radius of the circle since $|ir| = r$. Applying the formula for the power of a point to the center of the circle, it is found that $t = ir$, and hence $|t| = r$.

The locus definition of a circle. The circle is often defined as the locus of points a given distance from a given point. If z and w are real, then the distance d between points (z_1,w_1) and (z_2,w_2) is determined by the equation

$$d = \sqrt{(z_2 - z_1)^2 + (w_2 - w_1)^2}.$$

In the more general situation under consideration here where z and w are not both real, then d is not necessarily a distance. Again, though, certain observations can be made in comparing this to the idea of a distance. If the value of d is computed for two points when one is the center of a circle and the other is on the supplementary, then d is equal to the radius of the circle. For instance, the point $(5,4i)$ is on the supplementary of the circle of Figure 15, page 52, whose radius is 3. Then

$$d = \sqrt{(5-0)^2 + (4i-0)^2} = 3.$$

Certain difficulties arise in trying to consider the relation d as being a distance metric in the present context. Consider the line of slope i, that is, the line whose co-ordinates (z,w) have the form (a,ai) where a is real. Lines of this type are sometimes defined to be isotropic lines [8, p. 121]. Using d as a distance relation, the distance between two points on this line, say (a,ai) and (b,bi), $a \neq b$, is

$$d = \sqrt{(b-a)^2 + (bi-ai)^2} = 0.$$

Thus the distance between any two finite points on such a line is zero. Now one of the conditions for a metric, defined on a space, is that if the distance between two points is zero, the points are the same. As this is not true in this case, the relation d cannot be a distance metric. Still, it is interesting to see the results of comparing the relation to the idea of the distance between two points.

<u>Circles with imaginary radius</u>. Consider the relation

$$z^2 + w^2 = a, \; a < 0.$$

If the $y = 0$ cross-section is taken as before, then the expansion of the relation gives

$$x^2 + u^2 = a \tag{1}$$

and

$$x^2 - v^2 = a. \tag{2}$$

Now (1) is not possible since a is negative and x and u are real. Therefore (2) is the only possible locus. This is in the form of a hyperbola in the (x,v) plane conjugate to the hyperbolic supplementary of the circle of radius $r = \sqrt{-a}$, as shown in previous representations. There is no traçe at all in the real plane. It is a curve existing only in the complex regions.

Circles with zero radius. Consider the circle

$$z^2 + w^2 = 0.$$

This is normally thought of as a point at the origin. Resolving the function into its real and imaginary parts, and taking the same cross-section as before, it is found that

$$x^2 - v^2 = 0.$$

This is a pair of intersecting straight lines in the (x,v) plane, that is, the lines

$$x - v = 0 \text{ and } x + v = 0.$$

These lines intersect the (x,u) plane in the point (0,0), the only point of the curve existing in the real plane.

The radical axis of two circles. The radical axis of two intersecting circles is sometimes defined as the common secant of the two circles. If the two circles do not intersect in the real plane, they still have a radical axis, for the supplementarys will intersect in complex points. (Intersections in the infinite domain are not considered here.) Therefore the word "intersecting" in the definition is unnecessary, and the radical axis of two circles is the common secant of the two circles. Another definition sometimes given is, the radical axis of two circles is the locus of points such that the lengths of the tangents from it to the two circles are equal. The two definitions are equivalent, and either one requires the use of the supplementary in order to define what is intended. If the definition is taken as the common secant of the two circles, and if the real parts of the circles do not intersect, then the common secant joins the points of intersection in the supplementarys. The common secant line in the (x,v) plane is the intersection of the (x,v) plane with a plane with an equation of the form $x = k$. The intersection of the plane $x = k$ with the (x,u) plane will be the line which is the real radical axis to the two circles. Thus, the real radical axis is a real line perpendicular to the x-axis, and passing between the real circles.

If the definition is taken as the locus of points such that the lengths of the tangents to the two circles are equal, then supplementarys to the circles are still required. Otherwise, if the two circles intersect, there are points of the common secant lying within the circles. These points would not then have tangents to the circles.

Other conic sections. The procedures described here for the circle may by extended, and similar applications made, to conic sections other than the circle. The graph of the parabola has already been considered. The ellipse has a supplementary in the form of a hyperbola, but not necessarily equilateral. The hyperbola will have a supplementary in the form of an ellipse. If the hyperbola is equilateral, the ellipse will be a circle. Thus when looking at the graphs of an ellipse and a hyperbola, strict attention must be paid to the identities of the axes in order to identify the particular relation that has been graphed.

The circular and hyperbolic functions. Figure 16 is the graph of the real part of the unit circle

$$z^2 + w^2 = 1 \tag{1}$$

and the real part of the unit equilateral hyperbola

$$z^2 - w^2 = 1. \tag{2}$$

The circular and hyperbolic trigonometric functions are often identified with lines on the (x,u) plane. If θ is an angle with vertex at the center of the circle, initial side on the x-axis, and terminal side intersecting the circle at A and the hyperbola at C, with lines drawn from A and C perpendicular to the x-axis, meeting the x-axis at B and D, respectively, then

$$\sin \theta = AB \text{ and } \sinh \theta = CD.$$

Likewise,

$$\cos \theta = OB \text{ and } \cosh \theta = OD.$$

Figure 17 is the graph of the $y = 0$ cross-section of the supplementarys of (1) and (2), with the supplementary of (1) appearing as a hyperbola in the (x,v) plane and the supplementary of (2) appearing as a circle, also in the (x,v) plane.

On a complex plane, multiplication of a number by i is often depicted as a positive rotation of one right angle of a vector drawn

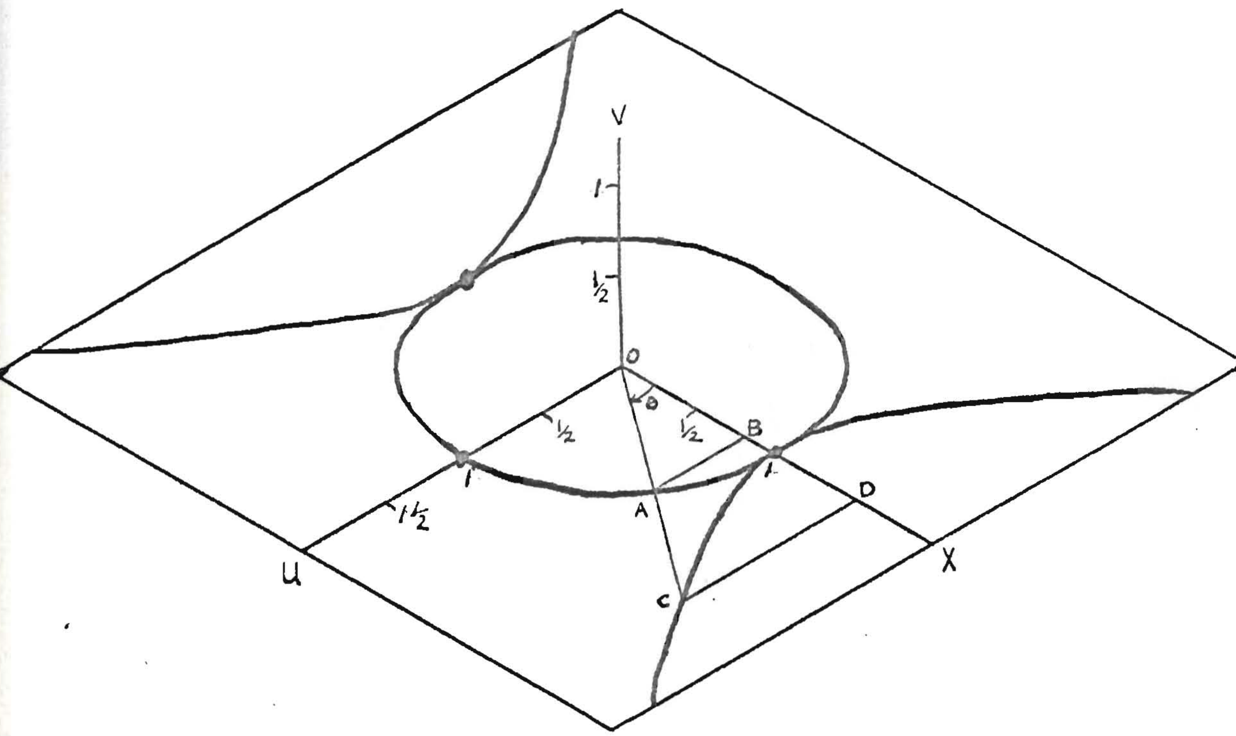

FIGURE 16

$$z^2 + w^2 = 1, \; y = 0, \; v = 0$$

$$z^2 - w^2 = 1, \; y = 0, \; v = 0$$

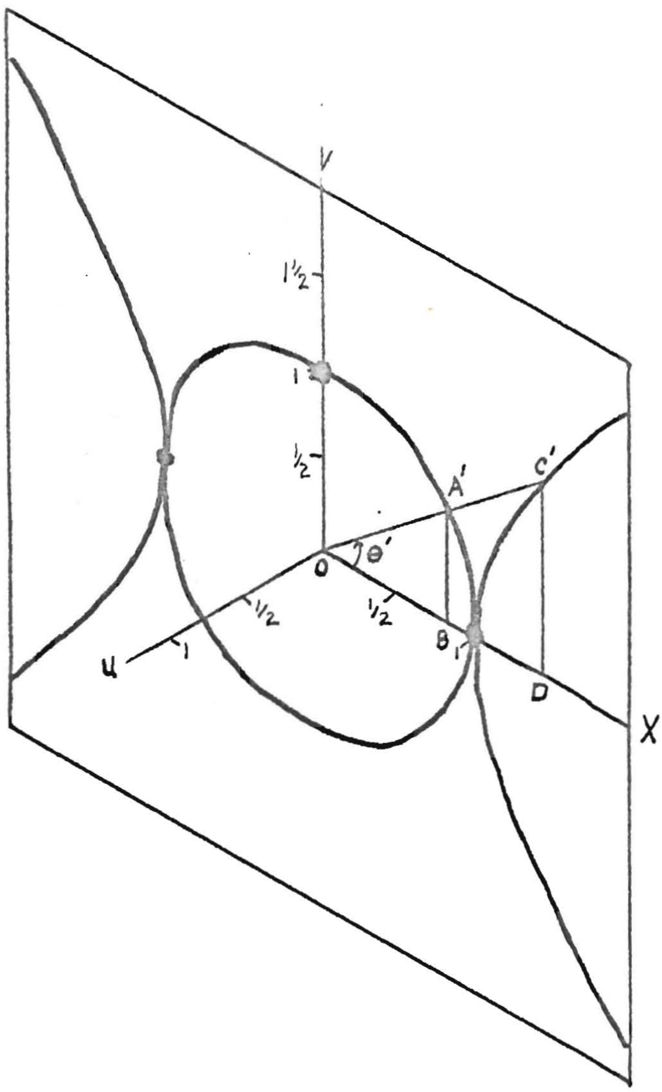

FIGURE 17

$z^2 + w^2 = 1,\ y = 0,\ u = 0$

$z^2 - w^2 = 1,\ y = 0,\ u = 0$

from the origin to the point representing the number, locating a new point corresponding to the product. In the three-space being considered here, multiplication by i will be represented as a positive rotation of one right angle of the space around the x-axis. Thus, $i\theta$ will be the angle θ' in the (x,v) plane, with the terminal side of the angle cutting the hyperbola and the circle in the points A' and C', respectively. This interpretation of multiplication results in the equalities

$$i(AB) = A'B, \qquad i(CD) = C'D,$$

$$i(OB) = OB, \qquad i(OD) = OD,$$

and

$$i(\theta) = \theta'.$$

Interpreting the circular and hyperbolic relations as before, but this time in the (x,v) plane in Figure 17, being careful as to which points belong to the circle and which points belong to the hyperbola,

$$\sin\theta' = C'D, \qquad \cos\theta' = OD,$$

$$\sinh\theta' = A'B, \text{ and } \qquad \cosh\theta' = OB.$$

The relation between the circular and hyperbolic functions, usually proved in terms of exponentials, or other abstract definitions, or derivations from definitions, of the functions, are here interpreted graphically. For $\sin i\theta = \sin\theta' = C'D = i(CD) = i(\sinh\theta)$. The other relations are similarly obtained and are summarized here. They are

$$\sin i\theta = i\sinh\theta, \qquad \sinh i\theta = i\sin\theta$$

$$\cos i\theta = \cos\theta, \text{ and } \qquad \cosh i\theta = \cos\theta.$$

Concluding Remarks. This chapter has presented some of the uses of graphs of circles using complex numbers as variables for w while

z is real in the circle $z^2 + w^2 = r^2$. It has been shown that many of the problems encountered in elementary work that normally are left without a suitable representation geometrically can be so represented. These usually concern the intersections of lines meeting at points not on the real plane. Also, it has been shown that some definitions made for real curves may be more general than is usually supposed. There has also been shown a way of graphing certain equations which, in the real plane, have no points whatsoever.

As a last observation, consider two conjugate hyperbolas, such as

$$\frac{z^2}{a^2} - \frac{w^2}{b^2} = 1$$

and

$$\frac{w^2}{b^2} - \frac{z^2}{a^2} = 1,$$

a and b real. These hyperbolas, while being separate in the real plane, share certain things in that plane. They have the same centers and common asymptotes, and they exchange transverse and conjugate axes. In the $y = 0$ cross-section of the graphs for z and w complex, the supplementarys of the two hyperbolas are identical, occupying the same locus. This locus, lying in the (x,v) plane, is the ellipse

$$\frac{x^2}{a^2} + \frac{v^2}{b^2} = 1.$$

From this emerges a central theme of an even stronger relation among the conic sections in the merging of the circle and ellipse with the hyperbola than can be brought out through the study of their real parts alone.

CHAPTER VI

DERIVATIVES

In the study of real functions, certain geometrical meanings are placed on the derivative of a function. If the derivative exists at a point, then it is known that the curve is continuous at that point. The value of the derivative is found to be the slope of the tangent line. When zero, a maximum point, a minimum point, or a point of inflection is indicated. To determine which of these is the case, the second derivative is used. If it is positive, it indicates a positive change of slope of the tangent line, that is, a minimum point, and if negative a maximum point is indicated. The concern of this chapter is the geometrical interpretation that can be made on the derivative when variables are complex.

Complex functions of a real variable. A function of the form

$$w = f(z)$$

where the domain of z is the real numbers and the range of w is the complex numbers is defined as a complex function of a real variable. Some use of such functions has already been made, in those situations where graphs with $y = 0$ were used. The function

$$w = \sqrt{r^2 - z^2}$$

together with

$$w = -\sqrt{r^2 - z^2}$$

gives the circle studied in the last chapter, where $w = u + iv$ and $z = x + iy$, with $y = 0$. Essentially, then, z was considered as a real variable with w as a complex function of that variable. Before anything can be done with the derivative, it must be made clear just what a derivative of a complex function is.

Just as was true in the calculus with real variables, the derivative of a complex function w of a real variable x is defined in terms of limits, with a similar geometrical interpretation. (See Figure 18.)

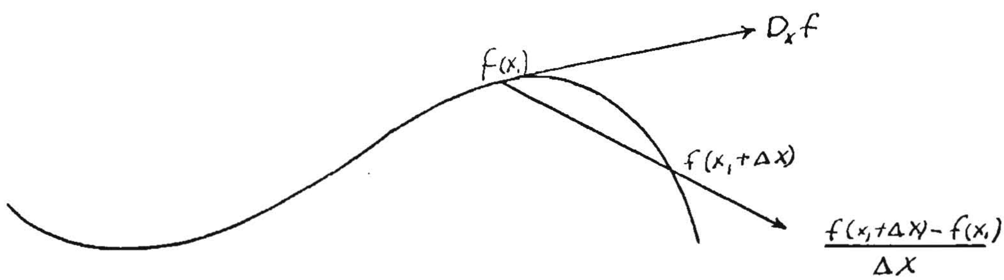

FIGURE 18

DERIVATIVE OF A COMPLEX FUNCTION OF A REAL VARIABLE

Given $w = f(x)$, $w = u + iv = \phi(x) + i\psi(x)$, the derivative of w at a point x_1 is defined by the following equation.

$$D_x w = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x},$$

provided such a limit exists. Resolving this into its component parts, it is found that

$$D_x u + iD_x v = \lim_{\Delta x \to 0}\left[\frac{\phi(x_1 + \Delta x) - \phi(x_1)}{\Delta x} + i\left(\frac{\psi(x_1 + \Delta x) - \psi(x_1)}{\Delta x}\right)\right]$$

which implies the two relations

$$D_x u = \lim_{\Delta x \to 0} \frac{\phi(x_1 + \Delta x) - \phi(x_1)}{\Delta x}$$

and

$$D_x v = \lim_{\Delta x \to 0} \frac{\psi(x_1 + \Delta x) - \psi(x_1)}{\Delta x}.$$

The derivative of a complex function is now reduced to a complex combination, $D_x u + iD_x v$, of two real derivatives of the real functions ϕ and ψ.

One geometrical result which follows immediately is that if the derivative exists and is not zero at a point, then a tangent exists at that point. Because there does not exist an order on the complex numbers, it is not possible for the value of the derivative to indicate anything that might be meant by an "increasing" or "decreasing" function. However, a meaning can be attached to the value of the derivative at a point, for this will be the slope of the curve at that particular point.

For instance, consider once more the function

$$w = \sqrt{r^2 - x^2}.$$

The part of the graph of Figure 15, page 52, for which $u > 0$ and $v > 0$ is the graph of this function for which $r = 3$. For any value of x such that $0 < x^2 < r^2$, the function is real valued and any meaning attached to derivatives for real functions applies equally well here. If $x^2 > r^2$, the derivative has an imaginary value. An imaginary slope, such as mi, is here interpreted as the slope of the line lying in the (x,v) plane, whose slope is m in that plane. In other words, the i in the imaginary number causes a rotation of one right angle around the x-axis of a line of slope m in the (x,u) plane into the (x,v) plane.

As an example of the derivative indicating an imaginary slope, and how this may be represented, consider the point on the above curve where $x = 5$. In the function being used, $w = u$ for all x such that $x^2 \leq 9$, and $w = iv$ for all x such that $x^2 \geq 9$. Obtaining the derivative of the function,

$$D_x w = \frac{-x}{\sqrt{9 - x^2}} .$$

When $x = 5$, $D_x w = 5i/4$. Then the slope of the tangent line at the point $(5,4)$ in the (x,v) plane is $5/4$. This is supported by observation of the graph of the function, on which the tangent line has been drawn.

The definition of the derivative does not guarantee the existence of the derivative at any point. Necessary and sufficient conditions for a derivative to exist will be given later. A tangent is not guaranteed at points where the derivative is zero because of the possibility of such special points as cusps and angular points [6, p. 30].

Certain intuitive ideas will usually be expected if a curve is said to be smooth; that is, it would not be expected to have any cusps, angular points, or other "sudden turns." On a closed real interval $x_1 \leq x \leq x_2$, a curve is defined to be smooth if it can be represented by a function $w = f(x)$ whose derivative is continuous and not zero at all points on the interval. Then the idea of smoothness is another property of the graph of a function which may be indicated by the derivative of the function. Again considering the function

$$w = \sqrt{r^2 - x^2},$$

with its derivative on any closed interval containing x such that $x^2 = r^2$, it is found that the derivative is not continuous, as there does not exist a derivative when $x^2 = r^2$. Observing the graph of the function, in Figure 15, page 52, it is immediately apparent that the curve is not smooth at those particular points, but rather undergoes a sudden turn of one right angle out of one plane into another.

Function of a complex variable. The definition of the derivative of a function of a complex variable takes the same form as the definition of the derivative for functions of other variables. The derivative $D_z w$, where $w = f(z)$, at a point $z_0 = x_0 + iy_0$, with $\Delta z = z - z_0$, is defined by the equation

$$D_z w = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

if this limit exists.

Three conditions must be satisfied for a function of a complex variable to be continuous at a point z_0. These are

$$f(z_0) \text{ exists,}$$

$$\lim_{z \to z_0} f(z) \text{ exists,}$$

and

$$\lim_{z \to z_0} f(z) = f(z_0).$$

In view of the definition of the derivative of a function of a complex variable,

$$\lim_{\Delta z \to 0} (f(z_0 + \Delta z) - f(z_0)) = \lim_{\Delta z \to 0} \left[\left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right) \Delta z \right]$$

$$= \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \lim_{\Delta z \to 0} \Delta z$$

$$= 0.$$

Then

$$\lim_{\Delta z \to 0} (f(z_0 + \Delta z) - f(z_0)) = \lim_{\Delta z \to 0} f(z_0 + \Delta z) - \lim_{\Delta z \to 0} f(z_0) = 0.$$

Therefore,

$$\lim_{\Delta z \to 0} f(z_0 + \Delta z) = \lim_{\Delta z \to 0} f(z_0).$$

Since $\Delta z = z - z_0$,

$$\lim_{z \to z_0} f(z) = f(z_0).$$

Thus, if the derivative exists at a point z_0, then the function is continuous at that point. The condition of the existence of a derivative, then, implies the continuity of the function, which, in turn, implies that the graph is connected.

Relation of the functions u and v. Necessary and sufficient conditions for the existence of a derivative at a point are important theorems in the study of complex variables. The conditions are given in terms of what are called the Cauchy-Riemann, or d'Alembert-Euler, conditions, which are equations in terms of partial derivatives of the real functions u and v. The equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Necessary conditions for a derivative to exist are given in the following theorem.

Theorem 1. If the derivative $f'(z)$ of a function $f = u + iv$ exists at a point z, then the partial derivatives of the first order, with respect to x and y, of each of the components u and v must exist at that point and satisfy the Cauchy-Riemann conditions [1, p. 35].

Sufficient conditions guaranteeing the existence of a derivative are given in the following theorem.

Theorem 2. Let u and v be real- and single-valued functions of x and y which, together with their partial derivatives of the first order, are continuous at a point (x_0, y_0). If those partial derivatives satisfy the Cauchy-Riemann conditions at that point, then the derivative $f'(z_0)$ of the function $f = u + iv$ exists, where $z = x + iy$ and $z_0 = x_0 + iy_0$ [1, p. 36].

Proofs of these theorems may be found in most textbooks on complex variables. Churchill gives a proof of each of them and includes a relation giving the value of the derivative in terms of partial derivatives of the real functions u and v,

$$D_z w = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \qquad [1, \text{pp. } 34\text{-}38]$$

The graph of a function in this paper has been made by assuming four mutually perpendicular axes, or a modification of this, as the basis of the model of the space the graph occurs in, the axes being x, y, u, and v. These axes determine six planes. However, there is nothing in the consideration of these planes to distinguish any of them as being basically different from any of the others except as this distinction has been specified in working with particular problems. The planes do have basic differences. The (x,y) plane and the (u,v) plane are complex planes in which a point on one of these planes represents a complex number, while a point on the (x,u) plane represents an ordered pair of real numbers. On other planes, such as the (x,v) plane, a point represents an ordered pair of numbers, the first real and the second imaginary, while on the (y,v) plane, a point represents an ordered pair of imaginary numbers. Certain equations among the partial derivatives of a function whose derivative exists, such as the Cauchy-Riemann conditions, provide some of the basis of the space being utilized. For instance, given u as a function of z, and knowing $w = f(z)$ to be a differentiable function, the corresponding function of v may be found, using the Cauchy-Riemann conditions, providing a full knowledge of w [4, p. 6].

As an illustrative example, suppose a function in u is known for which the corresponding function of w is known to have a derivative. Let

$$u = x^3 - 3xy^2 + 4x^2 - 4y^2 + 2x + 5.$$

Then

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 8x + 2 = \frac{\partial v}{\partial y},$$

$$v = 3x^2y - y^3 + 8xy + 2y + g(x), \qquad (1)$$

$$\frac{\partial u}{\partial y} = - 6xy - 8y = - \frac{\partial v}{\partial x},$$

and

$$v = 3x^2y + 8xy + h(y). \qquad (2)$$

Comparing the two values of v, it is found that there is no function of x in (2) not contained in (1). Therefore $g(x) = 0$. The equation (1) contains $2y - y^3$ not contained explicitly in (2). Therefore, $h(y) = 2y - y^3$. Then $v = 3x^2y - y^3 + 8xy + 2y$. The function $w = u + iv$ can, by proper factoring, be found to be

$$w = z^3 + 4z^2 + 2z + 5.$$

Conclusions. The derivative is found to be very important in working with graphs. The slope of a curve representing a complex function of a real variable can be found, and suggestions as to where the points might be where the graph leaves one plane for another are those points where the derivative is zero or does not exist--that is, points where the curve is not necessarily smooth. It has been shown that a curve is connected wherever the derivative of a function representing that curve exists. Finally, it is suggested that the whole problem of defining a space to properly represent the four variables here involved can be founded on certain relations among the partial derivatives of those functions whose derivatives exist.

CHAPTER VII

CONCLUSIONS

General conclusions. There is quite a break in continuity for the student of mathematics in going from the graphical methods of real analytic geometry to the conformal mappings of complex functions. The approach taken here helps to avoid this break by beginning with the simple Cartesian system, and moving step by step through familiar problems to the transformations of function theory.

Summary. Chapter III gave a method for finding the complex roots of a quadratic equation by the use of a dual plane, where a single plane is considered as having characteristics of both a Cartesian plane representing ordered number pairs of real numbers and a Gaussian plane representing complex numbers. A careful distinction was made as to when the plane was considered real and when it was considered complex. A generalized method for finding roots of any polynomial in one variable was then given which is dependent upon the effect the complex zeros of the polynomial function have on the graph of the function.

After this, the points of the graph were considered as ordered number pairs (z,w) where

$$z = x + iy \text{ and } w = u + iv.$$

For representing the roots of a polynomial $F(z) = 0$, the graph of

$$w = F(z), \quad v = 0$$

was made using the procedure developed carefully and with great skill by Phillips and Beebe in their book, Graphic Algebra [11, pp. 97-156].

Graphically, this represents a certain three-dimensional cross-section of the four-space determined by x, y, u, and v. At certain critical points the real curve intersects its supplementary, and the intersections of either the real curve or its supplementary with the plane $w = 0$ gives a point or points in the complex z-plane which correspond to the roots of the original polynomial equation.

In Chapter IV, the same cross-section, $v = 0$, was used to introduce intersections of curves, and then v was allowed to vary, giving, as v varies continuously, a surface in the (x,y,u) three-space. A particular value of v for each surface passing through a point in the space must be associated with that point. This procedure is similar to the methods given by Sophus Lie, whose work was mentioned in Chapter II, with the interpretation given there that the points of the space are weighted. In this way, each point of the surface is associated with all four variables. The points of any particular surface for which v is a constant then determine a curve on the surface. Any one of the four variables could serve as parameters on a surface that is the graph of the other three for various values of the fourth.

The remainder of the thesis was concerned more with the effect on the value of w with a change in x. Thus, the graph of z real and w complex, that is, the three-dimensional cross-section with $y = 0$ of the four-space, was taken as a model.

The graph involving the supplementary of a circle, introduced in Chapter III, and used extensively in Chapter IV, follows from the equations first obtained by Jean Poncelet, whose work in this field, and the results he obtained, were summarized in Chapter II. It was Poncelet who first suggested the word "supplementary" for the complex part of a curve.

In Chapter VI, the use of partial derivatives of functions whose derivatives exist was used to show the relation which exists between the functions u and v, showing that a knowledge of one of these functions determines the other, and hence determines w.

Questions for further study. There were a number of questions encountered in the study of this problem that could lead to further investigation. Some of these are summarized here.

In working with the four-space system required for the complete representation of an equation involving two complex variables, the four axes x, y, u, and v are used, or usually here, three of these four. From the viewpoint of a graph in four-space (or three-space) points on a plane such as that determined by the x and v axes seem to obey the usual Euclidean measure theory. But when it is remembered that the v-axis represents the imaginary part of the (usually) dependent variable w, this measure, when the points are considered in their larger context, does not necessarily hold. As was pointed out at the time, any two points on a line of slope i or -i have a distance of zero between them. Further investigation shows that, again assuming Euclidean measures, the distance from such a line to any point on the plane is always undefined, and that the line fails to make an angle with any other line. A study of the geometry of such a space should prove worthwhile.

A study of the graphs of equations in which the coefficients may be complex instead of restricting them to being real would also be of interest.

In working with the unit circle in Chapter V, the equation of the circle

$$z^2 + w^2 = 1$$

was resolved into its component parts,

$$(x^2 + u^2) - (y^2 + v^2) + 2i(xy + uv) = 1,$$

which gives

$$(x^2 + u^2) - (y^2 + v^2) = 1$$

and

$$xy + uv = 0.$$

If $y = 0$, then $u = 0$ or $v = 0$, giving the two parts of the circle graphed,

$$x^2 + u^2 = 1$$

and

$$x^2 - v^2 = 1.$$

Now if $x = 0$, then

$$u^2 - y^2 = 1 \tag{1}$$

and

$$y^2 + v^2 = -1. \tag{2}$$

The graph of (1) is a hyperbola in the (u,y) plane with the u-axis as a transverse axis. Up to this point, the graph of (2) does not exist, as y and v were defined as real. The case might be considered where y and v are not real, producing a number of the form $a + bi$

where a and b are not necessarily real. Numbers of this type have the complication that, in the equation

$$a + bi = c + di,$$

it is not necessarily true that

$$a = c \text{ and } b = d.$$

A study of a number system of this kind, which is one approach to the study of quaternions, might be very interesting.

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