CHARACTERIZATION OF AN ARC

A Thesis

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CHAPTER I

INTRODUCTION

Since the time of the Greeks, the concept of a curve has been a common mathematical term. The first "definition" of a curve was probably as follows: "A curve is the path of a continuously moving point."[1] This definition is usually accompanied with equally ambiguous ideas of "thinness" and "two-sidedness." A curve may also be thought of as given in its entirety as an infinite set of points.

It was not until the 1870's that attempts were first made to formulate a precise definition of a curve. Cantor was the first to be credited with such a definition.[2] What he actually determined was the set theoretic properties of a set of points whose cardinality is that of the real numbers, a continuum. Cantor also proved that there is a one-to-one correspondence between the points of an interval and the points of a square together with its interior points. Thus, according to Cantor's definition, a square together with its interior points is also a curve. This hardly agrees with

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[1] G. T. Whyburn, "What is a Curve?", American Mathematical Monthly, Vol. 49(October 1942), p. 493.

[2] R. L. Wilder, "The Origin and Growth of Mathematical Concepts", Bulletin of American Mathematical Society, Vol. 59 (September 1953), pp. 423-48.

the usual idea of a curve. If, however, this continuum that is to be called a curve lies in a plane and is required that the continuum contains no such square, the result is "cantorian lines." If cantorian lines could reasonably be defined for dimensions other than a plane, it probably would be an acceptable definition today.

The next important attempt to define a curve was made by Jordan. He defined a plane curve as given by any two functions $x = f(t)$ and $y = g(t)$ where the range of parameter t is the real number interval $[0,1]$. In the original publication, he made no mention of continuity. However, in a later publication, he noted that if the functions are continuous, the curve is continuous. This is usually cited as the "origin" of the concept of "continuous curve." Cantor had pointed out that if a curve is continuous, it need not have a one-to-one relationship with the points of a continuum. Jordan's definition was somewhere in between. It did not require one-to-oneness and did not require continuity.

It was only three years after Jordan's formulation that Peano showed it was possible to construct a continuous curve that would pass through every point of a square and its interior. It was also pointed out that some configurations that do form curves in the intuitive sense are excluded using Jordan's definition. Thus it became apparent that Jordan's definition, even with the condition that the function be continuous, was inadequate as a definition of the usual concept of a curve.

Later, Schoenflies, Brouwer and others studied curves in a topological structure and arrived at a characterization of a continuous curve to be mentioned later.

A continuous curve was said to be the image of continuous functions. If the relationship is also required to be one-to-one, the resulting configuration is called an arc. It is this special type of a curve with which this paper is concerned. Also, as was mentioned earlier, a curve may be considered as a path of a continuously moving point or as an infinite set of points. This paper will consider an arc as a set of points.

In Chapter II, some concepts from elementary topology are included. In Chapter III, some topological properties of the real number interval $[0,1]$ will be discussed. Necessary conditions for a topological space to be an arc will also be developed. In Chapter IV, sufficient conditions for a topological space to be an arc will be developed, and an arc will be characterized as a topological space. Chapter V will contain a summary and suggestions for further study.

CHAPTER II

DEFINITIONS AND BASIC CONCEPTS

Definition 2.1. A topological space (S,T) is a set S and a collection T of subsets of S that satisify the following axioms. (1) The union of any number of elements of T is an element of T. (2) The intersection of a finite number of elements of T is an element of T. (3) Both S and $\emptyset$ are elements of T. The collection T is called a topology for the set S and the members of T are called open sets.

Definition 2.2. The set G is a neighborhood of p iff G is an open set containing p.

Definition 2.3. Let S be a set and T' a collection of subsets of S. Then T' is said to generate the collection T of subsets of S defined as follows: A subset K of S is an element of T iff K is the union of a collection of elements of T'. The collection T' is said to be a basis for the collection T which it generates.

It is frequently more convenient to consider topological spaces in terms of a basis rather than the entire collection forming its topology. The following theorem gives a characterization of a topology of a space in terms of a basis.

Theorem 2.1. Let S be a set and T a collection of subsets of S. Let T' be a basis for the collection T. Then (S,T) is a topological space iff the following conditions are true. (a) For every p in S, there exists an element U in T' such that p is in U. (b) For every U and V in T' and any point p in $U \cap V$, there exists an element W of T' such that $p \in W \subset U \cap V$.

Proof: First, suppose that (S,T) is a topological space with basis T'. Since (S,T) is a topological space, the union of two open sets is open. Also, since T' is a basis, each open set in T is the union of elements of T'. Since every point p of S is in some open set U of S and U is the union of elements of T', p is in some open set U' of T'. Each element of T' is an element of T since it can be considered as the union of itself. Then considering elements U' and V' of T', $U' \cap V'$ is an element of T. Now, let p be an element of $U' \cap V'$. Then, since $U' \cap V'$ is an element of T containing p and T is the union of elements of T', there is some W' of T' such that W' is contained in $U' \cap V'$ and W' contains p. Thus conditions (a) and (b) are satisfied. Conversely, suppose that conditons (a) and (b) are true. Consider the union of any number of elements of T. Each of these is the union of elements of T'. Thus the union of any number of elements of T is the union of elements of T' which by definition is an element of T. The sets S and $\emptyset$ can be considered as the union of all and no elements of T respectively. Therefore, conditions (1) and (3) of definition 2.1 are satisfied. Let G and H be elements of T. Then each G and H is the union of a collection of elements of T'. Thus G = $\{G_\alpha\}$ and H = $\{H_\beta\}$ where G_α and H_β are elements of T' for each α and each β. Now G $\cap$ H = $(\cup G_\alpha) \cap (\cup H_\beta) = (G_{\alpha_1} \cap H_{\beta_1}) \cup (G_{\alpha_1} \cap H_{\beta_2}) \cup \ldots \cup (G_{\alpha_2} \cap H_{\beta_1}) \cup (G_{\alpha_2} \cap H_{\beta_2}) \cup \ldots \cup (G_{\alpha_K} \cap H_{\beta_1}) \cup (G_{\alpha_K} \cap H_{\beta_2}) \cup \ldots$ From condition (b), $G_{\alpha_x} \cap H_{\beta_y}$ is an element of T' for each α_x and each β_y. Thus G $\cap$ H is the union of elements of T' and an element of T. By mathematical induction, it can be shown that the intersection of any finite number of elements of T is an element of T. Thus condition (2) of definition 2.1 holds and (S,T) is a topological space.

<u>Definition</u> <u>2.4</u>. Let S be a topological space and X a subset of S. The subspace topology of X is that obtained by defining a subset U of X to be open in X if it is the intersection of X with some open subset of S.

<u>Definition</u> <u>2.5</u>. A set is closed iff its complement is open.

<u>Definition</u> <u>2.6</u>. A point x in S is a limit point of a subset A of S iff every open set containing x contains a point of A distinct from x.

Definition 2.7. A set together with all its limit points is called the closure of A and is denoted $\bar{A}$.

Definition 2.8. The mapping f of a topological space S into another topological space T is continuous at s in S iff for every open subset G of T such that s is in $f^{-1}(G)$, there exists an open subset G' of S such that s is in G' and G' is a subset of $f^{-1}(G)$. The mapping is continuous on S iff it is continuous at every point of S.

The following theorem gives an equivalent definition of continuity that is easier to apply in many instances.

Theorem 2.2. Let S and T be spaces and $f:S \rightarrow T$ a mapping. Then f is continuous on S iff for every open subset G of T, $f^{-1}(G)$ is an open subset of S.

Proof: First, suppose f is a continuous mapping and that G is any open subset of T. From definition 2.8, for each point x in $f^{-1}(G)$ there exists an open set containing x that is contained in $f^{-1}(G)$. Since $f^{-1}(G)$ is the union of all such neighborhoods of x, $f^{-1}(G)$ is an open subset of S by axiom (1) of definition 2.1. Conversely, suppose that for every open subset G of T, $f^{-1}(G)$ is an open subset of S. Let s be any element and G any open subset of T such that f(s) is in G. Define U = $f^{-1}(G)$. Then U is an open subset of S such that s is in U and U is a subset of $f^{-1}(G)$. Thus f is continuous at s. Since s was an arbitrary point of S, f is continuous on S.

Theorem 2.3. If $f:S\to T$ and $g:T\to W$ are continuous mappings, then $h:S\to W$ is a continuous mapping where h is defined by $h(x) = f(g(x))$ for every x in S.

Proof: Pick any open set G in W. By theorem 2.2, $g^{-1}(G) = H$ is an open set in T. Also, $f^{-1}(H)$ is an open set in S. Since $h^{-1}(G) = f^{-1}(H)$, $h^{-1}(G)$ is an open set and h is a continuous mapping.

Definition 2.9. A mapping is open iff the image of every open set is an open set.

Definition 2.10. A mapping is one-to-one iff $f(x) = f(y)$ implies $x = y$.

Definition 2.11. A mapping $f:S\to T$ is onto iff every element of T is the image of some point in S.

Definition 2.12. Space S is homeomorphic to space T iff there exists a one-to-one open continuous mapping of S onto T. The mapping f is called a homeomorphism.

Definition 2.13. A property of space S is a topological invariant iff every space T homeomorphic to S has the same property.

Definition 2.14. A set S is said to be metric iff there is associated with S a mapping $p:(S \times S) \rightarrow R$ (where R is the space of real numbers) having the following properties for every x, y and z in S.

(a) $p(x,y) \geq 0$

(b) $p(x,y) = 0$ iff $x = y$

(c) $p(x,y) = p(y,x)$

(d) $p(x,z) \leq p(x,y) + p(y,z)$

The mapping p is called a metric for the set S.

Definition 2.15. Let K be a metric set. Then, with each point x of K and each positive real number r, there is associated a subset $S_r(x)$ called a spherical neighborhood of x. A point y of K is in $S_r(x)$ iff $p(x,y) < r$.

Definition 2.16. A metric set S is said to be a metric space iff the topology which is generated by the collection of subsets of S consists of all spherical neighborhoods of S. This topology of S is said to be induced by the metric p of S.

Definition 2.17. Let (S,T) be a topological space. Then S is metrizable iff it is possible to define a metric on S which induces the topology T.

Definition 2.18. A topological space S is a Hausdorff space iff for every two distinct points p and q of S, there exists disjoint open sets U and V of S such that p is in U and q is in V.

Definition 2.19. A topological space is a T_1 space iff every point of S is a closed subset of S.

Definition 2.20. Let X be a subset of space S. Then X is said to be a cutting of S iff S - X is not connected. A single point of S is called a cut point of S iff it is a cutting of S; otherwise, it is called a non-cut point.

Definition 2.21. A set X is non-degenerate iff X contains at least two distinct elements.

Definition 2.22. A subset B of K is dense in K iff K is a subset of $\overline{B}$.

Definition 2.23. A subset B of S is separable iff there exists a countable subset H of B which is dense in B.

Definition 2.24. Two subsets A and B of a space S are separated iff $A \neq \emptyset$, $B \neq \emptyset$, $\overline{A} \cap B = \emptyset$, and $A \cap \overline{B} = \emptyset$.

Definition 2.25. A subset A of space S is connected iff there exists no continuous mapping $f:A \rightarrow R$ such that f(A) consists of exactly two elements.

The following theorems give useful relationships between cut points, connected sets, and separated sets.

<u>Theorem</u> <u>2.4</u>. Let S be a connected space and x a point of S such that $S - x = A \cup B$ where A and B are separated sets. Then $A \cup x$ and $B \cup x$ are connected.

Proof: Suppose $A \cup x$ is not connected. Then there exists a continuous mapping f such that $f(A \cup x) \rightarrow (a \cup b)$ where a and b are distinct. Now, define a mapping $f: S \rightarrow (a \cup b)$ such that $g(y) = f(y)$ if y is in $A \cup x$ and $g(y) = b$ if y is in B. Since $g(S) = a \cup b$ and g is a continuous mapping, S is not connected. This is a contradiction and therefore $A \cup x$ is connected. Similarily, $B \cup x$ is connected.

<u>Theorem</u> <u>2.5</u>. If A is a connected set that is contained in the union of two separated sets, then A is contained in one of these.

Proof: Consider a connected set A and separated sets G and H. Suppose A is not contained in G or in H. Then there exists an x in A such that x is in G and there exists a y in A such that y is in H. Let X be the set of all points s such that x is in A and in G. Let Y be the set of all points y such that y is in A and in H. Thus $A = X \cup Y$. Since G and H are separated, X and Y are separated. Therefore A is not connected since a connected set cannot be expressed as the union of two separated sets. This is a contradiction to the hypothesis that A is connected.

Therefore, if A is a connected set contained in the union of two connected sets, A is contained in one of these.

Theorem 2.6. Let A be a connected subset of space S and A^* any set such that $A \subset A^* \subset \bar{A}$. Then A^* is connected.

Proof: Suppose A^* is not connected. Then A^* can be written, $A^* = C \cup D$, where C and D are separated sets. From theorem 2.5, A is in C or in D. Let A be in C. By definition of A^*, every point of D is a limit point of C. This is a contradiction to the assumption that C and D are separated sets. Therefore, A^* is connected.

Definition 2.26. A subset H of space S is compact iff every open covering of H contains a finite subcovering of H.

Definition 2.27. A subset H of S is countably compact iff every infinite subset of H has at least one limit point in H.

Definition 2.28. A compact and connected space is called a continuum.

The following theorem gives a relationship of compact and countably compact sets.

Theorem 2.7. A compact set is countably compact.

Proof: Consider any space S and an infinite subset

K of S. Suppose K has no limit point in S. Then for each point x in K, there is an open set containing x that contains no other point of K. Also, for each point y in S - K, there is an open set containing y that contains no points of K. These open sets form an open covering of S. Since S is compact, there is a finite subcollection of these open sets that covers S. Thus K is finite as no two points of K lie in the same open set. This is a contradiction as K was given to be infinite. Therefore, if K is an infinite subset of S, it must have a limit point in S and S must be countably compact.

<u>Theorem</u> <u>2.8</u>. If x is a cut point of a compact connected space S and $S - x = A \cup B$, then A and B are connected sets and each contain at least one non-cut point of S.

Proof: Let x be an element of $S - (a \cup b)$ where a and b are the two non-cut points of S. Then $S - x = A \cup B$ is a separation of S. Also, $A \cup x$ and $B \cup x$ are each countably compact, connected, non-degenerate, separable subspaces of S. Thus $A \cup x$ and $B \cup x$ each have at least two non-cut points. Let z be a non-cut point of $A \cup x$ distinct from x. Then $S - z = ((A \cup x) - z) \cup (B \cup x)$ which is the union of two connected sets each containing x. Hence S - x is connected and z is a non-cut point of $A \cup x$. Thus z is either point a or point b. Let z be a. Then z is in $A \cup x$.

Similarily, let m be a non-cut point of $B \cup x$ distinct from x. Then $S - m = ((B \cup x) - m) \cup (A \cup x)$ which is the union of two connected sets with a common element x. Hence, $S - m$ is connected and m is a non-cut point of $B \cup x$. Therefore, m is b and A and B each contain at least one non-cut point of $B \cup x$. Since $A \cup x$ contained only one non-cut point of S, then x must be a non-cut point of $A \cup x$ and A is connected. Similarily $B \cup x$ is connected.

Definition 2.29. A Dedekind cut in the set of real numbers is a partition of the reals into two subsets A and B such that (a) neither A nor B is empty, (b) $A \cup B = R$, and (c) every number in A is less than any number in B. Under this definition there is either a maximum in A or a minimum in B, but not both.

Definition 2.30. Consider a set A and a binary relation * defined between elements of A. The relation * is a simple-order relation, and A is simply ordered by *, provided that (a) for each two elements x and y in A, either x*y or y*x, (b) if x*y, then y*x is false, and (c) if x*y and y*z, then x*z.

Definition 2.31. Consider a set A and a binary relation * defined between elements of A. The relation * is a partial-order relation, and A is partially-ordered by *, provided that (a) for each x in A, $x*x$, (b) if $x*y$ and $y*x$, then $x = y$, and (c) if $x*y$ and $y*z$, then $x*z$.

Definition 2.32. Let S and T be ordered sets and $f:S \rightarrow T$ a mapping. Then f is order preserving iff for any two elements x and y in S, $f(x)$ precedes $f(y)$ in T iff x precedes y in S.

Definition 2.33. Two simply ordered sets A and B are of the same order type iff there exists a one-to-one order preserving correspondence between the elements of the two sets. Such a correspondence is called an order-isomorphism.

Definition 2.34. A topological space is an arc iff it is homeomorphic with J, the interval $[0,1]$ of the space of real numbers.

Definition 2.35. Let S be a set partially ordered by $<$, and K a subset of S. Then K is said to be a maximal simply ordered subset of S iff K is simply ordered and there exists no element x in $S - K$ such that $y < x$ for every y in K.

The following theorem is equivalent to the axiom of choice and will be stated without proof.

Theorem 2.9. (Zorn's lemma) Let S be a partially ordered set. Then there exists at least one maximal simply ordered subset K of S.[3]

Using the previously defined terms from elementary topology, it is possible to discuss some of the properties of the interval $[0,1]$ and develop a characterization of an arc as a topological space.

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[3]Dick Wick Hall and Guilford L. Spencer II, Elementary Topology (New York: John Wiley & Sons, Inc., 1955), p. 280.

CHAPTER III

PROPERTIES OF THE INTERVAL J

Definition 2.34 states that an arc is homeomorphic with J. Thus the topological properties common to all arcs are exactly those possessed by J which are preserved under a homeomorphism--topological invariants. Therefore, to determine properties of an arc, it is possible to consider properties of J and determine if they are topologically invariant.

<u>Theorem 3.1</u>. J is a connected space.

Proof: Suppose there exists a continuous mapping $f:J \rightarrow (a \cup b)$. Then there exists distinct points p and q in J such that $f(p) = a$ and $f(q) = b$ where notation is chosen in such a way that $p < q$. Then $[p,q]$ is contained in J. Now define a mapping $f:R \rightarrow (a \cup b)$ as follows:

$$g(x) = f(p) \quad \text{if } x < p$$

$$g(x) = f(x) \quad \text{if } p \leq x \leq q$$

$$g(x) = f(q) \quad \text{if } x > q.$$

Since f(p) and f(q) are constant mappings, they are continuous. Also f(x) was given continuous on J and thus is continuous on $[p,q]$. Therefore $g:R \rightarrow (a \cup b)$ is a continuous mapping. Let G_1 be the open set consisting of all points x where $x < \frac{a+b}{2}$ and G_2 the open set consisting of all points x where $x > \frac{a+b}{2}$. It follows from the definition of g and the definition of continuity that R is the union of two disjoint non-empty open sets $g^{-1}(G_1)$ and $g^{-1}(G_2)$. This is a contradiction as it is impossible for R to be expressed as the union of two disjoint open sets. Therefore, J is a connected space.

Theorem 3.2. J is a non-degenerate space.

Proof: Consider 0 and 1 which are elements of J. Thus J contains at least two distinct points and is non-degenerate by definition 2.21.

Theorem 3.3. J is a separable space.

Proof: Consider the set of rational numbers which are elements of J, call them K. The closure of K is J. Therefore, J is a subset of $\overline{K}$ and K is dense in J. Since K is countable, J is a separable space by definition 2.23.

Theorem 3.4. J has exactly two non-cut points.

Proof: To show that 0 and 1 are non-cut points of J, consider $(0,1]$ and $[0,1)$. These two intervals can be shown to be connected in a manner similar to the proof of theorem 3.1. Hence from definition 2.20, 0 and 1 are non-cut points of J. Now suppose there exists some point p in J other than 0 and 1 that is a non-cut point of J. Hence $J - p = [0,p) \cup (p,1]$ is connected. Define a mapping f such that $f(x) = a$ if $0 \leq x < p$ and $f(x) = b$ if $p < x \leq 1$.

Since $f:(J - p) \rightarrow (a \cup b)$ is continuous, $J - p$ is not connected and p is a cut point of J. This contradicts the assumption that p was a non-cut point of J. Therefore, J has exactly two non-cut points.

Theorem 3.5. J is a compact space.

Proof: Let $\{G_\alpha\}$ be a collection of open sets covering J. Construct a Dedekind cut (L,R) of E^1 as follows. A point p is put in L if $p < 0$ or if $0 \leq p \leq 1$ and a finite number of open sets G_α cover $[0,p]$. A point is in R otherwise. By the definition of a Dedekind cut, there is a point m that is either a maximum of L or a minimum of R. In either case m is in $[0,1]$, so that there is some $G_{\alpha'}$ that contains m. Since open intervals constitute a basis for E^1, there is some interval (x,y) in G such that $0 < x < m < y$. Regardless of whether m is in L or in R, x is in L so that a finite number of open sets $G_{\alpha_1}, G_{\alpha_2}, G_{\alpha_3}, \ldots, G_{\alpha_n}$ cover $[0,x]$. Hence the open sets $G_{\alpha_1}, G_{\alpha_2}, G_{\alpha_3}, \ldots, G_{\alpha_n}, G_{\alpha'}$ cover $[0,y]$ and y is in L. But $y > m$ contradicting the hypothesis that m be a maximum of L or a minimum of R. Therefore, J is a compact space.

Theorem 3.6. J is a countably compact space.

Proof: From theorem 2.7, since J is compact it is also countably compact.

Theorem 3.7. J is a metric space.

Proof: Define $p(x,y) = |x - y|$. From the definition of absolute value, it can easily be seen that p satisfies the properties of definition 2.14 and thus is a metric for J.

Thus J is a compact, countably compact, connected, non-degenerate, separable metric space with two non-cut points. The fact that J has these properties is not sufficient to say that an arc also has these properties. For example, J is bounded but R is not bounded. Therefore, it remains to be shown that the previously mentioned properties of J are preserved under a homeomorphism before it can be concluded that an arc must also have these properties.

Theorem 3.8. The property of being connected is a topological invariant.

Proof: Consider any connected set S and a mapping $f:S \rightarrow T$ such that f is a homeomorphism. Suppose that T is not connected. Then there exists a continuous mapping $g:T \rightarrow R$ such that $g(T) = a \cup b$ where a and b are distinct. Consider also a mapping $h:S \rightarrow R$ where h is defined by $h(x) = g(f(x))$ for each x in S. This mapping is continuous by theorem 2.3. Since $h(S) = a \cup b$, S is not connected. This contradicts the assumption that S is connected. Hence, if S is connected, so is T under homeomorphism f.

Theorem 3.9. The property of being a non-cut point is a topological invariant.

Proof: Consider space S with non-cut point x and a mapping $f:S \rightarrow T$ such that f is a homeomorphism. Thus $S - x$ is connected. By theorem 3.8, $f(S - x)$ is also connected. But $f(S - x) = f(S) - f(x) = T - f(x)$ since f is one-to-one. Hence, $f(x)$ is a non-cut point of T.

Theorem 3.10. Compactness is a topological invariant.

Proof: Consider any compact set S and a mapping $f:S \rightarrow T$ such that f is a homeomorphism. Suppose T is not compact. Let $\{G_\alpha\}$ be any open covering of T. Then $V_\alpha = f^{-1}(G_\alpha)$ is an open covering of S. Since S is compact, there exists a finite subcollection $V_{\alpha_1}, V_{\alpha_2}, \ldots, V_{\alpha_n}$ that covers S. Now $f(V_{\alpha_i} \cap S) \subset G_{\alpha_i}$ and hence $f(\bigcup_{i=1}^{n}(V_{\alpha_i} \cap S)) \subset \bigcup_{i=1}^{n} G_{\alpha_i}$. Since $S \subset \bigcup_{i=1}^{n}(V_{\alpha_i} \cap S)$, $f(S) \subset \bigcup_{i=1}^{n} G_{\alpha_i}$. Thus $f(S)$ is covered by a finite subcollection $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ of any open covering $\{G_\alpha\}$ of $f(S)$ and, therefore, $f(S)$ is compact.

Theorem 3.11. Countably compactness is a topological invariant.

Proof: Let S be any countably compact space and f a homeomorphism such that $f:S \rightarrow T$. Consider an infinite subset X of T. For each point x in X, select a point y in S such that $f(y) = x$. The set Y of all such points y is an infinite set since f is a mapping. Also, since S is countably compact, the set Y has some limit point p in S. For each open set G containing f(p), $f^{-1}(G)$ is an open set of S containing p. Since p is a limit point of S, every neighborhood of p contains a point y distinct from p. Also, since f is one-to-one, f(y) is in G and distinct from f(p). Thus every neighborhood of f(p) contains a point distinct from f(y) and f(p) is a limit point of T. Therefore T is countably compact.

<u>Theorem</u> <u>3.12</u>. The property of being non-degenerate is a topological invariant.

Proof: Consider any set S which is non-degenerate and a mapping $f:S\rightarrow T$ which is a homeomorphism. Since S is non-degenerate, S has at least two distinct points, call them x and y. Also, since f is a mapping, there exists at least one element of T, call it a. Suppose a is the only element of T. Then f(x) = f(y) = a. Since f is one-to-one, x = y by definition 2.10. But x and y were given distinct. Therefore T has at least two distinct elements and is non-degenerate.

<u>Theorem</u> <u>3.13</u>. The property of being separable is a topological invariant.

Proof: Let f be a homeomorphism such that $f:S\rightarrow T$ and let D be a dense subset of S. Then $T = f(S) \subset f(\overline{D})$.

Also, $f(\overline{D}) = f(D) \cup f(D')$ where D' is the set of all limit points of D. Let x be any point of $f(\overline{D})$. Then x is either in f(D) or in f(D'). If x is in f(D), it is in $\overline{f(D)}$ by the definition of closure. If x is in f(D'), it can be assumed that x is not in f(D). Then there is some point y in D' - D such that f(y) = x. Let U be any open set containing x. By theorem 2.2, $f^{-1}(U)$ is an open set containing y. Since y is a limit point of D, there is some point z of D in $f^{-1}(U)$ such that z and y are distinct. Then the point f(z) is in $U \cap f(D)$ and is distinct from x since the mapping is one-to-one. Thus x is a limit point of f(D) and is in $\overline{f(D)}$. Thus $f(\overline{D}) \subset \overline{f(D)}$ and $T \subset \overline{f(D)}$. Since f is one-to-one, f(D) is countable. Hence f(D) is a countable subset of T dense in T. Therefore T is separable.

<u>Theorem 3.14</u>. The property of being a metric space is a topological invariant.

Proof: Consider the homeomorphism $h:(X,T)\rightarrow(S,T')$ where T' is induced by the metric d. Define d(h(x), h(y)) = p(x,y). Since h is one-to-one, p is a metric on the set S. Also, h is a homeomorphism of (X,T*) and (S,T') where T* is induced by p. Therefore T = T* and the image of a metric space under a homeomorphism is a metric space.

As a result of the previous theorems about space J and topological invariants, the following theorem may be stated giving a necessary condition for a topological space to be an arc.

<u>Theorem</u> <u>3.15</u>. If a topological space is an arc, then it is a compact, countably compact, connected, non-degenerate separable metric space with at most two non-cut points.

CHAPTER IV

CHARACTERIZING AN ARC

In order to arrive at a characterization of an arc as a topological space, a set of conditions that are both necessary and sufficient must be established. In Chapter III necessary conditions for a topological space to be an arc were discussed. The next question to consider is then an obvious one. Given a topological space, what conditions are sufficient to make it an arc? For a topological space to be an arc, it must be homeomorphic with the interval J. Thus a homeomorphism between a space with certain properties and J must be exhibited. Before this can be done several ideas must be developed.

Definition 4.1. A subset X of a space S is said to separate a subset Y of S in S iff S - X can be expressed as the union of two separated sets $M_1(X)$ and $M_2(X)$, such that $M_1(X) \cap Y \neq \emptyset$ and $M_2(X) \cap Y \neq \emptyset$. The union of two such sets is called a separation of Y in S. A subset X of a space S is said to separate two subsets A and B of S in S iff S - X can be expressed as the union of two separated sets $M_1(X)$ and $M_2(X)$ such that A is contained in one of these and B is contained in the other. The union of two such sets is called a separation of A and B in S.

This definition makes it possible to obtain the existance of non-cut points in a topological space with certain properties. It should be noted that the $M_1(X)$ and $M_2(X)$ described in definition 4.1 are by no means unique.

<u>Theorem</u> <u>4.1</u>. Let S be a countably compact, connected, non-degenerate, separable T_1 space. Then S has at least two non-cut points.

Proof: Since S is separable, there exists a countable subset H of S which is dense in S. Suppose S has at most one non-cut point. Since S is non-degenerate, H must contain at least two points. But by assumption, there exists at most one non-cut point. Thus there must be at least one cut point. Define n_1 to be the least integer such that the element p_{n_1} of H is a cut point of S. Since p_{n_1} is a cut point of S, there is a separation of S given by $S - p_{n_1} = A_1 \cup B_1$ where the notation is chosen in such a way that the non-cut point of S, if there is one, lies in B_1. Thus every point of the countably compact set $\overline{A}_1$ is a cut point of S. The set A_1 may be expressed by $A_1 = S - \overline{B}_1$. Therefore A_1 is an open subset of S. Now there must exist a least integer n_2 such that the point p_{n_2} of H lies in A_1. Define A_2 and B_2 to be a separation of S such that $S - p_{n_2} = A_2 \cup B_2$ where the notation is chosen in such a way that the connected set $\overline{B}_1 = B_1 \cup p_{n_1}$ is contained in B_2. Hence $\overline{A}_2$ is a subset of A_1. In general, for $k \geq 2$, define p_{n_k} to be the element of H with the least index that lies in A_{k-1}, and define A_k and B_k to be the sets of a separation of S such that $S - p_{n_k} = A_k \cup B_k$ where the notation is chosen in such a manner that the connected set $\overline{B}_{k-1} = B_{k-1} \cup p_{n_{k-1}}$ is contained in B_k, and consequently, $\overline{A}_k \subset A_{k-1}$. Now, define $B = \bigcup \overline{B}_k = \bigcup (B_k \cup p_{n_k})$. Since B is the union of connected sets each containing the point p_{n_1}, B is connected. But for each index k, all elements of H having index less than n_{k+1} lie in B_{k+1}. Therefore, B contains H and B is dense in S. Then define $A = \bigcup \overline{A}_k$. For each index k, the set $\overline{A}_{k+1}$ is a countably compact set contained in $\overline{A}_k$. Thus by the Cantor intersection theorem, A is not empty.[4] Since A contains at least one element and A is a subset of S - B, there is at least one element p of S not in B. Hence, B is a proper subset of S. Since $B \subset S - p \subset \overline{B}$, then by theorem 2.6, S - p is connected. Therefore, p is a non-cut point of S. This is a contradiction since the set B was constructed so that it contained all non-cut points. Therefore, S has at least two non-cut points.

Since the property of being a non-cut point was shown to be a topological invariant in theorem 3.9, if f is to be a homeomorphism from S to J, then the non-cut

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[4] Hall and Spencer, _op_. _cit_., p. 69.

points of S must map to the non-cut points of J. Thus in defining such a mapping f, $f(a) = 0$ and $f(b) = 1$ or $f(a) = 1$ and $f(b) = 0$ must be true. In order to define f for other points of S, additional ideas and notation must be discussed.

<u>Definition</u> <u>4.2</u>. Let p and q be points of a connected space S. Then $E(p,q)$ will denote the subset of S consisting of the points p and q together with all the cut points of S that separate p and q.

<u>Definition</u> <u>4.3</u>. The separation order in $E(p,q)$ is defined as follows. Let x and y be two points in $E(p,q)$. Then x precedes y, $x < y$, in $E(p,q)$ if either $x = p$ or if x separates p and y in S.

<u>Theorem</u> <u>4.2</u>. Let r and s be two points of $E(p,q) - (p \cup q) = E(p,q)^*$. Also, let r have the separation $S - r = A_r \cup B_r$ and s the separation $S - s = A_s \cup B_s$. If s is in B_r, then A_s contains $A_r \cup r$ and B_r contains $B_s \cup s$. If s is in A_r, then A_r contains $B_s \cup s$ and B_s contains $B_r \cup r$.

Proof: Case 1. Let s be in B_r. $A_r \cup r$ is connected by theorem 2.4. Also, from theorem 2.5, since $A_r \cup r$ is contained in the union of two separated sets, it must be in one of these. Now, $A_r \cup r$ contains p but not s. Thus $A_r \cup r$ lies in A_s. The set $(B_s \cup s) \cap (A_r \cup r)$ is then empty. Thus $B_s \cup s$ must lie in B_r.

Case 2. Let s be in A_r. As in case 1, $B_r \cup r$ is connected and lies in one of the separated sets. The set $B_r \cup r$ contains the point p but not the point s. Thus $B_r \cup r$ lies in B_s. The set $(B_r \cup r) \cap (B_s \cup s)$ is then empty. Thus $B_s \cup s$ must lie in A_r.

<u>Theorem 4.3</u>. The separation order in E(p,q) is a simple order.

Proof: For each point x in E(p,q), $x \neq p$ and $x \neq q$, there is a separation $S - x = A_x \cup B_x$ where p is in A_x and q is in B_x. By theorem 2.4, $A_x \cup x$ and $B_x \cup x$ are connected sets. Now let r and s be two points of E(p,q)*. Then either s is in B_r or s is in A_r. If s is in B_r, then $r < s$ in E(p,q). If s is in A_r, then r is in B_s and $s < r$ in E(p,q). Thus for any two elements of E(p,q), either $r < s$ or $s < r$. From theorem 4.2, if $r < s$ and $s < t$, then B_r contains $B_s \cup s$ which contains B_s. Also, B_s contains $B_t \cup t$. Thus B_r contains $B_t \cup t$ and $r < t$. Therefore, the relation < is transitive. If $r < s$ and $s < r$, then using the transitive property of <, $r < r$ is a true statement. But r cannot separate p and r. Thus $r < r$ is a false statement. Therefore, if $r < s$ is true, $s < r$ cannot be true. The case where $E(p,q) = p \cup q$ must also be considered. If $x = p$, then $y = q$ and $x < y$. If $x = q$, then $y = p$ and $y < x$. If $x < y$, then $x = p$ and $y = q$. Then y cannot equal p, and it cannot be true that $y < x$. The transitive property is satisfied vacuously. Hence, the conditions of definition 2.30 are satisfied and the separation order in $E(p,q)$ is a simple order.

<u>Theorem</u> <u>4.4</u>. If A is a countable simply-ordered set such that (1) A has no least element and no greatest element in its order, and (2) for any two elements a and b of A with $a < b$, there is an element c in A such that $a < c < b$, then A has the same order type as the rationals.

Proof: The proof of this theorem will make use of the fact that there is an order isomorphism between the set of rationals and the set of dyadic fractions. The set of dyadic fractions is the set of fractions of the form $k/2^n$ where $n = 1, 2, 3, \ldots$ and k is any odd number less than 2^n for each n.

Let $A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ where $a_i \neq a_j$ when $i \neq j$. Define $f(a_1) = \frac{1}{2}$. Let n_1 be the first integer such that $a_{n_1} < a_1$ in the order of A and n_2 be the first integer such that $a_1 < a_{n_2}$ in the order of A. That n_1 and n_2 exist follows from condition (1). Define $f(a_{n_1}) = 1/4$ and $f(a_{n_2}) = 3/4$. Now let n_3, n_4, n_5, n_6 be the first integers such that $a_{n_3} < a_{n_1} < a_{n_4} < a_1 < a_{n_5} < a_{n_2} < a_{n_6}$. Define $f(a_{n_3}) = 1/8$, $f(a_{n_4}) = 3/8$, $f(a_{n_5}) = 5/8$, and $f(a_{n_6}) = 7/8$. Similarily determine a_{n_7}, a_{n_8}, ... and define $f(a_{n_7})$, $f(a_{n_8})$, ... Thus A has an order isomorphic to the set of dyadic fractions under the isomorphism described above. Since the relation "is an order isomorphism of" is a transitive relation, A is an order isomorphism of the set of rational numbers.

From theorem 4.4 it is noted that for a set to be order isomorphic to the set of rationals, the set must first be simply ordered. Thus it is necessary to define an order on a topological space S and show that this is a simple order.

Definition 4.4. Let A be a simply ordered set. The order topology in A is the topology given by a basis whose elements are (1) the set A, (2) for each element x in A, the set of all y such that $y < x$, (3) for each element x in A, the set of all y such that $x < y$, and (4) for each pair x and y in A with $x < y$, the set of all z such that $x < z < y$.

Theorem 4.5. Let S be a connected space and let p and q be two points of S such that $E(p,q)$ contains a point of S distinct from p and q. Let $E(p,q)$ have the subspace topology and let E^* denote the set $E(p,q)$ with its order topology. Then the mapping $h:E(p,q) \rightarrow E^*$, defined by $h(x) = x$ is continuous.

Proof: For any point x in $E(p,q) - (p \cup q)$, let $S - x = A_x \cup B_x$ where A_x and B_x are disjoint open sets such that A_x contains p and B_x contains q. A basis element of E* may be of type (2), (3) or (4) as described in definition 4.4. If a basis element is of type (2), then it is in the form $A_x \cap E(p,q)$. If a basis element of E* is of type (3), it is of the form $B_x \cap E(p,q)$. If a basis element of E* is of type (4), it is of the form $(B_x \cap A_y) \cap E(p,q)$. In each case, the basis element is open in E(p,q) as they are the intersection of a finite number of open sets. Thus for each open set G in E*, $h^{-1}(g)$ is an open set in E(p,q). Therefore h is continuous by theorem 2.2.

Theorem 4.6. Let S be a compact, connected, Hausdorff space with exactly two non-cut points, a and b. Then E(a,b) = S and the order topology defined by the points in E(a,b) is the same as the topology in S.

Proof: Suppose there is some x in S that is not in $E(a,b) - (a \cup b)$. Therefore, x is a cut point and $S - x = U \cup V$, where U and V are separated sets. If a is in U and b is in V, then x is in E(a,b) by definition. If the non-cut points a and b are in U, then by theorem 2.8, V must contain a third non-cut point of S. This is a contradiction of exactly two non-cut points. Therefore, S is a subset of E(a,b). By definition 4.2, E(a,b) is defined to be a subset of S. Thus E(a,b) = S.

From the proof of theorem 2.8, it is known that open sets in the order topology are open in S. Thus, to show that the order topology defined by points in E(a,b) is the same as the topology in S, it is sufficient to show that open sets in S are unions of basis elements of the order topology. If this is not true, then for some open set U in S there is a point x in U such that no basis element of the order topology lies in U. If $x \neq a$ and $x \neq b$, then basis elements of type (4) of definition 4.4 need to be considered. Let (y,z) denote a basis element determined by y and z where $y < z$. By use of Zorn's lemma, a collection of sets $\{(y_\alpha, z_\alpha)\}$ is obtained. Each $(y_{\alpha_K}, z_{\alpha_K})$ is picked so that $(y_{\alpha_K}, z_{\alpha_K})$ is contained in $(y_{\alpha_{K-1}}, z_{\alpha_{K-1}})$. This collection is simply ordered by set inclusion and the intersection of this collection is x. The previous statement is also true for the closed sets $[y_\alpha, z_\alpha] = (y_\alpha, z_\alpha) \cup y_\alpha \cup z_\alpha$. Now, $[y_\alpha, z_\alpha] \cap (S - U)$ is closed in S and has a non-empty intersection for each α. It is also simply ordered by set inclusion. Thus there is a point w in the intersection of the collection $\{(y_\alpha, z_\alpha) \cap (S - U)\}$. Then w is also in the intersection of the collection $\{(y_\alpha, z_\alpha)\}$. Since x is in U, x is not in S - U and w cannot be x. Thus a point w distinct from x is in the intersection of the collection $\{(y_\alpha, z_\alpha)\}$. This is a contradiction. Therefore, x must be in some basis element of the order topology. If $x = a$ or $x = b$, then basis elements of type (2) or (3) in definition 4.4 need to be considered. Since the order topology of $E(a,b)$ is a simple order relation, either $x < y$ or $y < x$. In either case, x is in some basis element of the order topology. Therefore, the order topology of $E(a,b)$ is the same as the topology of S.

The next theorem gives the sufficient conditions for a topological space to be an arc.

Theorem 4.7. If S is a non-degenerate, countably compact, connected, separable metric space having at most two non-cut points, then S is an arc.

Proof: Since S is separable, there is a countable dense subset R contained in S. It can be assumed that R does not contain the two non-cut points a and b of S. Thus R is a subset of $E(a,b)$. As a subset of $E(a,b)$, R has an order that satisfies the conditions of theorem 4.4. Hence there is an order isomorphism h of R onto K where K is the set of rationals in J. Since h is an order isomorphism of R onto K, open sets of R map to open sets of K and the pre-image of open sets in K are open sets of R. Thus h is also a homeomorphism of R onto K. Now, let x be a point of S other than the non-cut points. Also, let K_A be the set of all points y of K such that $y < x$ and let K_B be the set of all points y of K such that $x < y$. Then the sets $h(K_A)$ and $h(K_A)$ determine a partition of K. This partition of K is also a partition of J. From the Dedekind cut theorem, such a partition determines a unique point y. Define $h'(x) = y$. Then h' defines a mapping of S onto J. Since a unique point is determined by $h'(x)$, h' is one-to-one. For any open set of J, say $(h'(x),h'(y))$, $h'^{-1}(h'(x),h'(y))$ is the set of all z such that $x < z < y$ in S. This is a basis element in S and hence an open set of S. Also, any open set in S is of the form $\{x \mid x < y\}$, $\{x \mid y < x\}$ or $\{z \mid x < z < y\}$. The images of these sets are $\{h'(x) \mid h'(x) < h'(y)\}$, $\{h'(x) \mid h'(y) < h'(x)\}$ and $\{h'(z) \mid h'(x) < h'(z) < h'(y)\}$ respectively. Each of these are open in J. Hence, h' is a homeomorphism of S onto J and S is an arc.

As a result of the preceding theorem and theorem 3.15 a characterization of an arc as a topological space may be stated.

<u>Theorem 4.8</u>. A topological space is an arc iff it is a non-degenerate, countably compact, connected, separable metric space with at most two non-cut points.

CHAPTER V

SUMMARY AND CONCLUSIONS

An arc was defined to be a topological space homeomorphic to J, the real number interval $[0,1]$. A characterization of an arc in topological terms was developed in Chapter III and Chapter IV. This was done by first considering properties of J which are invariant under a homeomorphism. Then a topological space with these properties was shown to be a homeomorphic to J. The result was stated in theorem 4.7.

It was mentioned in Chapter I that an arc was a special type of a continuous curve. A continuous curve can also be defined with respect to the interval J.

Definition 5.1. A topological space S is a continuous curve iff it is the continuous image of the interval J.

Continuous curves are sometimes called Peano spaces. A characterization of Peano spaces could be derived in a manner similar to deriving a characterization of an arc. Instead of considering properties of a topological space invariant under a homeomorphism, properties invariant under a continuous mapping would need to be considered. A characterization of a Peano space was first developed by Hahn and Mazurkiewicz at about the same time although they worked independently. The result is given in the following theorem.

Theorem 5.1. A topological space S is a Peano space iff S is non-empty, compact, connected, locally connected and metrizable.[5]

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[5]Hall and Spencer, *op. cit.*, p. 204.

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