Extension of a Quasi

Topological Space

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Particular gratitude is expressed to my wife for her encouragement and to Dr. Bruyr for his suggestions which helped lead me in this direction.

PREFACE

"To qualify as pure a mathematical topic had to be useless; if useless it was not only pure, but beautiful. If useful--which is to say impure--it was ugly, and the more useful, the more ugly."[1] These words echo the ideas of a pure mathematician, Godfrey Harold Hardy, referenced to the world outside the art of mathematics. To be sure, his words reflect my sentiment toward mathematics. For me it is sufficient to study mathematics for its own value, not seeking an application in the physical world. With this philosophy I embarked on a study which, as I see it, has no relationship with the physical world.

The realm of my quasi topological spaces was my undeveloped imagination, heavily slanted by my background in topology. My tools for this work were my mind and my prior work in topological spaces.

Using these ingredients I constructed a concept that for the non-mathematical world seems useless. Thus, by Mr. Hardy's standards if it is useless, then it is pure and beautiful. This, of course, does not necessarily justify a mathematical work. Accordingly, Mr. Hardy felt that a

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[1] James R. Newman, "Commentary on G.H. Hardy", (Vol. IV of The World of Mathematics, ed. James R. Newman. 4 vols.; New York: Simon and Schuster, 1956), p. 2024.

mathematical work must also be "serious".[2] By "serious" he meant that a mathematical theorem or work should tie together significant mathematical ideas. Mathematical ideas become significant if they "can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas."[3]

Naturally, to me the paper appears serious. For another the subject may appear worthless. That matters not; for I found the paper not only a worthwhile endeavor from an educational viewpoint but also quite enjoyable.

Endeavoring to approach a topological space from my more fundamental quasi topological space, I felt an accomplishment in relating a topological space more fundamentally to the basics of set theory.

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[2] G.H. Hardy, "A Mathematicians Apology", (Vol. IV of The World of Mathematics, ed. James R. Newman. 4 vols.; New York: Simon and Schuster, 1956), p. 2029.

[3] G.H. Hardy, "A Mathematicians Apology", (Vol. IV of The World of Mathematics, 4 vols.; New York: Simon and Schuster, 1956), p. 2029.

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CHAPTER I

INTRODUCTION

This chapter introduces a few elementary definitions and concepts from set theory. An index of notation immediately follows the last chapter.

Throughout this paper collections of "points" are studied. These collections of "points" will be referred to as "sets". At any point during the paper the discussion is restricted to a certain set of points and considers no other points. Ensuing discussions are then relative to this set, which is the "universe". Naturally, the "universe" varies from time to time. Throughout the discussion sets will be denoted by capital letters or with the familiar bracket notation. For example, the set that contains the points x, y, and z may be denoted by $\{x, y, z\}$. Another common notation to be used is a qualitative description of a set. When a set consists only of those integers x, where x is greater than 3, then it can be denoted by $\{x \mid x \in I \text{ and } x > 3\}$. A point p in a set A is denoted by $p \in A$. Now the concept of a subset can be made clear.

Definition 1.1. A set A is a subset of a set B, written $A \subset B$, if and only if for all $p \in A$, $p \in B$.

Thus, the set that contains no points is a subset of every set. This set, denoted by $\emptyset$, vacuously satisfies the definition of a subset. Since the set contains no points, it satisfies the definition for all its points.

Henceforth, the symbol "$\forall$" may be used to stand for "for all" and the letters "iff" for "if and only if".

The logic used eliminates the possibility of having a point in a subset of a universe and not in the subset. The latter case is denoted by $p \notin A$, where p is the point and A a subset of the universe. This is the idea of a complement.

Definition 1.2. Relative to a universe S, $p \notin A$ where $A \subset S$ iff p is an element of the complement (-A) of A.

Two other very important concepts from set theory are union and intersection. For the purpose of discussion let S be the universe, $A \subset S$, and $B \subset S$.

Definition 1.3. The union of A and B (written $A \cup B$) is $\{x \mid x \in A \text{ or } x \in B\}$.

Definition 1.4. The intersection of A and B (written $A \cap B$) is $\{x \mid x \in A \text{ and } x \in B\}$.

Another way to obtain a set by operating on two sets is known as the Cartesian product of sets. Let A and B be sets.

Definition 1.5. The Cartesian product of A and B is $\{(a,b) \mid a \in A \text{ and } b \in B\}$ where (a,b) denotes the ordered pair of a and b.

If $A = \emptyset$ or $B = \emptyset$, then by definition the Cartesian product (written A x B) of A and B is the empty set. The Cartesian product is important for defining function, quasi topological product, and topological product.

It should be emphasized that these basic ideas and many other underlying concepts in the field of set theory and the topology of real numbers are assumed to be prior knowledge. However, the definitions presented in this chapter are designed to orient the reader and the "underlying" concepts in later chapters will be mentioned to keep the reader on the right path.

The path is to define a quasi topological space, discuss some of its more general characteristics, and examine the immediate outgrowths of the definitions and the single axiom of the space. Later, a definition for a topological space is presented in comparison to a quasi topological space, a quasi topological space is extended to a topological space, and consequences of this extension are discussed. Then, invariant properties under this extension are discussed and finally, consequences of this work, relative solely to a topological space, are explored.

CHAPTER II

A QUASI TOPOLOGICAL SPACE

"One way to establish a theorem is to prove it, and that means to show how it follows from previous theorems, i.e., theorems we already regard as established. If now we demand that these theorems be proved, we have to go back to still earlier theorems, and so on. It becomes clear that if we are going to prove anything, there must also be propositions that we regard as true but for which we demand no proof. In order to go forward, we must stop going backward. When certain propositions are laid down as the starting point of a deductive theory, and no proofs are required for these propositions, then these propositions are called 'axioms'."[1]

"Just as it is with propositions, so it is with definitions. To define an object or term is to give its meaning in terms of other objects and terms, and to define these would mean to relate them to still other object and terms, and so on. Again it is clear that if anything is to be

[1]A. Seidenberg, Lectures in Projective Geometry (D. Van Nostrand Company, Inc., Princeton, 1955), p. 42.

defined, there must also be undefined terms."[2] For a quasi topological space, point and subset are undefined. Subset is undefined in the sense that one must know of what it is a subset.

The propositional origin of a quasi topological space is the following axiom.

Axiom 1. For all points p, $\exists$ (there exists) at least one subset U $\ni$ (such that) $p \in U$.

Axiom 1 is needed to define a quasi topological space, which depends on the axiom for its meaning.

Definition 2.1. A set S, with a collection τ of subsets of S, is a quasi topological space iff τ satisfies Axiom 1 for all points of S.

Speaking of a quasi topological space in terms of a set and a collection of its subsets needs notation. Thus, a quasi topological space with S as the set and τ the collection of subsets that satisfies Definition 2.1 is denoted by (S,τ).

Definition 2.2. The collection of subsets τ is called the quasi topology of the quasi topological space.

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[2] A. Seidenberg, Lectures in Projective Geometry (D. Van Nostrand Company, Inc., Princeton, 1955), p. 42.

Definition 2.3. A set V is a neighborhood of a point p, $p \in S$, iff $p \in V$ and $V \in \mathcal{T}$.

A neighborhood of a point p will be denoted by N_p. Then, $\{N_p \mid p \in S\}$ represents the collection of all neighborhoods in $(S,\mathcal{T})$.

Theorem 2.4. Let $(S,\mathcal{T})$ be a quasi topological space. Then, $\mathcal{T} - \{\emptyset\} = \{N_p \mid p \in S\}$.

Proof. Let $V \in \mathcal{T} - \{\emptyset\}$ where $\mathcal{T} - \{\emptyset\} = \mathcal{T} \cap - \{\emptyset\}$. Then, $V \in \mathcal{T}$ but $V \neq \emptyset$. Thus, $\exists\, p \in S \ni p \in V$. This satisfies Definition 2.3. Thus, $\mathcal{T} - \{\emptyset\} \subset \{N_p \mid p \in S\}$. Let $V \in \{N_p \mid p \in S\}$. Then, V is a neighborhood of some point $p \in S$. By Definition 2.3, $V \in \mathcal{T}$ and $p \in V$. Thus, $V \neq \emptyset$ and $V \in \mathcal{T} - \{\emptyset\}$. Thus, $\{N_p \mid p \in S\} \subset \mathcal{T} - \{\emptyset\}$. Therefore, $\mathcal{T} - \{\emptyset\} = \{N_p \mid p \in S\}$. ||

The collection of all neighborhoods of a point p will be denoted by $\{N_p\}$.

Definition 2.5. A subset G of S is open iff $\forall\, p \in G\ \exists$ a neighborhood $N_p \ni N_p \subset G$.

Theorem 2.6. A set U is open iff $\forall\, p \in U\ \exists$ an open set $V \ni p \in V \subset U$.

Proof. (Sufficiency) Let U be open. Then, $\forall\, p \in U\ \exists$ an open set V, namely U itself, $\ni p \in V \subset U$.

(Necessity) Let, $\forall p \in U$, $\exists$ an open set $V \ni p \in V \subset U$. Then, consider any $p \in U$. Then, $\exists$ an open set $V \ni p \in V \subset U$. Since V is open, $\exists N_p \ni p \in N_p \subset V$. Since $V \subset U$, $N_p \subset U$. Thus, U is open by the definition of open sets.||

Let $(S,\mathcal{T})$ be a quasi topological space. By Definition 2.5, if $V \in \{N_p \mid p \in S\}$, then V is open.

Before proceeding, the concepts of indexed sets, the union of a collection of sets, and the intersection of a collection of sets must be understood. Let S be a universe and σ a collection of subsets of S. Let each set have a name and consider the set that contains these names as elements. Then, σ is indexed by this set of names. For example, let N (the set of natural numbers) be the index set. To each $n \in N$ associate a set A_n. Then, $\{A_n\}_{n \in N}$ denotes a collection of sets indexed by N.

Definition 2.7. Let S be the universe, Λ an index set, and $\{A_\alpha\}_{\alpha \in \Lambda}$ a collection of subsets of S indexed by Λ. Then, $\bigcup_{\alpha \in \Lambda} A_\alpha$ is the union of all elements in $\{A_\alpha\}_{\alpha \in \Lambda}$. Thus, $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$ iff $\exists \alpha \in \Lambda \ni x \in A_\alpha$. If $\Lambda = \emptyset$, then $\bigcup_{\alpha \in \Lambda} A_\alpha = \emptyset$.

Definition 2.8. Let S again be the universe, Λ an index set, and $\{A_\alpha\}_{\alpha \in \Lambda}$ a collection of subsets of S indexed by Λ.

Then, $\bigcap_{\alpha \in \Lambda} A_\alpha$ is the intersection of all elements in $\{A_\alpha\}_{\alpha \in \Lambda}$. Thus, $x \in \bigcap_{\alpha \in \Lambda} A_\alpha$ iff $x \in A_\alpha \ \forall\ \alpha \in \Lambda$. If $\Lambda = \emptyset$, then $\bigcap_{\alpha \in \Lambda} A_\alpha = S$.

A mathematical theorem is often stated biconditionally, that is, in the form "p iff q". The statement is interpreted as "if p, then q and if q, then p". The sufficient part of the statement is "if p, then q". Necessity is "if q, then p". To avoid confusion in the proof of a biconditional theorem, the words "sufficiency" and "necessity" will be enclosed in parentheses prior to the beginnings of the proofs of the respective parts of the theorem. This was done for Theorem 2.6.

An open set in $(S,\mathcal{T})$ is characterized as follows.

Theorem 2.9. A set O is open in $(S,\mathcal{T})$ iff $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $A_\alpha \in \{N_p \mid p \in S\}\ \forall\ \alpha \in \Lambda$.

Proof. (Sufficiency) Let O be open and suppose $O \neq \emptyset$. Then, for $p \in O\ \exists\ V_p \in \{N_p\} \ni p \in V_p \subset O$ by Definition 2.5. Thus, $O = \bigcup_{p \in O} V_p$ for if $x \in O$, then $x \in V_x$ by the way V_x is defined and $x \in \bigcup_{p \in O} V_p$. Thus, $O \subset \bigcup_{p \in O} V_p$. If $x \in \bigcup_{p \in O} V_p$, then $x \in O$ since $V_p \subset O\ \forall\ p \in O$. Thus, $\bigcup_{p \in O} V_p \subset O$. Consequently, $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $\Lambda = O$ and $A_\alpha \in \{V_p\}_{p \in O}$ $\forall\ \alpha \in \Lambda$. If $O = \emptyset$, then $\bigcup_{\alpha \in \Lambda} A_\alpha = \emptyset$ where $\Lambda = \emptyset$.

(Necessity) Let $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $A_\alpha \in \{N_p \mid p \in S\}$

$\forall \alpha \in \Lambda$. If $x \in O$, then $x \in A_\alpha$ for some $\alpha \in \Lambda$. For some p, $A_\alpha = N_p$. However, $N_p \in \{N_x\}$ since $x \in N_p$. Since $A_\alpha \subset O$ and A_α as described is a neighborhood of x, then O is open by Definition 2.5. If $\Lambda = \emptyset$, then $\bigcup_{\alpha \in \Lambda} A_\alpha = \emptyset$ and $\emptyset$ is open vacuously. ||

Another characterization of an open set in a quasi topological space $(S, \mathcal{T})$ is the following.

Theorem 2.10. A set O is open iff $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $A_\alpha \in \mathcal{T}$ $\forall \alpha \in \Lambda$.

Proof. (Sufficiency) Let O be open and $O \neq \emptyset$. Then, by Theorem 2.9 $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $A_\alpha \in \{N_p \mid p \in S\} \forall \alpha \in \Lambda$. By Definition 2.3, $A_\alpha \in \mathcal{T} \; \forall \alpha \in \Lambda$. If $O = \emptyset$, then $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $\Lambda = \emptyset$.

(Necessity) Let $O = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $A_\alpha \in \mathcal{T} \; \forall \alpha \in \Lambda$. If $O \neq \emptyset$, $\exists\, p \in O$ and hence $p \in A_\alpha$ for some $\alpha \in \Lambda$. Thus, $\exists\, A_\alpha \neq \emptyset$ for some $\alpha \in \Lambda$. Since $A_\alpha \neq \emptyset$ and $p \in A_\alpha$, $A_\alpha \in \{N_p \mid p \in S\}$. Thus, by Definition 2.5 O is open. If $\Lambda = \emptyset$, then $\bigcup_{\alpha \in \Lambda} A_\alpha = \emptyset$ and $\emptyset$ is open as before. If $A_\alpha = \emptyset \; \forall \alpha$, then $\bigcup_{\alpha \in \Lambda} A_\alpha = \emptyset$ and $\emptyset$ is open. ||

Theorem 2.11. The union of any collection of open sets in a quasi topological space is open.

Proof. Let $\{O_\alpha \mid \alpha \in \Lambda\}$ be a collection of open sets. If $\Lambda = \emptyset$, then $\bigcup_{\alpha \in \Lambda} O_\alpha = \emptyset$ and $\emptyset$ is vacuously open. Now, let

$p \in \bigcup_{\alpha \in \Lambda} O_\alpha$. Then, $p \in O_\alpha$ for some $\alpha \in \Lambda$. Since O_α is open, $\exists N_p \ni p \in N_p \subset O_\alpha$. Since $N_p \subset \bigcup_{\alpha \in \Lambda} O_\alpha$, $\bigcup_{\alpha \in \Lambda} O_\alpha$ is open by Definition 2.5. If $O_\alpha = \emptyset \; \forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} O_\alpha = \emptyset$ and $\emptyset$ is open vacuously.||

Corollary 2.12. If $(S, \mathcal{T})$ is a quasi topological space, then S is open.

Proof. By Axiom 1, $\forall p \in S \; \exists V \in \mathcal{T} \ni p \in V \subset S$ and V is open since by Definition 2.3, $V \in \{N_p \mid p \in S\}$. Thus, $\forall p \; \exists N_p \ni p \in N_p \subset S$. Therefore, S is open.||

As stated in Theorem 2.11, $\emptyset$ vacuously satisfies the definition of an open set.

Definition 2.13. A subset C of S in a quasi topological space $(S, \mathcal{T})$ is closed iff S-C is open.

Theorem 2.14. If $(S, \mathcal{T})$ is a quasi topological space, then S and $\emptyset$ are closed.

Proof. The set $\emptyset$ is open and $S - \emptyset = S$. Thus, S is closed. Since S is open and $S - S = \emptyset$, $\emptyset$ is closed.||

The following is DeMorgan's Theorem.

Theorem 2.15. If Λ is an index set, S the universe, and $\{A_\alpha\}_{\alpha \in \Lambda}$ a collection of subsets indexed by Λ, then

1.) $S - \bigcap_{\alpha \in \Lambda} A_\alpha = \bigcup_{\alpha \in \Lambda} S - A_\alpha$

2.) $S-\bigcup_{\alpha\in\Lambda} A_\alpha = \bigcap_{\alpha\in\Lambda} S-A_\alpha$

Proof. For part 1, let $x \in S-\bigcap_{\alpha\in\Lambda} A_\alpha$. Then $x \in S$ and $x \notin \bigcap_{\alpha\in\Lambda} A_\alpha$. Thus, $x \notin A_\alpha$ for some $\alpha \in \Lambda$. Thus, $x \in S-A_\alpha$ for this α and $x \in \bigcup_{\alpha\in\Lambda} S-A_\alpha$. Therefore, $S-\bigcap_{\alpha\in\Lambda} A_\alpha \subset \bigcup_{\alpha\in\Lambda} S-A_\alpha$. Let $x \in \bigcup_{\alpha\in\Lambda} S-A_\alpha$. Then, $x \in S-A_\alpha$ for some $\alpha \in \Lambda$. Then, $x \in S$ but $x \notin A_\alpha$ for this same $\alpha \in \Lambda$. Thus, $x \notin \bigcap_{\alpha\in\Lambda} A_\alpha$ and x is then an element of $S-\bigcap_{\alpha\in\Lambda} A_\alpha$. Therefore, $\bigcup_{\alpha\in\Lambda} S-A_\alpha \subset S-\bigcap_{\alpha\in\Lambda} A_\alpha$ and $S-\bigcap_{\alpha\in\Lambda} A_\alpha = \bigcup_{\alpha\in\Lambda} S-A_\alpha$.

For part 2, let $x \in S-\bigcup_{\alpha\in\Lambda} A_\alpha$. Then, $x \in S$ but $x \notin \bigcup_{\alpha\in\Lambda} A_\alpha$. Thus, $x \notin A_\alpha\ \forall\ \alpha \in \Lambda$. Thus, $x \in S-A_\alpha\ \forall\ \alpha \in \Lambda$ and $x \in \bigcap_{\alpha\in\Lambda} S-A_\alpha$. Therefore, $S-\bigcup_{\alpha\in\Lambda} A_\alpha \subset \bigcap_{\alpha\in\Lambda} S-A_\alpha$. Let $x \in \bigcap_{\alpha\in\Lambda} S-A_\alpha$. Then, $x \in S$ but $x \notin A_\alpha\ \forall\ \alpha \in \Lambda$. Thus, $x \notin \bigcup_{\alpha\in\Lambda} A_\alpha$. Therefore, $x \in S-\bigcup_{\alpha\in\Lambda} A_\alpha$, $\bigcap_{\alpha\in\Lambda} S-A_\alpha \subset S-\bigcup_{\alpha\in\Lambda} A_\alpha$, and $S-\bigcup_{\alpha\in\Lambda} A_\alpha = \bigcap_{\alpha\in\Lambda} S-A_\alpha$. ||

Theorem 2.16. The intersection of any collection of closed sets in $(S,\mathcal{T})$ is closed.

Proof. Let $\{C_\alpha \mid \alpha \in \Lambda\}$ be a collection of closed sets. Consider $\bigcap_{\alpha\in\Lambda} C_\alpha$. Then, $S-\bigcap_{\alpha\in\Lambda} C_\alpha = \bigcup_{\alpha\in\Lambda} S-C_\alpha$ by DeMorgan's Theorem. Since C_α is closed, by the definition of closed sets, $S-C_\alpha$ is open for all $\alpha \in \Lambda$. By Theorem 2.11, $\bigcup_{\alpha\in\Lambda} S-C_\alpha$ is open. Thus, $\bigcap_{\alpha\in\Lambda} C_\alpha$ is closed. ||

Another characterization of an open set comes from the definition of a closed set.

Theorem 2.17. A set O is open iff S-O is closed.

Proof. (Sufficiency) Let O be open. Consider S-O. Then, $S-(S-O) = S \cap -(S \cap -O)$ and $S \cap -(S \cap -O) = S \cap (-S \cup O)$ by DeMorgan's Theorem. Then, $S \cap (-S \cup O) = (S \cap -S) \cup (S \cap O)$ $= \emptyset \cup O = O$. Since O is open, S-O is closed.

(Necessity) Let S-O be closed. Since S-O is closed, its complement is open. The complement of S-O is O. Thus, O is open. ||

Definition 2.18. A point p is a limit point of a set A iff every open set containing p contains a point $q \in A$, $q \neq p$.

It is feasible to have a notation to express an open set containing p. This is O_p. If A is a set in $(S,\mathcal{T})$, then the set of limit points of A is denoted by A'. Using Definition 2.18 a characterization for a closed set can be presented.

Theorem 2.19. In $(S,\mathcal{T})$ a set A is closed iff $A' \subset A$.

Proof. (Sufficiency) Let A be closed and let p be a limit point of A. Suppose (§) $p \in S-A$, i.e., A' not a subset of A. If $p \in S-A$, then $\exists N_p \ni N_p \subset S-A$, since S-A is open. This contradicts the fact that p is a limit point of A, since N_p is open. Thus, $A' \subset A$.

(Necessity) Let A contain all its limit points. Let $p \in S-A$. Then, $\exists O_p \ni O_p \subset S-A$, since p is not a limit point. Since O_p is open, $\exists N_p \subset O_p$. Thus, S-A is open by the definition of an open set. Therefore, A is closed. ||

Another characterization of a closed set in a quasi topology is an outgrowth of the closure of a set.

Definition 2.20. Let $(S,\mathcal{T})$ be a quasi topological space. The closure of a set A is $A \cup A'$. This set is denoted by $\bar{A}$.

Theorem 2.21. The closure of a set A is closed.

Proof. It must be shown that $\bar{A}$ contains all its limit points. Suppose $\bar{A}$ doesn't contain all its limit points, i.e., $\S\ \exists\ p \notin A \cup A' \ni \forall\, O_p \ \exists\ q \in O_p$ where $q \in A \cup A'$ but $p \neq q$. Consider any O_p and the point $q \in A \cup A'$. Since $q \in A \cup A'$, $q \in A$ or $q \in A'$. If $q \notin A$, then $q \in A'$. Since $q \in A'$, q is a limit point of A. Thus, O_p contains a point $x \in O_p$ where $x \in A$ but $x \neq q$. Thus, O_p is an open set containing $x \in A$ where $x \neq p$, since $p \notin A$.

If $q \in A$, then O_p contains a point $q \in A$, $q \neq p$. Thus, in either case O_p contains a point $x \in A$, $x \neq p$. Therefore, p is a limit point of A. This contradicts that $p \notin A \cup A'$. Thus, the assumption was wrong and $p \in A \cup A'$ and $\bar{A}$ contains all its limit points and hence is closed. ||

Theorem 2.22. Let B be any closed set in $(S,\mathcal{T})$ that contains A, then $\bar{A} \subset B$.

Proof. It must be shown that $A' \subset B$. If $x \in A'$, then $\forall\, O_x$

$\ni O_x \in \{O_x\}$, $\{O_x\}$ denoting the collection of open sets about x, $\exists\, y \in O_x$, $y \in A$, but $y \neq x$. Since $\forall\, y$, $y \in A$, $y \in B$ and thus $x \in B'$. Since B is closed, by Theorem 2.19 $x \in B$. Thus, $\bar{A} \subset B$. ||

Theorem 2.23. Let $A \subset S$ in $(S,\mathcal{T})$, then $\bar{A} = \bigcap_{\alpha \in \Lambda} A_\alpha$ where $\{A_\alpha\}_{\alpha \in \Lambda}$ is the collection of closed sets $\ni A \subset A_\alpha\ \forall\, \alpha \in \Lambda$.
Proof. Since $\bar{A}$ is a closed set that contains A, $\bar{A} \in \{A_\alpha\}_{\alpha \in \Lambda}$. Thus, if $x \in \bigcap_{\alpha \in \Lambda} A_\alpha$, then $x \in \bar{A}$. So, $\bigcap_{\alpha \in \Lambda} A_\alpha \subset \bar{A}$. By Theorem 2.16, $\bigcap_{\alpha \in \Lambda} A_\alpha$ is closed. Thus, $\bar{A} \subset \bigcap_{\alpha \in \Lambda} A_\alpha$ by Theorem 2.22 and the fact that $A \subset \bigcap_{\alpha \in \Lambda} A_\alpha$. Therefore, $\bar{A} = \bigcap_{\alpha \in \Lambda} A_\alpha$. ||

Theorem 2.24. In $(S,\mathcal{T})$ where $A \subset S$, A is closed iff $A = \bar{A}$.
Proof. (Sufficiency) Let A be closed, then $A' \subset A$ by Theorem 2.19. Thus, $A \cup A' = A$. Since $A \cup A' = \bar{A}$, $A = \bar{A}$.

(Necessity) If $A = \bar{A}$, $A' \subset A$ and by Theorem 2.19, A is closed. ||

As a prelude to Chapter 3 two elementary theorems regarding a quasi topological space are now presented.

Theorem 2.25. In $(S,\mathcal{T})$ $\forall\, p \in S$ $\exists$ at least one open set containing p.
Proof. Let $p \in S$. By Axiom 1 $\exists\, V \in \mathcal{T} \ni p \in V \subset S$. By Theorem 2.4, $V \in \{N_p \mid p \in S\}$. Thus, $\forall\, x \in V$ $\exists\, N_x$, namely V itself, $\ni x \in N_x \subset V$. Therefore, V is open. ||

Theorem 2.26. Every non-empty open set in $(S,\mathcal{T})$ is a set of points.

Proof. Let U be an open set. Let $x \in U$. Then, $\exists\, N_x \subset U$. Since $N_x \in \mathcal{T}$, $N_x \subset S$. Thus, $x \in S$. ||

Before proceeding to Chapter 3 it would be well to review the concept of a function and ideas dependent on this concept.

Definition 2.27. A rule f that assigns to each point p in a set S one and only one point q in a set T is a function from S to T, $f: S \longrightarrow T$. The image of x under f is denoted by $f(x)$.

Definition 2.28. In Definition 2.27, S is the domain of f. The range of f, denoted by $f(S)$, is $\{x \mid x \in T \text{ and } \exists\, y \in S \ni f(y) = x\}$.

Definition 2.29. A function $f: S \longrightarrow T$ is one-to-one iff $\forall$ x and y in S, if $f(x) = f(y)$, then $x = y$.

Definition 2.30. A function $f: S \longrightarrow T$ is onto T iff $f(S) = T$.

Definition 2.31. Two sets S and T are in one-to-one correspondence iff $\exists\, f: S \longrightarrow T \ni f$ is one-to-one and onto T.

The word "mapping" will be used synonimously with "function".

CHAPTER III

EXTENSION OF A QUASI TOPOLOGICAL SPACE TO A TOPOLOGICAL SPACE

In the framework of a topological space the undefined terms are "point" and "open set". A sufficient set of axioms for a topological space are the following.

1. Every non-empty open set is a set of points.
2. The empty set $\emptyset$ is an open set.
3. For all p, $\exists$ an open set $\ni$ p is in this open set.
4. The union of any collection of open sets is open.
5. The intersection of any finite collection of open sets is open.

Definition 3.1. A set S, together with a collection of subsets called open sets, is a topological space iff the collection of open sets satisfies the above five axioms.

A definition is in order for the intuitive concept of finiteness.

Definition 3.2. A set is infinite iff $\exists$ a one-to-one function $f: S \longrightarrow S \ni f(S)$ is a proper subset of S.

Definition 3.3. A set is finite iff it is not infinite.

Considering the collection of open sets in a quasi topological space $(S,\mathcal{T})$, the set S, Theorem 2.26, the remark following Corollary 2.12, Theorem 2.25, and Theorem 2.11, it is seen that the first four axioms stated above are satisfied for the collection of open sets and the set of points S.

Example 3.4. Let $S = \{a,b,c,d\}$ and $\mathcal{T} = \{\{a,b\}, \{b,c\}, \{c,d\}\}$, then $\{N_p \mid p \in S\} = \{\{a,b\}, \{b,c\}, \{c,d\}\}$ and $\{O_p \mid p \in S\} = \{\emptyset, \{a,b\}, \{b,c\}, \{c,d\}, \{a,b,c\}, \{b,c,d\}, S\}$. The set $\{O_p \mid p \in S\}$ represents the collection of open sets in $(S,\mathcal{T})$. Clearly, $\{a,b\} \cap \{b,c\} = \{b\}$ and $\{b\} \notin \{O_p \mid p \in S\}$.

Thus, Axiom 5 of a topological space is the sole axiom that is unsatisfied by a quasi topological space.

The question naturally arose as to extending a quasi topological space to a topological space. Since this naturally hinged on the collection of open sets, it was desired to extend the existing collection of open sets in the quasi topological space and yet stay within the framework of the definition of open sets, i.e., use neighborhoods to obtain additional open sets. It will now be shown that, by constructing a new collection of neighborhoods from the existing collection of neighborhoods, a quasi topological space was extended to a topological space; by taking all possible finite collections of neighborhoods, taking the intersection of all sets in each collection, letting the collection of these intersections form the new collection of neighborhoods, and extending the collection of open sets using this new collection of neighborhoods and the definition of open sets in a quasi topological space, a topological space was obtained where the extended collection of open sets satisfies the axioms for a topological space.

To state and prove the theorem that describes this extension, some extensive discussion related to finite and infinite sets is necessary.

Definition 3.5. Consider $f: A \longrightarrow B$ and $g: B \longrightarrow C$. The composition of f and g, written gf, is a function from A to C, $gf: A \longrightarrow C$. Then $gf(x) = g(f(x))$. The domain of gf is A and the range is $\{z \mid z \in C$, where $\exists\, y \in B \ni y = f(x)$ for some $x \in A$, and $g(y) = z\}$.

Definition 3.6. Consider $f: A \longrightarrow B$. Let $C \subset A$. Then, $f \mid C$ is a function $g \ni g: C \longrightarrow B$ and $g(x) = f(x)\ \forall\, x \in C$. The function g is called the restriction of f to C, $C \subset A$.

Definition 3.7. Let $f: A \longrightarrow B$ and $y \in B$. Then, $f^{-1}(y) = \{x \mid x \in A$ and $f(x) = y\}$. It can be shown that f^{-1} is a function on B iff f is onto and one-to-one.

Theorem 3.8. Let A and B be sets where $f: A \longrightarrow B$ is an onto function. If A is finite, then B is finite.

Proof. The proof consists of two parts: (i) f is one-to-one and (ii) f is not one-to-one.

Suppose B is infinite. Then, $\exists$ $g: B \longrightarrow B \ni g(B)$ is a proper subset of B and g is one-to-one. The composition of onto functions is onto and the composition of one-to-one functions is one-to-one.

(i) Since f is one-to-one and onto, f^{-1} is a function on B. Consider $f^{-1}gf(A)$. However, f^{-1} has to be one-to-one, otherwise f would violate the definition of a function. If $D \subset B$, then $f^{-1}|D$ is a one-to-one function $\ni (f^{-1}|D): D \longrightarrow A$. Since f, f^{-1}, and g are one-to-one, $f^{-1}gf(A)$ is one-to-one on A. Since $f(A) = B$, $f^{-1}gf(A) = f^{-1}g(B)$. Thus $f^{-1}g(B) = f^{-1}(E)$ where E is a proper subset of B by the definition of g. Suppose $f^{-1}(E) = A$. Since f is onto B, $\forall$ $y \in B$ $\exists$ $x \in A \ni f(x) = y$. Since E is a proper subset of B, $\exists$ $s \in B$ but $s \notin E$. Since $s \in B$, $\exists$ $t \in A \ni f(t) = s$. Since $t \in A$, $\exists$ $s_1 \in E \ni f(t) = s_1$. Since $s \notin E$, $s_1 \neq s$. Thus, f is not a function. Therefore, $f^{-1}(E)$ is a proper subset of A and A is infinite. This contradicts the fact that A is finite. Thus, B is infinite.

(ii) If f is not one-to-one, $\forall$ $y \in B$ consider one and only one $x \in A \ni y = f(x)$. Then, the set E of these points is a proper subset of A. Denote $f \mid E$ by h. Clearly, h is one-to-one and onto. Thus, h^{-1} is one-to-one. Therefore,

$h^{-1}gh$ is one-to-one on E. The rest of the proof for this part duplicates that above, except that E is substituted for A and h for f. ||

Definition 3.9. A sequence is a set A indexed by the set N of all natural numbers. Thus, it is a mapping $f: N \longrightarrow A$, $\ni f$ is an onto mapping. The sequence is denoted by $\{a_n\}$ where a_n is the element of A indexed by n, i.e., the image of n under f.

To continue the development of the prerequisites for the extension, the Axiom of Choice has been assumed.

Axiom of Choice. Let S be a set and β a non-empty collection of non-empty subsets of S. Then, $\exists f: \beta \longrightarrow S \ni f(A) \in A \;\forall\; A \in \beta$.

Theorem 3.10. If S is an infinite set and N the set of natural numbers, then $\exists f: N \longrightarrow S \ni f$ is one-to-one.

Proof. Let S be infinite and β the collection of all non-empty subsets of S. Then, by the Axiom of Choice, $\exists h: \beta \longrightarrow S \ni h(A) \in A \;\forall\; A \in \beta$. Define $f: N \longrightarrow S$ as follows.

$$f(1) = h(S)$$
$$f(n+1) = h(S - \bigcup_{j=1}^{n} f(j)).$$

However, $S \neq \emptyset$ and $S - \bigcup_{j=1}^{n} f(j) \neq \emptyset$, for if $S - \bigcup_{j=1}^{n} f(j) = \emptyset$, then $S = \bigcup_{j=1}^{n} f(j)$ and there could not exist a one-to-one mapping of $\bigcup_{j=1}^{n} f(j)$ onto a proper subset of $\bigcup_{j=1}^{n} f(j)$. Thus, the mapping is well defined. If $f(m) = f(n)$, where $m \neq n$, then without loss of generality, assume $m < n$. Since $m < n$, $f(n) = h(S-\bigcup_{i=1}^{m-1} f(i)-\bigcup_{i=m+1}^{n-1} f(i)- \{f(m)\})$. Thus, if $f(n) = f(m)$, then $f(m) \in S-\bigcup_{i=1}^{m-1} f(i)-\bigcup_{i=m+1}^{n-1} f(i)- \{f(m)\}$ but clearly it is not. Thus, $f(m) \neq f(n)$, if $m \neq n$. Therefore, f is one-to-one.||

Theorem 3.11. If $\exists$ a sequence $\{p_n\}$ of distinct points of a set K, then K is infinite.

Proof. By the definition of a sequence, $\exists\, f:N \longrightarrow K \ni f$ is one-to-one, since the points are distinct. Since f is one-to-one, f^{-1} is a function on $f(N)$. Suppose K is finite. Define $h:K \longrightarrow N \ni h(k) = 1$ if $k \notin f(N)$ and $h(k) = f^{-1}(k)$ if $k \in f(N)$. However, $K \neq \emptyset$, since $\exists$ a sequence $\{p_n\}$ of distinct points of K. The function h is onto N, since $f^{-1}(f(N)) = N$. Thus, by Theorem 3.8, N is finite. However, $\exists$ a one-to-one function $g \ni g(x) = 2x\ \forall\ x \in N$ from N onto a proper subset of N which makes N infinite. Therefore, the assumption that K is finite is wrong and thus K is infinite.||

Theorem 3.12. Let $K \neq \emptyset$. The set K is finite iff $\exists$ a finite subset L of N and $f:L \longrightarrow K \ni f$ is onto.

Proof. (Necessity) Let $\exists$ a finite subset L of N and $f:L \longrightarrow K \ni f$ is onto. Then, K is finite by Theorem 3.8.

(Sufficiency) To prove sufficiency, the contrapositive will be proved, i.e., if $\forall$ finite subset L of N $\nexists$ (there does not exist) $f: L \longrightarrow K \ni f$ is onto, then K is infinite. Suppose $\forall$ L $\nexists$ $f: L \longrightarrow K \ni f$ is onto. A sequence $\{p_n\}$ of distinct points of K will be constructed which will imply K is infinite by Theorem 3.11. Let β be the collection of all non-empty subsets of K. By the Axiom of Choice $\exists\ g: \beta \longrightarrow K \ni g(A) \in A$ $\forall A \in \beta$. Let p_i be defined as follows.

$$p_1 = g(K)$$
$$p_2 = g(K - p_1)$$
$$p_{k+1} = g(K - \bigcup_{j=1}^{k} p_j).$$

The function g is well defined since $K \neq \emptyset$ and if $K - \bigcup_{j=1}^{k} p_j = \emptyset$ then, $K = \bigcup_{j=1}^{k} p_j$ and $\exists\ f: \{1, 2, \ldots, k\} \longrightarrow K \ni$

$$f(1) = p_1$$
$$f(2) = p_2$$
$$\vdots$$
$$f(k) = p_k \text{ and } f \text{ is onto.}$$

If $K = \bigcup_{j=1}^{k} p_j$, f contradicts the non-existence of an onto function $\ni f: L \longrightarrow K$ where L is a finite subset of N. Thus, the function g is well defined and, as in Theorem 3.10, is one-to-one.

Therefore, the sequence consists of distinct points and K is infinite by Theorem 3.11. ||

Let I_n denote the set of all natural numbers less than or equal to n.

Definition 3.13. A set K, $K \subset N$, is bounded iff $\exists\, n_0 \in N \ni x \leq n_0 \ \forall\, x \in K$.

Theorem 3.14. Let $K \neq \emptyset$. Then, K is finite iff $\exists$ a natural number n_0 and a one-to-one function from I_{n_0} onto K.

Proof. (Necessity) Let $\exists$ a natural number n_0 and a one-to-one function from I_{n_0} onto K. By Theorem 3.12, K is finite since I_{n_0} is finite.

(Sufficiency) Let K be finite. By Theorem 3.12, $\exists$ a finite set of natural numbers L and $f_1 : L \longrightarrow K \ni f_1$ is onto. It must be shown that if L is finite, L is bounded. Suppose L is not bounded. Then, given any natural number n, $\exists\, x \in L \ni x > n$. If L is not bounded, define $f : N \longrightarrow L \ni f$ is onto as follows.

$$f(1) = \text{the least natural number in } L$$
$$f(n+1) = \text{the least natural number in } L - \bigcup_{j=1}^{n} f(j).$$

Since $\forall$ subset K of N $\exists\, x \in K \ni x \leq n \ \forall\, n \in K$, the fact that $L \neq \emptyset$ since f is onto, and the fact that $L - \bigcup_{j=1}^{n} f(j) \neq \emptyset$, the function f is well defined. For if $L - \bigcup_{j=1}^{n} f(j) = \emptyset$, then $\exists$ an $n_0 \ni L - \bigcup_{j=1}^{n_0} f(j) = \emptyset$. Thus, if $x \in L$ $x \leq f(n_0)$ and L would be bounded. The function f is one-to-one as in Theorem 3.10. The set $\{f(j)\}$ is a sequence of distinct points of L. Thus, by Theorem 3.11 L is infinite. This is a contradiction, thus L is bounded. Therefore, $\exists\, n_0$

$\ni x \leq n_0 \; \forall \; x \in L$. $I_{n_0} = \{1,2,\ldots,n_0\}$. By definition, $L \subset I_{n_0}$. Define $g: I_{n_0} \longrightarrow L \ni$

$$g(i) = i \text{ if } i \in L \text{ where } L = \{l_1, l_2, \ldots, l_l\}$$
$$g(i) = l_1 \text{ if } i \notin L.$$

This function is well defined, since $I_{n_0} \neq \emptyset$ and $L \neq \emptyset$. Clearly, $f_1 g: I_{n_0} \longrightarrow K$ is an onto mapping.

Define $S = \{1;\ 2 \text{ iff } f_1g(2) \neq f_1g(1);\ 3 \text{ iff } f_1g(3) \neq f_1g(2) \text{ and } f_1g(3) \neq f_1g(1); \ldots;\ n_0 \text{ iff } f_1g(n_0) \neq f_1g(n_0-1), \ldots, \text{ and } f_1g(n_0) \neq f_1g(1)\}$. Now $(f_1g|S)$ is a function $(f_1g|S): S \longrightarrow K$ and $(f_1g|S)$ is onto. Let $l \in K$. If $l \in K$, then $\exists\, n \in I_{n_0} \ni f_1g(n) = l$. However, $n \in S$ iff $\nexists\, n_1 < n \ni f_1g(n_1) = l$. If $\exists\, n_1 < n \ni f_1g(n_1) = l$, then if $f_1g(1) = l$, $1 \in S$; if $f_1g(1) \neq l$ but $f_1g(2) = l$, $2 \in S; \ldots;$ if $f_1g(1) \neq l$, $f_1g(2) \neq l, \ldots, f_1g(n_1-2) \neq l$ but $f_1g(n_1-1) = l$, $n_1-1 \in S$; if $f_1g(1) \neq l, \ldots, f_1g(n_1-1) \neq l$, $n_1 \in S$. Thus, $\exists\, k \in I_{n_0} \ni f_1g(k) = l$, $k \in S$, and $(f_1g|S)(k) = l$. Denote $(f_1g|S)$ by h. Let $m \in S$ and $n \in S$. If $h(m) = h(n)$ but $m \neq n$, then without loss of generality assume $m < n$. Then, $f_1g(m) = f_1g(n)$. But by definition of S, $n \notin S$. Thus, h is one-to-one. Since $S \subset I_{n_0}$, the number of elements in S is less than or equal to n_0. Let k denote this number. Then, $S = \{k_1, k_2, \ldots, k_k\}$, where $k_i = k_j$ iff $i = j$. Then, $k_i \leq n_0 \; \forall \; i \in I_k$. Define $t: I_k \longrightarrow S$ by

$$t(1) = k_1$$

$t(2) = k_2$

$\cdot$

$\cdot$

$\cdot$

$t(k) = k_k$.

This function is well defined since $1 \in S$. The function t is one-to-one since $t(i) = t(j)$ iff $k_i = k_j$ which is true iff $i = j$. Since $t(I_k) = S$, t is onto. Consider $ht: I_k \longrightarrow K$. Since h and t are both one-to-one and onto ht is one-to-one and onto. ||

The notation $(S,\mathcal{T})$ has been used to denote a quasi topological space where $\mathcal{T}$ is the quasi topology and $\{N_p \mid p \in S\}$ the collection of neighborhoods in $(S,\mathcal{T})$. Henceforth, when speaking in reference to the extension of $(S,\mathcal{T})$ to a topological space, the collection of open sets in the topological space will be denoted by σ and $\{N'_p \mid p \in S\}$ will represent the collection of extended neighborhoods. Thus, σ is the topology for the topological space which is denoted by $[S,\sigma]$ where $[S,\sigma]$ is the extension of $(S,\mathcal{T})$.

Theorem 3.15. Let $(S,\mathcal{T})$ be a quasi topological space, $\{N_p \mid p \in S\}$ the collection of neighborhoods in $(S,\mathcal{T})$, and $\{O_p \mid p \in S\}$ the collection of open sets in $(S,\mathcal{T})$. Consider all possible non-empty finite collections of sets from $\{N_p \mid p \in S\}$, i.e., by Theorem 3.14 consider all possible sets of the form $\{U_i\}_{i \in I_k}$, $U_i \in \{N_p \mid p \in S\}$ $\forall\, i \in I_k$ where $k \in N$. For all sets of the form $\{U_i\}_{i \in I_k}$ consider $\bigcap_{i \in I_k} U_i$, i.e., consider all possible intersections of non-empty finite collections from $\{N_p \mid p \in S\}$. The collection of these intersections is $\{N'_p \mid p \in S\}$. Then, the open sets, using Definition 2.5 relative to $\{N'_p \mid p \in S\}$ as neighborhoods, satisfy the axioms for a topological space with S as the set of points and where σ is used to denote this collection of open sets.

Proof. For Axiom 1, (Every non-empty open set is a set of points, i.e., a subset of S.) let $V \in \sigma$, $V \neq \emptyset$. Let $p \in V$, then $\exists\, N'(p) \in \{N'_p \mid p \in S\} \ni p \in N'(p) \subset V$. The notation $N'(p)$ is used rather than N'_p to avoid confusion with the set of limit points of N_p. For some $k \in N$, $N'(p) = \bigcap_{i \in I_k} U_i$. Since $p \in N'(p)$, $p \in U_i\ \forall\, i \in I_k$. For all $i \in I_k$, $U_i \in \mathcal{T}$. Since $U_i \in \mathcal{T}\ \forall\, i \in I_k$, $U_i \subset S\ \forall\, i \in I_k$. For all $i \in I_k$ $p \in U_i$ and $U_i \subset S\ \forall\, i \in I_k$. Thus, $p \in S$. Therefore, $V \subset S$ and every non-empty open set is a set of points.

For Axiom 2, (The empty set $\emptyset$ is an open set.) consider $\emptyset$. The definition (Definition 2.5) of an open set is vacuously satisfied.

For Axiom 3, (For all p, $\exists$ an open set $\ni$ p is in this open set.) let $p \in S$. Since $p \in S$, by Axiom 1 for a quasi topological space, $\exists\, V \in \mathcal{T} \ni p \in V \subset S$. Since $V \in \mathcal{T}$ and $p \in V$, $V \in \{N_p \mid p \in S\}$. In the definition of $\{N'_p \mid p \in S\}$ let $I_k = \{1\}$. Then, $V = \bigcap_{i \in \{1\}} U_i$ where $U_1 = V$. Even though $\{U_i\}_{i \in \{1\}}$ contains only one set, $\bigcap_{i \in \{1\}} U_i$ is consistent with Definition 2.8. Thus, $V \in \{N'_p \mid p \in S\}$ and $\forall p \in S$, $\exists$ an open set $\ni p$ is in this open set.

For Axiom 4, (The union of any collection of open sets is open.) consider $\bigcup_{\alpha \in \Lambda} U_\alpha$ where U_α is open $\forall \alpha \in \Lambda$, i.e., $U_\alpha \in \sigma \ \forall \alpha \in \Lambda$. If $\Lambda = \emptyset$, then $\bigcup_{\alpha \in \Lambda} U_\alpha = \emptyset$ and $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open since $\emptyset$ was shown to be open by a previous part of this proof. Suppose $\Lambda \neq \emptyset$. Let $p \in \bigcup_{\alpha \in \Lambda} U_\alpha$. Then $p \in U_\alpha$ for some $\alpha \in \Lambda$. Since $p \in U_\alpha$, $\exists N'(p) \ni p \in N'(p) \subset U_\alpha$. Since $U_\alpha \subset \bigcup_{\alpha \in \Lambda} U_\alpha$ and $N'(p) \subset U_\alpha$, $N'(p) \subset \bigcup_{\alpha \in \Lambda} U_\alpha$. Therefore, $\bigcup_{\alpha \in \Lambda} U_\alpha$ is open. If $U_\alpha = \emptyset \ \forall \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} U_\alpha = \emptyset$ and $\emptyset$ is open from above.

For Axiom 5, (The intersection of any finite collection of open sets is open.) consider the empty collection. By Definition 2.8 the intersection of the empty collection is S; as in Corollary 2.12, S is open. Now consider all other finite collections given by $\{V_i\}_{i \in I_k} \ni V_i \in \sigma \ \forall i \in I_k$, where $k \in N$. It must be shown that $\bigcap_{i \in I_k} V_i$ is open where $k \in N$. Thus, a proof by induction will be used, i.e., it will be shown that it is true for 1 and if it is true for $n \in N$, then it is true for $n+1$, i.e., the intersection of $n+1$ open sets is open. When considering any $\{V_i\}_{i \in I_k}$ the case where $V_i = \emptyset$ for some $i \in I_k$ is eliminated since then

$\bigcap_{i \in I_k} V_i = \emptyset$ and $\emptyset$ is open by previous remarks in this proof. Now let k = 1 and consider all possible sets $\{V_i\}_{i \in I_1}$.

However, $\bigcap_{i \in I_1} V_i = V_1$ and $\bigcap_{i \in I_1} V_i$ is open since V_1 is open where V_1 is variable over σ. Now let k = 2 and consider all possible sets $\{V_i\}_{i \in I_2}$. However, $\bigcap_{i \in I_2} V_i = V_1 \cap V_2$. Let $p \in V_1 \cap V_2$. Then, $p \in V_1$ and $p \in V_2$. Since $p \in V_1$, $\exists N_1'(p) \ni N_1'(p) \subset V_1$ since V_1 is open. Since $p \in V_2$ $\exists N_2'(p) \ni N_2'(p) \subset V_2$ since V_2 is open. Again V_1 and V_2 are variables over σ. However, $N_1'(p) = \bigcap_{i \in I_m} U_i$, where $U_i \in \{N_p \mid p \in S\}$ $\forall i \in I_m$ and $N_2'(p) = \bigcap_{i \in I_n} W_i$ where $W_i \in \{N_p \mid p \in S\}$ $\forall i \in I_n$. Thus, $N_1'(p) \cap N_2'(p) = (\bigcap_{i \in I_m} U_i) \cap (\bigcap_{i \in I_n} W_i)$ and $N_1'(p) \cap N_2'(p) \neq \emptyset$ since $p \in N_1'(p)$ and $p \in N_2'(p)$.

Thus, $(\bigcap_{i \in I_m} U_i) \cap (\bigcap_{i \in I_n} W_i) = \bigcap_{i \in I_{m+n}} S_i$ where $S_i = U_i$ $\forall i \ni 1 \leq i \leq m$ and $S_i = W_{i-m}$ $\forall i \ni m+1 \leq i \leq m+n$. Thus, $N_1'(p) \cap N_2'(p) \in \{N'_p \mid p \in S\}$ and in particular $N_1'(p) \cap N_2'(p) \in \{N'_p\}$ which is used to denote the collection of extended neighborhoods of a particular point p. Therefore, $N_1'(p) \cap N_2'(p) = N_3'(p)$.

If $x \in N_1'(p) \cap N_2'(p)$, then $x \in N_1'(p)$ and $x \in N_2'(p)$. Thus, $x \in V_1$ and $x \in V_2$ since $N_1'(p) \subset V_1$ and $N_2'(p) \subset V_2$.

Thus, $x \in V_1 \cap V_2$ and $N_3'(p) \subset V_1 \cap V_2$. Therefore, $V_1 \cap V_2$ is open.

Now assume $\bigcap_{i \in I_n} V_i$ is open for all possible sets of the form $\{V_i\}_{i \in I_n}$ for $n \in N$.

Consider $\bigcap_{i \in I_{n+1}} V_i$. Then, $\bigcap_{i \in I_{n+1}} V_i = (\bigcap_{i \in I_n} V_i) \cap V_{n+1}$. By the induction hypothesis, $\bigcap_{i \in I_n} V_i$ is open. Thus, let $\bigcap_{i \in I_n} V_i = S_1$. Let $V_{n+1} = S_2$. Then, $\bigcap_{i \in I_2} S_i$ is open by the case above for $n = 2$. Therefore, the intersection of any finite collection of open sets is open. ||

Definition 3.16. The collection of open sets in $(S_1,\mathcal{T}_1)$ is the same as the collection of open sets in $(S_2,\mathcal{T}_2)$ iff $(S_1,\mathcal{T}_1) = (S_2,\mathcal{T}_2)$.

Definition 3.17. Let S be a set and σ a collection of subsets of S. Then σ is a basis for a collection $\mathcal{T}$ of subsets of S iff $K \in \mathcal{T}$ iff $K = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \sigma$ $\forall \alpha \in \Lambda$.

Theorem 3.18. If $(S,\mathcal{T})$ is a quasi topological space where $\{N_p \mid p \in S\}$ and $\{O_p \mid p \in S\}$ are the collections of neighborhoods and open sets respectively, then $\{N_p \mid p \in S\}$ is a basis for $\{O_p \mid p \in S\}$.

Proof. Let $K \in \{O_p \mid p \in S\}$. If $K = \emptyset$, then $K = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $\Lambda = \emptyset$. If $K \neq \emptyset$, then $\forall p \in K \ \exists N_p \ni N_p \in \{N_p \mid p \in S\}$

and $N_p \subset K$. Consider the union of these N_p, $\bigcup_{p \in K} N_p$. If $x \in \bigcup_{p \in K} N_p$, then $x \in N_p$ for some p. Since $N_p \subset K$, $x \in K$ and $\bigcup_{p \in K} N_p \subset K$. If $x \in K$, then $\exists\, N_x \ni x \in N_x \subset K$ and $N_x \in \{N_p\}_{p \in K}$. Thus, $K \subset \bigcup_{p \in K} N_p$. Therefore, $K = \bigcup_{p \in K} N_p$. Let $K = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \{N_p \mid p \in S\}\ \forall\, \alpha \in \Lambda$. Let $x \in K$, then $x \in U_\alpha$ for some $\alpha \in \Lambda$. Thus, $U_\alpha \subset \bigcup_{\alpha \in \Lambda} U_\alpha$ and $U_\alpha \subset K$. Thus, K is open. Therefore, $K \in \{O_p \mid p \in S\}$. Thus, $\{N_p \mid p \in S\}$ is a basis for $\{O_p \mid p \in S\}$. ||

By the same token the collection of "extended" neighborhoods obtained from an extension of a quasi topological space is a basis for the topology in the topological space obtained under the extension.

Corollary 3.19. If $(S,\mathcal{T})$ is a quasi topological space where $\{N_p \mid p \in S\}$ and $\{O_p \mid p \in S\}$ are the collections of neighborhoods and open sets respectively and $[S,\sigma]$ is the extension of $(S,\mathcal{T})$, then $\{N'_p \mid p \in S\}$ is a basis for σ.

Proof. The proof follows that for Theorem 3.18. ||

Quite obviously, since $\mathcal{T} - \{\emptyset\} = \{N_p \mid p \in S\}$ in $(S,\mathcal{T})$, $\mathcal{T}$ is also a basis for the open sets in $(S,\mathcal{T})$. The addition of $\emptyset$, if that is the case, makes no difference as far as having a basis.

A collection of sets σ can be shown to be a basis for $\mathcal{T}$ iff $\forall\, U \in \mathcal{T}$ it can be shown that $\forall\, p \in U\ \exists\, V \in \sigma \ni p \in V \subset U$.

Lemma 3.20. Let S be a set. Two collections σ_1 and σ_2 of subsets of S generate the same collection $\mathcal{T}$, i.e., they are both a basis for $\mathcal{T}$, iff:

1) $U \in \sigma_1$ and $p \in U$ imply $\exists\, V \in \sigma_2 \ni p \in V \subset U$ and.

2) $V \in \sigma_2$ and $p \in V$ imply $\exists\, U \in \sigma_1 \ni p \in U \subset V$.

Proof. (Sufficiency) Let σ_1 and σ_2 generate $\mathcal{T}$.

1) If $U \in \sigma_1$, then $U \in \mathcal{T}$. Thus, $U = \bigcup_{\alpha \in \Lambda} V_\alpha$ where $V_\alpha \in \sigma_2\ \forall\, \alpha \in \Lambda$. If $p \in U$, then $p \in V_\alpha$ for some $\alpha \in \Lambda$ and $V_\alpha \subset U$.

2) The proof for this part is the same as above except for obvious substitutions.

(Necessity) Let 1) and 2) hold. Let σ_1 generate $\mathcal{T}_1$ and σ_2 generate $\mathcal{T}_2$. If $W \in \mathcal{T}_1$, then $W = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \sigma_1\ \forall\, \alpha \in \Lambda$. For any $\alpha \in \Lambda$, let $p \in U_\alpha$. Then, $\exists\, V_p \ni p \in V_p \subset U_\alpha$ and $\forall\, p$, $V_p \in \sigma_2$. Just as in Theorem 3.18, for any $\alpha \in \Lambda$, $\bigcup_{p \in U_\alpha} V_p = U_\alpha$. Then, $\bigcup_{p \in W} V_p = W$. Thus, $W \in \mathcal{T}_2$. Therefore, $\mathcal{T}_1 \subset \mathcal{T}_2$. Likewise, $\mathcal{T}_2 \subset \mathcal{T}_1$ and thus, $\mathcal{T}_1 = \mathcal{T}_2$. ||

Definition 3.16 was stated in terms of S_1 and S_2. However, the definition forces $S_1 = S_2$.

Theorem 3.21. If $(S_1,\mathcal{T}_1) = (S_2,\mathcal{T}_2)$, then $S_1 = S_2$.

Proof. Let $p \in S_1$. Since $p \in S_1$, $\exists\, U \in \mathcal{T}_1 \ni p \in U$. Since $U \in \mathcal{T}_1$, $U \in \{N_p \mid p \in S_1\}$ in $(S_1,\mathcal{T}_1)$. Since $U \in \{N_p \mid p \in S_1\}$, U is open. Since U is open in $(S_1,\mathcal{T}_1)$ and $(S_1,\mathcal{T}_1) = (S_2,\mathcal{T}_2)$, U is open in $(S_2,\mathcal{T}_2)$. By Theorem 2.26, $U \subset S_2$. Thus, $p \in S_2$ and $S_1 \subset S_2$. The symmetry of the argument dictates that if $p \in S_2$, then $p \in S_1$ and $S_2 \subset S_1$. Therefore, $S_1 = S_2$. The case where $(S_1, \{\emptyset\}) = (S_2, \{\emptyset\})$ forces $S_1 = S_2 = \emptyset$. ||

Theorem 3.22. Let β denote the collection of neighborhoods in $(S_1,\mathcal{T}_1)$ and γ the collection of neighborhoods in $(S_2,\mathcal{T}_2)$. Then, $(S_1,\mathcal{T}_1) = (S_2,\mathcal{T}_2)$ iff β and γ generate the same collection of open sets.

Proof. (Sufficiency) If $(S_1,\mathcal{T}_1) = (S_2,\mathcal{T}_2)$, then the collection of open sets in $(S_1,\mathcal{T}_1)$ is the same as that in $(S_2,\mathcal{T}_2)$ by Definition 3.16. Since β and γ generate this collection, they generate the same collection of open sets.

(Necessity) If β and γ do generate the same collection of open sets, then the definition of equality for quasi topological spaces is satisfied.||

Corollary 3.23. For quasi topological spaces $(S_1,\mathcal{T}_1)$ and $(S_2,\mathcal{T}_2)$, $(S_1,\mathcal{T}_1) = (S_2,\mathcal{T}_2)$ iff $\mathcal{T}_1$ and $\mathcal{T}_2$ generate the same collection of open sets.

Proof. The proof holds by Theorem 3.22 and the remark following Corollary 3.19.||

A question that needs consideration is the uniqueness of an extension of a quasi topological space. It will be shown in the following proof that given a quasi topological space $(S,\mathcal{T})$, then $(S,\mathcal{T})$ extends to a unique topological space $[S,\sigma]$.

Definition 3.24. For topological spaces $[S_1,\sigma_1]$ and $[S_2,\sigma_2]$, $[S_1,\sigma_1] = [S_2,\sigma_2]$ iff $S_1 = S_2$ and $\sigma_1 = \sigma_2$.

Theorem 3.25. Given $(S,\mathcal{T})$, then $(S,\mathcal{T})$ extends to a unique topological space $[S,\sigma]$.

Proof. Suppose the contrary. Suppose $(S,\mathcal{T})$ extends to $[S,\sigma_1]$ and to $[S,\sigma_2]$ where $\sigma_1 \neq \sigma_2$. Thus, $\exists\ V \in \sigma_1 \ni V \notin \sigma_2$ or $\exists\ V \in \sigma_2 \ni V \notin \sigma_1$. Let $V \in \sigma_1$ but $V \notin \sigma_2$. The symmetry of the argument handles the other case. For all $p \in V\ \exists\ N'(p) \ni p \in N'(p) \subset V$ where $N'(p) \in \{N'_p \mid p \in S\}$. Clearly, $V = \bigcup_{p \in V} N'(p)$. By the definition of extending $(S,\mathcal{T})$ to $[S,\sigma_2]$, $\bigcup_{p \in V} N'(p) \in \sigma_2$. Thus, $V \in \sigma_2$. Therefore, the assumption was wrong and $\sigma_1 = \sigma_2$. Thus, $[S,\sigma_1] = [S,\sigma_2]$ by Definition 3.24. ||

Since a topological space is also a quasi topological space, a quasi topological space may be a topological space.

Theorem 3.26. Let $(S,\mathcal{T})$ be a quasi topological space and let σ denote the collection of open sets in $(S,\mathcal{T})$. Then, $(S,\mathcal{T})$ is also a topological space with σ the topology, $[S,\sigma]$, iff $\forall U \in \{N_p \mid p \in S\}$ and $V \in \{N_p \mid p \in S\}$ and $\forall p \in U \cap V$ $\exists W \in \{N_p \mid p \in S\} \ni p \in W \subset U \cap V$.

Proof. (Sufficiency) Since U and V are open $U \cap V$ is open. Thus, $\forall p \in U \cap V \; \exists W \in \{N_p \mid p \in S\} \ni p \in W \subset U \cap V$.

(Necessity) It must be shown that $(S,\mathcal{T})$ satisfies the axioms for a topological space. By Theorem 2.26, the remark following Corollary 2.12, Theorem 2.25, and Theorem 2.11 a quasi topological space satisfies the first four axioms for a topological space. To prove that axiom 5 holds, consider $A,B \in \sigma$ and $A \cap B$. Since $A \in \sigma$, $A = \bigcup_{\alpha \in \Lambda} A_\alpha$ where $A_\alpha \in \{N_p \mid p \in S\}$ $\forall \alpha \in \Lambda$. Likewise, $B = \bigcup_{\gamma \in \beta} B_\gamma$. Consider $(\bigcup_{\alpha \in \Lambda} A_\alpha) \cap (\bigcup_{\gamma \in \beta} B_\gamma)$. Let $x \in (\bigcup_{\alpha \in \Lambda} A_\alpha) \cap (\bigcup_{\gamma \in \beta} B_\gamma)$. Then, $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$ and $x \in \bigcup_{\gamma \in \beta} B_\gamma$. Thus, $x \in A_\alpha$ for some α and $x \in B_\gamma$ for some γ. Therefore, $x \in A_\alpha \cap B_\gamma$ for these α and γ. Thus, $x \in \bigcup_{\substack{\alpha \in \Lambda \\ \gamma \in \beta}} (A_\alpha \cap B_\gamma)$. Consequently,

$(\bigcup_{\alpha \in \Lambda} A_\alpha) \cap (\bigcup_{\gamma \in \beta} B_\gamma) \subset \bigcup_{\substack{\alpha \in \Lambda \\ \gamma \in \beta}} (A_\alpha \cap B_\gamma)$. Let $x \in \bigcup_{\substack{\alpha \in \Lambda \\ \gamma \in \beta}} (A_\alpha \cap B_\gamma)$. Then, $x \in A_\alpha \cap B_\gamma$ for some α and γ. Thus, $x \in A_\alpha$ for this α and $x \in B_\gamma$ for this γ. Therefore, $x \in \bigcup_{\alpha \in \Lambda} A_\alpha$ and $x \in \bigcup_{\gamma \in \beta} B_\gamma$. Then, $\bigcup_{\substack{\alpha \in \Lambda \\ \gamma \in \beta}} (A_\alpha \cap B_\gamma) \subset (\bigcup_{\alpha \in \Lambda} A_\alpha) \cap (\bigcup_{\gamma \in \beta} B_\gamma)$ and $(\bigcup_{\alpha \in \Lambda} A_\alpha) \cap (\bigcup_{\gamma \in \beta} B_\gamma) = \bigcup_{\substack{\alpha \in \Lambda \\ \gamma \in \beta}} (A_\alpha \cap B_\gamma)$. Since, given α

and γ, by the hypothesis $A_\alpha \cap B_\gamma = \bigcup_{\eta \in \Sigma} C$ where $C_\eta \in \{N_p \mid p \in S\} \forall \eta \in \Sigma$, $A \cap B$ is the union of a collection of sets from $\{N_p \mid p \in S\}$. By induction, as in the proof for Theorem 3.15, the intersection of any finite collection of open sets is open. Thus, $(S, \mathcal{T})$ is a topological space. ||

The question now to be discussed is that of what topological spaces can be considered as extensions of quasi topological spaces. It will be shown that any topological space can be considered as the extension of at least one quasi topological space. The uniqueness of the question will also be considered, i.e., can more than one quasi topological space be extended to the same topological space.

Of course, every topological space can be considered as the extension of at least one quasi topological space since a topological space is also a quasi topological space.

Theorem 3.27. Every topological space can be obtained by the extension of at least one quasi topological space.

Using a trivial example it can be seen that given a topological space there can exist more than one quasi topological space that extends to this topological space.

Example 3.28. Let $\mathcal{T}_1 = \{\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ and $\mathcal{T}_2 = \{\{a,b\}, \{a,c\}, \{a,b,c\}\}$ where $S = \{a,b,c\}$. Thus, $(S, \mathcal{T}_1) \neq (S, \mathcal{T}_2)$ but $(S, \mathcal{T}_2)$ and $(S, \mathcal{T}_1)$ both extend to $[S, \sigma]$

where $\sigma = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$.

The question then is under what conditions, if any, does more than one quasi topological space extend to a given topological space.

Lemma 3.29. If $[S,\sigma]$ is a topological space where $\mathcal{T}$ is a basis for σ, let A denote the set of all non-empty finite intersections of sets from $\mathcal{T}$, i.e., $K \in A$ iff $K = \bigcap_{i \in I_n} U_i$ where $U_i \in \mathcal{T}\ \forall\, i \in I_n$, for some $n \in N$. Let $B \subset A$, then $\mathcal{T} \cup B$ is a basis for σ.

Proof. By definition $\mathcal{T}$ generates σ. Let $\mathcal{T} \cup B$ generate Σ. It must be shown that $\Sigma = \sigma$. If $V \in \sigma$, then $V = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \mathcal{T}\ \forall\, \alpha \in \Lambda$. Since $\mathcal{T} \subset \mathcal{T} \cup B$, $U_\alpha \in \mathcal{T} \cup B\ \forall\, \alpha \in \Lambda$. Thus, $\bigcup_{\alpha \in \Lambda} U_\alpha \in \Sigma$. Therefore, $\sigma \subset \Sigma$. Let $V \in \Sigma$. Then, $V = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \mathcal{T} \cup B\ \forall\, \alpha \in \Lambda$. If $U_\alpha \in \mathcal{T}\ \forall\, \alpha \in \Lambda$, then $\bigcup_{\alpha \in \Lambda} U_\alpha \in \sigma$ and $V \in \sigma$. Were this always the case, $\Sigma \subset \sigma$. Suppose for $V = \bigcup_{\alpha \in \Lambda} U_\alpha\ \exists\ U_\alpha \ni U_\alpha \in B$. For these α, $U_\alpha = \bigcap_{i \in I_m} W_i$ where $W_i \in \mathcal{T}\ \forall\, i \in I_m$, for some $m \in N$, Since $W_i \in \sigma\ \forall\, i \in I_m$, then $\bigcap_{i \in I_m} W_i \in \sigma$. Thus, $\bigcap_{i \in I_m} W_i = \bigcup_{\gamma \in \beta} V_\gamma$ where $V_\gamma \in \mathcal{T}\ \forall\, \gamma \in \beta$. Therefore, $\forall$ these $\alpha \ni U_\alpha = \bigcap_{i \in I_m} W_i$ for some $m \in N$, $U_\alpha = \bigcup_{\gamma \in \beta} V_\gamma$ where $V_\gamma \in \mathcal{T}\ \forall\, \gamma \in \beta$. Then $V \in \sigma$ since $V = \bigcup_{\mu \in \Phi} S_\mu$ where $S_\mu \in \mathcal{T}\ \forall \mu \in \Phi$. ||

Theorem 3.30. If $[S,\sigma]$ is a topological space where $\mathcal{T}$ is a basis for σ, let $P = \{A_p \mid \forall\, p \in S$ pick one and only one $A_p \in \mathcal{T} \ni p \in A_p\}$. Let A denote the set of all non-empty finite intersections of sets from P, i.e., $K \in A$ iff $K = \bigcap_{i \in I_n} A_{p_i}$ where $A_{p_i} \in P\ \forall\, i \in I_n$, for some $n \in N$. Let $B \subset A$. Then, $P \cup (\mathcal{T}-B)$ is a quasi topology Σ on S and (S,Σ) extends to $[S,\sigma]$.

Proof. Since $P \subset \Sigma$ and $\forall\, p \in S\ \exists\, V \in \Sigma$, namely A_p, $\ni p \in V \subset S$, Σ is a quasi topology on S.

If it can be shown that the extension of Σ, i.e., the extended neighborhoods (Σ_1) is a set $\mathcal{T} \cup D$ where D is a subset of the collection of all finite intersections of sets from $\mathcal{T}$, then by Lemma 3.29 Σ_1 is a basis for σ and thus (S,Σ) will be shown to extend to $[S,\sigma]$ by the definition of an extension of a quasi topological space to a topological space.

Let $E \in \Sigma_1$, then $E = \bigcap_{i \in I_n} V_i$ where $V_i \in P \cup (\mathcal{T}-B)\ \forall\, i \in I_n$, for some $n \in N$. Since $V_i \in P \cup (\mathcal{T}-B)$, $V_i \in P$ or $V_i \in \mathcal{T}-B$. If $V_i \in P$, then $V_i \in \mathcal{T}$ since $P \subset \mathcal{T}$. If $V_i \in \mathcal{T}-B$, then $V_i \in \mathcal{T}$. Thus, $E = \bigcap_{i \in I_n} V_i$ where $V_i \in \mathcal{T}\ \forall\, i \in I_n$. Now it must be shown that $\mathcal{T} \subset \Sigma_1$. Let $E \in \mathcal{T}$, then $E \in P$ or $E \in \mathcal{T}-P$. If $E \in P$, then $E \in \Sigma_1$, since $P \subset \Sigma \subset \Sigma_1$. If $E \in \mathcal{T}-P$, $E \in B$ or $E \in -B$. If $E \in -B$, then $E \in \mathcal{T}-B$ and $E \in \Sigma_1$, since $\mathcal{T}-B \subset \Sigma \subset \Sigma_1$. If $E \in B$, then $E = \bigcap_{i \in I_m} U_i$

where $U_i \in P \ \forall \ i \in I_m$ and $m \in N$, since $B \subset A$. Thus, $E \in \Sigma_1$ by the definition of Σ_1. Thus, $\mathcal{T} \subset \Sigma_1$. Therefore, $\mathcal{T} \subset \Sigma_1 \subset \left\{E \mid E = \bigcap_{i \in I_n} V_i \text{ where } V_i \in \mathcal{T} \ \forall \ i \in I_n \text{ and } n \in N\right\}$. Thus, $\mathcal{T} \subset \Sigma_1 \subset \mathcal{T} \cup F$ where $F \subset \left\{E \mid E = \bigcap_{i \in I_n} V_i \text{ where } V_i \in \mathcal{T} \ \forall \ i \in I_n \text{ and } n \in N\right\}$. Precisely, $F = \left\{E \mid E = \bigcap_{i \in I_n} V_i \text{ where } V_i \in \mathcal{T} \ \forall \ i \in I_n \text{ and } n \in N - I_1\right\}$. Obviously, $\Sigma_1 = \mathcal{T} \cup D$ where $D \subset F$. ||

An immediate question is whether or not this theorem considered all possible quasi topological spaces that extend to a given topological space. The answer is yes.

Lemma 3.31. Given $(S,\mathcal{T}) \ni \emptyset \in \mathcal{T}$, then $(S,\mathcal{T}-\{\emptyset\}) = (S,\mathcal{T})$.

Proof. Let the collection of neighborhoods for $(S,\mathcal{T})$ be denoted here by σ. By the definition of neighborhoods $\sigma = \mathcal{T}-\{\emptyset\}$. Since $(\mathcal{T}-\{\emptyset\})-\{\emptyset\} = \mathcal{T}-\{\emptyset\} = \sigma$, the collection of neighborhoods in $(S,\mathcal{T}-\{\emptyset\})$ is the same as that in $(S,\mathcal{T})$. Thus, the collection of open sets in $(S,\mathcal{T}-\{\emptyset\})$ is the same as that in $(S,\mathcal{T})$. ||

Theorem 3.32. Given that $(S,\mathcal{T})$ extends to $[S,\sigma]$, where α and β denote the set of neighborhoods in $(S,\mathcal{T})$ and the set of extended neighborhoods respectively, then $(S,\mathcal{T}) = (S,\alpha \cup (\beta - (\beta-\alpha)))$.

Proof. If $\emptyset \notin \mathcal{T}$, then $\mathcal{T}-\{\emptyset\} = \mathcal{T} = \alpha$ and $\alpha \cup (\beta-(\beta-\alpha))$ $= \alpha \cup (\beta \cap -(\beta \cap -\alpha)) = \alpha \cup (\beta \cap (-\beta \cup \alpha)) = \alpha \cup ((\beta \cap -\beta)$ $\cup (\alpha \cap \beta)) = \alpha \cup (\emptyset \cup (\alpha \cap \beta)) = \alpha \cup (\alpha \cap \beta) = (\alpha \cup \alpha) \cap (\alpha \cup \beta)$ $= \alpha \cap \beta = \alpha$, since $\alpha \subset \beta$. Thus, $(S,\mathcal{T}) = (S,\alpha \cup (\beta-(\beta-\alpha)))$, since $\alpha \cup (\beta-(\beta-\alpha)) = \alpha$ and $\mathcal{T} = \alpha$.

If $\emptyset \in \mathcal{T}$, then $\alpha = \mathcal{T}-\{\emptyset\}$ and $(S,\mathcal{T}) = (S,\alpha)$ by Lemma 3.31. Thus, $(S,\mathcal{T}) = (S,\alpha \cup (\beta-(\beta-\alpha)))$, since $\alpha = \alpha \cup (\beta-(\beta-\alpha))$.||

Since β is a basis for σ, $\alpha \cup (\beta-(\beta-\alpha))$ is of the form expressed in Theorem 3.30 for a quasi topology. Thus, given a topological space all quasi topological spaces that extend to it can be expressed by Theorem 3.30.

Since every quasi topological space extends to a topological space, all quasi topological spaces can be expressed by Theorem 3.30.

Now a topological space is related directly to a structure, a quasi topological space, more fundamental to the foundation of topology, set theory. The quasi topological space, which rests solely on a single axiom, through two definitions and an extension was extended to a topological space. The relationship between the two structures having been established, a further study of the properties of a quasi topological space and simultaneously a study of invariant properties under the extension from the quasi topological space to the topological space will be undertaken.

CHAPTER IV

CLASSIFICATION OF SPACES

By classification of spaces is meant that the spaces are classified as a certain type iff they satisfy certain conditions.

Definition 4.1. A quasi topological space $(S,\mathcal{T})$ is Hausdorff iff $\forall\ p \in S$ and $q \in S \ni p \neq q\ \exists$ disjoint open sets in $(S,\mathcal{T})$, U and V $\ni p \in U$ and $q \in V$. A topological space $[S,\sigma]$ is Hausdorff iff $\forall\ p \in S$ and $q \in S \ni p \neq q\ \exists$ disjoint open sets in $[S,\sigma]$, U, and V $\ni p \in U$ and $q \in V$.

Then, any space that satisfies this condition is Hausdorff.

By invariant under the extension of $(S,\mathcal{T})$ to $[S,\sigma]$ is meant that the property under consideration in $(S,\mathcal{T})$ also holds in $[S,\sigma]$.

Since an open set in $(S,\mathcal{T})$ is also an open set in $[S,\sigma]$, Hausdorff is invariant.

Theorem 4.2. Hausdorff is invariant under an extension from a quasi topological space to a topological space.

Henceforth the definition for a type of quasi topological space will be used for the corresponding type of topological space with the obvious substitution of $[S,\sigma]$ for $(S,\mathcal{T})$.

Definition 4.3. A space $(S,\mathcal{T})$ is discrete iff every subset of S is open.

Since a discrete space is by definition a topological space, "discreteness" is invariant.

Theorem 4.4. "Discreteness" is invariant under an extension from a quasi topological space to a topological space.

Definition 4.5. A space $(S,\mathcal{T})$ is regular iff $\forall$ closed set C and all points p, $p \notin C$, $\exists$ disjoint open sets U and V $\ni$ $C \subset U$ and $p \in V$.

Theorem 4.6. A regular quasi topological space $(S,\mathcal{T})$ extends to a regular topological space $[S,\sigma]$.

Proof. It must be shown that if $(S,\mathcal{T})$ is regular, then its extension $[S,\sigma]$ is regular. Let A be closed in $[S,\sigma]$. Then S-A is open, by definition. Thus, $S-A = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \{N_p' \mid p \in S\} \forall \alpha \in \Lambda$. If $A = \emptyset$, then $S-A = S$ and $\forall p \in S \exists$ disjoint open sets S and $\emptyset \ni \emptyset \subset \emptyset$ and $p \in S$. If $A = S$, then $S-A = \emptyset$ and the definition, Definition 4.5., is vacuously satisfied. Thus, $\ni A \neq \emptyset$ and $S-A \neq \emptyset$. If $p \in S-A$, then $p \in U_\alpha$ for some $\alpha \in \Lambda$. Since $p \in U_\alpha$, $p \in \bigcap_{i \in I_n} V_i$ where $V_i \in \{N_p \mid p \in S\} \forall i \in I_n$, for some $n \in N$. Thus, V_i is open in $(S,\mathcal{T}) \forall i \in I_n$. Thus $S-V_i = B_i$ is closed $\forall i \in I_n$. Since

$p \in V_i \ \forall \ i \in I_n$, $p \notin B_i \ \forall \ i \in I_n$. Thus, B_i is closed in $(S,\mathcal{T}) \ \forall \ i \in I_n$, since $V_i \in$ the open sets in $(S,\mathcal{T}) \ \forall \ i \in I_n$. Thus, B_i is closed and $p \notin B_i$. Then $\exists$ disjoint open sets C_i and $D_i \ni B_i \subset C_i$ and $p \in D_i \ \forall \ i \in I_n$, since $(S,\mathcal{T})$ is regular. Consider $\bigcup_{i \in I_n} C_i$ and $\bigcap_{i \in I_n} D_i$. Since $p \in D_i \ \forall \ i \in I_n$, $p \in \bigcap_{i \in I_n} D_i$. Is $A \subset \bigcup_{i \in I_n} C_i$? Let $x \in A$ but $\nexists \ B_i \ni x \in B_i$. Thus, $\forall \ B_i \ x \notin B_i$. Then, $x \in V_i \ \forall \ i \in I_n$ and $x \in \bigcap_{i \in I_n} V_i$. Thus, $x \in U_\alpha$ for some $\alpha \in \Lambda$ and $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$. If $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$, $x \notin A$ since $\bigcup_{\alpha \in \Lambda} U_\alpha = S-A$. This contradicts that $x \in A$. Thus, $\exists \ B_i \ni x \in B_i$. Since $B_i \subset C_i$, $x \in C_i$ and $A \subset \bigcup_{i \in I_n} C_i$. Suppose $(\bigcap_{i \in I_n} D_i) \cap (\bigcup_{i \in I_n} C_i) \neq \emptyset$. Then, $\exists \ x \in \bigcap_{i \in I_n} D_i$ and $x \in \bigcup_{i \in I_n} C_i$. Then for some $k \in I_n$, $x \in D_k$ and $x \in C_k$. This contradicts the fact that $C_k \cap D_k = \emptyset$. Thus, $\bigcap_{i \in I_n} D_i$ and $\bigcup_{i \in I_n} C_i$ are disjoint. Since C_i and D_i are open in $(S,\mathcal{T}) \ \forall \ i \in I_n$, C_i and D_i are open in $[S,\sigma] \ \forall \ i \in I_n$. Thus, by axiom 4 for a topological space $\bigcup_{i \in I_n} C_i$ is open in $[S,\sigma]$. By axiom 5, $\bigcap_{i \in I_n} D_i$ is open in $[S,\sigma]$. Therefore, the conditions of Definition 4.5 are satisfied for the extension of $(S,\mathcal{T})$, $[S,\sigma]$. ||

The following example illustrates that non-discrete, non-Hausdorff, and non-regular quasi topological spaces can extend to discrete, Hausdorff, and regular topological spaces.

Example 4.7. Let $S = \{a, b, c\}$ and $\mathcal{T} = \{\{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Definition 4.8. A space $(S, \mathcal{T})$ is T_0 iff given any two points p and q, $p \neq q$, $\exists M \ni M$ is open and $p \in M$, $q \notin M$ or $q \in M$, $p \notin M$.

It is clear from the definition alone that a T_0 quasi topological space does extend to a T_0 topological space.

Theorem 4.9. If $(S, \mathcal{T})$ is T_0, then $[S, \sigma]$ is T_0.

In contrast to discrete, Hausdorff, and regular spaces the following holds for T_0 spaces.

Theorem 4.10. Given any T_0 topological space $[S, \sigma]$, then any quasi topological space $(S, \mathcal{T})$ that extends to $[S, \sigma]$ is also a T_0 space.

Proof. Given any two points p and q, $p \neq q$, $\exists$ A open in $[S, \sigma] \ni p \in A$, $q \notin A$ or $q \in A$, $p \notin A$. Suppose without loss of generality $p \in A$ and $q \notin A$. Consider any quasi topological space $(S, \mathcal{T})$ that extends to $[S, \sigma]$. If $A \in \mathcal{T}$, the proof is essentially done. Suppose $A \notin \mathcal{T}$. Then $A = \bigcup_{\alpha \in \Lambda} U_\alpha$ where $U_\alpha \in \{N_p' \mid p \in S\} \; \forall \alpha \in \Lambda$. If $p \in A$, then $p \in U_\alpha$ for some $\alpha \in \Lambda$. If $p \in U_\alpha$, then $p \in \bigcap_{i \in I_n} N_i$ where $U_\alpha = \bigcap_{i \in I_n} N_i$ for some $n \in N$ and $N_i \in \{N_p \mid p \in S\} \; \forall i \in I_n$. If $p \in \bigcap_{i \in I_n} N_i$, then $p \in N_i \; \forall \; i = 1,\ldots,n$. Now, if $n = 1$ then $U_\alpha = N_1$, and the proof would be finished. For $n \neq 1$ if $\exists \; N_k$ where $1 \leq k \leq n \ni q \notin N_k$, then the proof would be finished. Thus $\S$ that $q \in N_k \; \forall \; k = 1,\ldots,n$. Then $q \in \bigcap_{i \in I_n} N_i$ and $q \in U_\alpha \subset A$ and this contradicts the fact that $q \notin A$. Thus, $\exists \; N_k$ where $1 \leq k \leq n \ni q \notin N_k$ but $p \in N_k$.||

Definition 4.11. A space $(S, \mathcal{T})$ is T_1 iff every point is a closed set.

The following theorem is a direct consequence of Definition 4.11.

Theorem 4.12. A T_1 quasi topological space extends to a T_1 topological space.

Example 4.7 illustrates a quasi topological space that is not T_1, however one which does extend to a T_1 topological space.

Definition 4.13. A set K is countable iff $K = \emptyset$ or $\exists$ a mapping of N onto K.

Definition 4.14. A space $(S,\mathcal{T})$ is second countable iff $\exists$ a countable basis for the open sets in $(S,\mathcal{T})$.

If, for a second countable space $(S,\mathcal{T})$ with a countable basis β for the open sets, it can be shown that

$\left\{A_\alpha \mid A_\alpha = \bigcap_{i \in I_n} U_i \text{ where } U_i \in \beta\ \forall\, i \in I_n,\ \forall\, n \in N\right\}$ is countable, then second countability is invariant under an extension from $(S,\mathcal{T})$ to $[S,\sigma]$ for the following reason. The set $\left\{A_\alpha \mid A_\alpha = \bigcap_{i \in I_n} U_i \text{ where } U_i \in \beta\ \forall\, i \in I_n;\ \forall\, n \in N\right\}$ is the set of extended neighborhoods and is thus a basis for σ.

To do this the mapping required by Definition 4.13 must be exhibited or a previously proved lemma must be used. A lemma will be used and the lemma states that the union of a countable collection of open sets is open. To prove this lemma, however, N x N must be shown to be countable and a subset of a countable set must be shown to be countable.

Lemma 4.15. A subset of a countable set is countable.

Proof. Let K be countable and $S \subset K$. If $S = \emptyset$, then S is countable by definition. If $S \neq \emptyset$, let $p \in S$. Since K is countable $\exists f: N \xrightarrow{\text{onto}} K$.

Define $g: K \longrightarrow S \ni$

$$g(x) = x \text{ if } x \in S$$

$$g(x) = p \text{ if } x \notin S \text{ but } x \in K.$$

Clearly, $g: K \xrightarrow{\text{onto}} S$. Thus, since g and f are onto $gf: N \xrightarrow{\text{onto}} S$ and S is countable. ||

Lemma 4.16. The set N x N is countable.

Proof. Let $K = \left\{2^p 3^q \mid p, q \in N\right\}$. The set K is countable, since N is countable by Definition 4.15 and $K \subset N$ (Lemma 4.15). Define $f: K \longrightarrow N \times N \ni f(2^p 3^q) = (p,q)$. This mapping is onto for given $(p,q) \in N \times N \; \exists \; x \ni x = 2^p 3^q$ and thus $x \in K$. Since K is countable, $\exists \; g: N \xrightarrow{onto} K$. As in Lemma 4.15, since g and f are onto, $fg: N \xrightarrow{onto} N \times N$ and $N \times N$ is countable.//

Lemma 4.17. Union of a countable collection of countable sets is countable.

Proof. If the collection is empty, the union is countable by definition. If the collection is not empty index it by N, $\{A_i\}_{i \in N}$, i.e., $\exists \; f: N \longrightarrow \{A_i\}_{i \in N} \ni f(n) = A_n \; \forall \; n \in N$. By Lemma 4.15, the subcollection consisting of the non-empty sets from $\{A_i\}_{i \in N}$ is countable. If this subcollection is empty, then $\bigcup_{i \in N} A_i = \emptyset$ since $A_i = \emptyset \; \forall \; i \in N$ and thus $\bigcup_{i \in N} A_i$ is countable. Suppose this subcollection is non-empty. Then, it can be indexed by N since it is countable. Thus, denote it by $\{B_i\}_{i \in N}$, i.e., $\exists \; g: N \longrightarrow \{B_i\}_{i \in N}$ $\ni g(n) = B_n \; \forall \; n \in N$. Note that $\bigcup_{i \in N} B_i = \bigcup_{i \in N} A_i$, since $\{B_i\}_{i \in N}$ consists of just the non-empty sets from $\{A_i\}_{i \in N}$. Since each B_n is countable, $\exists$ a function $h_n: N \xrightarrow{onto} B_n \ni h_n(m) = p_{(n,m)}$, i.e., $p_{(n,m)}$ is the image of $h_n(m)$. Since $N \times N$ is countable $\exists \; l: N \xrightarrow{onto} N \times N$. Define, $k: N \times N \longrightarrow \bigcup_{i \in N} B_i$ $\ni k(n,m) = p_{(n,m)}$. This function is onto for a given $p \in \bigcup_{i \in N} B_i$, $p \in B_n$ for some $n \in N$. Since B_n is countable

$\exists h_n$ and $m \in N \ni h_n(m) = p$. Thus $p = p_{(n,m)}$ and $k(n,m) = p_{(n,m)}$. Therefore, k is onto. Since k is onto and l is onto $kl: N \xrightarrow{\text{onto}} \bigcup_{i \in N} B_i$ and thus $\bigcup_{i \in N} B_i$ is countable.

Therefore, $\bigcup_{i \in N} A_i$ is countable since $\bigcup_{i \in N} A_i = \bigcup_{i \in N} B_i$ and $\bigcup_{i \in N} B_i$ is countable. ||

Theorem 4.18. Let $\{U_i\}_{i \in N}$ be a countable collection of sets. Then, $\{A_\alpha \mid A_\alpha = \bigcap_{j \in I_n} V_j$ where $V_j \in \{U_i\}_{i \in N}$ $\forall j \in I_n, \forall n \in N\}$ is countable.

Proof. This theorem is proved using Lemma 4.17. For each $n \in N$ let $\{\bigcap_{j \in I_n} V_j \mid V_j \in \{U_i\}_{i \in N} \forall j \in I_n$ for one and only one $n \in N\} = \{\bigcap_{j \in I_n} V_j\}$. Then $\{\bigcap_{j \in I_n} V_j\}$ will be shown to be countable. Then, the lemma is applied to conclude the proof. Induction is used to show that $\{\bigcap_{j \in I_n} V_j\}$ is countable $\forall n \in N$.

Let $n = 1$. Then, $\{\bigcap_{j \in I_1} V_j\} = \{V_1\}$ and since V_1 is a variable over $\{U_i\}_{i \in N}$, then $\{\bigcap_{j \in I_1} V_j\} = \{V_1\} = \{U_i\}_{i \in N}$ and $\{U_i\}_{i \in N}$ is countable by the definition of $\{U_i\}_{i \in N}$.

Let $n = 2$. Then, $\{\bigcap_{j \in I_2} V_j\} = \{V_1 \cap V_2\}$ and consider the "2" intersections which conform to the following form.

$\{V_1 \cap V_2\} = \{U_1 \cap U_1,\ U_1 \cap U_2,\ \ldots,\ U_1 \cap U_k,\ \ldots;\ U_2 \cap U_1,$

$U_2 \cap U_2, \ldots, U_2 \cap U_k, \ldots; \ldots; U_m \cap U_1, U_m \cap U_2, \ldots, U_m \cap U_m, \ldots, U_m \cap U_k, \ldots; \ldots\}$. All "two" intersections are thus obtained. For if $U_n \cap U_m$ is under consideration $U_n \cap U_m \in \{U_n \cap U_i \mid i = 1,2, \ldots, m, \ldots\}$ and since $\{U_n \cap U_i \mid i = 1,2, \ldots m, \ldots\} \subset V_1 \cap V_2$ all "two" intersections are under consideration.

Now let A_i be defined $\forall\, i \in N$ as follows.

$A_1 = \{U_1 \cap U_i \mid i = 1,2, \ldots, n, \ldots\}$

$A_2 = \{U_2 \cap U_i \mid i = 1,2, \ldots, n, \ldots\}$

.
.
.

$A_n = \{U_n \cap U_i \mid i = 1,2, \ldots, n, \ldots\}$

.
.
.

Clearly, by the way the A's are defined there is a countable collection of them, i.e., $\{A_k\}_{k \in N}$ is a countable set. Thus, $\exists\, f: N \longrightarrow \{A_k\}_{k \in N} \ni f(n) = A_n$.

For all A_i define $f_i: N \longrightarrow A_i$ by $f_i(n) = U_i \cap U_n$. Clearly, this is a function onto A_i, for given $U_i \cap U_k$, f_i maps k to $U_i \cap U_k$. Thus, A_i is countable $\forall\, i \in N$ and thus by Lemma 4.17 $\bigcup_{i \in N} A_i$ is countable.

Now assume $\left\{\bigcap_{j \in I_n} V_j\right\}$ for a given $n \in N$ is countable. Thus, $\left\{\bigcap_{j \in I_n} V_j\right\}$ for a given n can be indexed by N. Let $\left\{\bigcap_{j \in I_n} V_j\right\} = \{B_k\}_{k \in N}$.

Define $C_i \; \forall \; i \in N$ as follows.

$C_1 = \{B_1 \cap U_i \mid i = 1,2,\ldots\}$

$C_2 = \{B_2 \cap U_i \mid i = 1,2,\ldots\}$

$\vdots$

$C_n = \{B_n \cap U_i \mid i = 1,2,\ldots\}$

$\vdots$

All $\bigcap_{j \in I_{n+1}} V_j$ are obtained in $\{C_k\}_{k \in N}$. For $\bigcap_{j \in I_{n+1}} V_j = (\bigcap_{j \in I_n} V_j) \cap V_{n+1} = (\bigcap_{j \in I_n} V_j) \cap U_k$ for some $k \in N$. Clearly, $\bigcap_{j \in I_n} V_j \in \{\bigcap_{j \in I_n} V_j\}$. Thus, $(\bigcap_{j \in I_n} V_j) \cap U_k = B_h \cap U_k$ for some $h \in N$. However, $B_h \cap U_k \in C_h$ and $\{C_i\}_{i \in N}$ is countable since $\exists g: N \longrightarrow \{C_i\}_{i \in N} \ni g(n) = C_n$ and g is onto since given C_k, $g(k) = C_k$. Each C_i is countable as are the A_i. Thus, by Lemma 4.17 $\bigcup_{i \in N} C_i$ is countable. Therefore, $\{\bigcap_{j \in I_{n+1}} V_j\}$ is countable.

Thus, given any $n \in N$ $\{\bigcap_{j \in I_n} V_j\}$ is countable. Since N is countable $\exists$ a countable collection of $\{\bigcap_{j \in I_n} V_j\}$. Thus by Lemma 4.17 their union is countable. Then, $\{A_\alpha \mid A_\alpha = \bigcap_{j \in I_n} V_j$ where $V_j \in \{U_i\}_{i \in N} \; \forall \; j \in I_n, \forall \; n \in N\}$ is countable. ||

As a result of Theorem 4.18 the following is true.

Theorem 4.19. Any second countable space $(S,\mathcal{T})$ extends to a second countable topological space $[S,\sigma]$.

Proof. If $S \neq \emptyset$, let $(S,\mathcal{T})$ have a countable basis for the open sets. Denote this countable basis by $\mathcal{T}_2$. Then, $(S,\mathcal{T}) = (S,\mathcal{T}_2)$. Also by Lemma 3.31, $(S,\mathcal{T}) = (S,\mathcal{T}_2-\{\emptyset\})$. So let $\mathcal{T}_2-\{\emptyset\} = \mathcal{T}_1$. Since $\emptyset \notin \mathcal{T}_1$, $\mathcal{T}_1$ is a set of neighborhoods for the open sets in $(S,\mathcal{T})$. By extension the collection of extended neighborhoods is $\{A_\alpha \mid A_\alpha = \bigcap_{j \in I_n} U_j$ where $U_j \in \mathcal{T}_1$ $\forall\, j \in I_n, \forall\, n \in N\}$. By Theorem 4.18 and the fact that $\mathcal{T}_1$ is countable, $\{A_\alpha \mid A_\alpha = \bigcap_{j \in I_n} U_j$ where $U_j \in \mathcal{T}_1$ $\forall\, j \in I_n, \forall\, n \in N\}$ is countable and thus $[S,\sigma]$ is second countable. If $S = \emptyset$, then $(S,\{\emptyset\})$ extends to $[S,\{\emptyset\}]$ and both are second countable. ||

Using Lemma 4.15 any quasi topological space that extends to a second countable topological space can be shown to be second countable since the set of neighborhoods in the quasi topological space is a subset of the set of neighborhoods in the topological space.

A concept similar to that of second countability is that of first countability where instead of referring to a basis for the entire collection of open sets the concept rests on a basis of a point.

Definition 4.20. A collection β of neighborhoods of a point p in a space $(S,\mathcal{T})$ is a basis at p iff $\forall$ open set

$U \ni p \in U$, $\exists\, V \in \beta \ni V \subset U$.

Definition 4.21. A space $(S,\mathcal{T})$ is first countable iff $\forall p \in S$, $\exists$ a countable basis at p.

Theorem 4.22. A first countable quasi topological space $(S,\mathcal{T})$ extends to a first countable topological space $[S,\sigma]$

Proof. Let $p \in S$ and γ a countable basis at p. By Theorem 4.18, the extension of γ, denoted here by γ', is countable. It must now be argued that given any open set U in the topology $\sigma \ni p \in U$, $\exists\, V \in \gamma' \ni V \subset U$. Of course, $\forall\, V \in \gamma'$ $p \in V$ since $p \in W\ \forall\, W \in \gamma$. Thus, consider $U \in \sigma \ni p \in U$. Thus, $U = \bigcup_{\alpha \in \Lambda} A_\alpha \ni A_\alpha \in \{N_p' \mid p \in S\}\ \forall\, \alpha \in \Lambda$. Since $p \in U$, $p \in A_\alpha$ for some $\alpha \in \Lambda$. Since $A_\alpha \in \{N_p' \mid p \in S\}\ \forall\, \alpha \in \Lambda$, $A_\alpha = \bigcap_{i \in I_n} B_i$ where $B_i \in \{N_p \mid p \in S\}\ \forall\, i \in I_n$. Since $p \in A_\alpha$ $p \in B_i\ \forall\, i \in I_n$. Since B_i is open in $(S,\mathcal{T})$ $\forall\, i \in I_n$, $\exists$ for all $i \in I_n$ a $C_i \ni C_i \in \gamma$ and $p \in C_i \subset B_i$. Consider $\bigcap_{i \in I_n} C_i$. Thus, $\bigcap_{i \in I_n} C_i \in \gamma'$ and $p \in \bigcap_{i \in I_n} C_i$ since $p \in C_i\ \forall\, i \in I_n$. However, $\bigcap_{i \in I_n} C_i \subset \bigcap_{i \in I_n} B_i$ and hence a subset of A_α and of U. For if $x \in \bigcap_{i \in I_n} C_i$, then $x \in C_i\ \forall\, i \in I_n$. If $x \in C_i$ $\forall\, i \in I_n$, then $x \in B_i\ \forall\, i \in I_n$ since $C_i \subset B_i\ \forall\, i \in I_n$. Thus, $x \in \bigcap_{i \in I_n} B_i$ and hence $x \in A_\alpha$ and $x \in U$. Therefore, $p \in \bigcap_{i \in I_n} C_i \subset U$ and since $\bigcap_{i \in I_n} C_i \in \gamma' \subset \{N_p' \mid p \in S\}$, $[S,\sigma]$

is first countable. ||

As an example of a non-first countable space $(S,\mathcal{T})$ that extends to a first countable space $[S,\sigma]$ consider the following example.

Example 4.23. Let $S = R$ where R denotes the set of real numbers. Let $\mathcal{T} = \{\{a,b\} \mid a, b \in R \text{ and } a \neq b\}$.

Consideration shows that this is a non-first countable quasi (topological) space that extends to a discrete topological space which is first countable.

Since first countability and T_1 have been shown to be invariant properties their combination called an I space is invariant.

From here on quasi space and quasi topological space are used interchangeably.

CHAPTER V

SUBSPACES AND CARTESIAN QUASI PRODUCT

With a mathematical structure in hand a way was sought to generate the same type structure from that on hand. Such is the case with a subspace of a quasi space.

Definition 5.1. If $\mathcal{T}$ is a collection of subsets of a set S and $K \subset S$ then $K \cap \mathcal{T} = \{K \cap A \mid A \in \mathcal{T}\}$.

Theorem 5.2. If $(S,\mathcal{T})$ is a quasi space and $K \subset S$ then $K \cap \mathcal{T}$ is a quasi topology on K.

Proof. Let $p \in K$. Then, $p \in S$ since $K \subset S$. If $p \in S$, then $\exists A \in \mathcal{T} \ni p \in A$. Thus, $p \in K \cap A$ and $K \cap A \in K \cap \mathcal{T}$. If $K = \emptyset$, then $K \cap \mathcal{T} = \{\emptyset\}$ and the definition of a quasi topology is vacuously satisfied. ||

Definition 5.3. If $(S,\mathcal{T})$ is a quasi space and $K \subset S$, then $(K, K \cap \mathcal{T})$ is a sub quasi space.

Theorem 5.4. If $(S,\mathcal{T}_1) = (S,\mathcal{T}_2)$, then $(K, K \cap \mathcal{T}_1) = (K, K \cap \mathcal{T}_2)$.

Proof. Let A be open in $(K, K \cap \mathcal{T}_1)$, then $\forall\, p \in A\ \exists\, B_p \in \mathcal{T}_1 \ni K \cap B_p \subset A$ where $p \in B_p$. Since $B_p \in \mathcal{T}_1$, B_p is open in $(S,\mathcal{T}_2)$. Thus, $\forall\, q \in B_p\ \exists\, C_q \ni q \in C_q \subset B_p$ where $C_q \in \mathcal{T}_2$ $\forall\, q \in B_p$. For $p \in B_p\ \exists\, C_p \in \mathcal{T}_2 \ni p \in C_p \subset B_p$. Thus,

$K \cap C_p \subset K \cap B_p$ for if $x \in K \cap C_p$, $x \in K$ and $x \in C_p$. If $x \in C_p$, then $x \in B_p$, thus $x \in K \cap B_p$. Thus, $\forall\, p \in A\ \exists\, K \cap C_p \ni p \in K \cap C_p \subset A$ and $K \cap C_p \in K \cap \mathcal{T}_2$. Thus, A is open in $(K, K \cap \mathcal{T}_2)$. Likewise, if A is open in $(K, K \cap \mathcal{T}_2)$, then A is open in $(K, K \cap \mathcal{T}_1)$. Therefore, $(K, K \cap \mathcal{T}_2) = (K, K \cap \mathcal{T}_1)$.||

Lemma 5.5. If $(S,\mathcal{T})$ extends to $[S,\sigma]$, then considering the extension of $(K, K \cap \mathcal{T})$ where $K \subset S$, $A \in$ the extended neighborhoods of $(K, K \cap \mathcal{T})$ iff $A = (\bigcap_{i \in I_n} N_i) \cap K$ where $N_i \in$ the set of neighborhoods of $(S,\mathcal{T})$ $\forall\, i \in I_n$ and $n \in N$.

Proof. What must be proved here is that $\bigcap_{i \in I_n}(K \cap N_i) = K \cap (\bigcap_{i \in I_n} N_i)$. Let $a \in \bigcap_{i \in I_n}(K \cap N_i)$, then $a \in K \cap N_i$ $\forall\, i \in I_n$. If $a \in K \cap N_i$ $\forall\, i \in I_n$, then $a \in K$ and $a \in N_i$ $\forall\, i \in I_n$. Then, $a \in K$ and $a \in \bigcap_{i \in I_n} N_i$. Thus, $a \in K \cap (\bigcap_{i \in I_n} N_i)$ and $\bigcap_{i \in I_n}(K \cap N_i) \subset K \cap (\bigcap_{i \in I_n} N_i)$. Let $a \in K \cap (\bigcap_{i \in I_n} N_i)$. Then, $a \in K$ and $a \in (\bigcap_{i \in I_n} N_i)$ and if $a \in \bigcap_{i \in I_n} N_i$, then $a \in N_i$ $\forall\, i \in I_n$. Thus, $a \in K$ and $a \in N_i$ $\forall\, i \in I_n$. Therefore, $a \in \bigcap_{i \in I_n}(K \cap N_i)$. Finally, $K \cap (\bigcap_{i \in I_n} N_i) \subset \bigcap_{i \in I_n}(K \cap N_i)$ and $K \cap (\bigcap_{i \in I_n} N_i) = \bigcap_{i \in I_n}(K \cap N_i)$.||

Lemma 5.6. Given A and $\{B_p\}_{p \in \Lambda}$, then $A \cap (\bigcup_{p \in \Lambda} B_p) = \bigcup_{p \in \Lambda}(A \cap B_p)$.

Proof. Let $x \in A \cap (\bigcup_{p \in \Lambda} B_p)$, then $x \in A$ and $x \in \bigcup_{p \in \Lambda} B_p$. Since $x \in \bigcup_{p \in \Lambda} B_p$ $x \in B_p$ for some $p \in \Lambda$. Then $x \in A$ and

$x \in B_p$ for some $p \in \Lambda$. Thus, $x \in A \cap B_p$ and $x \in \bigcup_{p \in \Lambda} (A \cap B_p)$. Therefore, $A \cap (\bigcup_{p \in \Lambda} B_p) \subset \bigcup_{p \in \Lambda} (A \cap B_p)$.

Let $x \in \bigcup_{p \in \Lambda} (A \cap B_p)$, then $x \in A \cap B_p$ for some $p \in \Lambda$. Since $x \in A \cap B_p$ $x \in A$ and $x \in B_p$ for this $p \in \Lambda$. If $x \in B_p$, then $x \in \bigcup_{p \in \Lambda} B_p$. Thus $x \in A$ and $x \in \bigcup_{p \in \Lambda} B_p$. Then, $x \in A \cap (\bigcup_{p \in \Lambda} B_p)$. Thus, $\bigcup_{p \in \Lambda} (A \cap B_p) \subset A \cap (\bigcup_{p \in \Lambda} B_p)$.

Therefore, $A \cap (\bigcup_{p \in \Lambda} B_p) = \bigcup_{p \in \Lambda} (A \cap B_p)$. ||

Following Lemmas 5.5 and 5.6 the following theorem can now be stated and proved.

Theorem 5.7. If $(S, \mathcal{T})$ extends to $[S, \sigma]$, then $(S, K \cap \mathcal{T})$ extends to $[S, K \cap \sigma]$.

Proof. It must be shown that the extended neighborhoods of $(S, K \cap \mathcal{T})$ do form a basis for the topology $K \cap \sigma$. Let $A \in K \cap \sigma$, i.e., A is open in $[S, K \cap \sigma]$. Then, $\exists\, B \in \sigma \ni A = K \cap B$. If $B \in \sigma$, then $B = \bigcup_{\alpha \in \Lambda} U_\alpha \ni U_\alpha \in \{N'_p \mid p \in S\}$. Thus, $A = K \cap (\bigcup_{\alpha \in \Lambda} U_\alpha)$. By Lemma 5.6 $K \cap (\bigcup_{\alpha \in \Lambda} U_\alpha) = \bigcup_{\alpha \in \Lambda} (K \cap U_\alpha)$. If $p \in A$, then $p \in K \cap U_\alpha$ for some $\alpha \in \Lambda$. If $p \in K \cap U_\alpha$, then $p \in K$ and $p \in U_\alpha$. Since $U_\alpha \in \{N'_p \mid p \in S\}$ $U_\alpha = \bigcap_{i \in I_m} N_i$ where $N_i \in \{N_p \mid p \in S\}$ $\forall\, i \in I_m$ and $m \in N$. Thus, $p \in K \cap (\bigcap_{i \in I_m} N_i)$. By Lemma 5.5 $K \cap (\bigcap_{i \in I_m} N_i)$ is an element of the set of extended neighborhoods of $(K, K \cap \mathcal{T})$. Thus, $A = \bigcup_{\beta \in \gamma} V_\beta$ where $V_\beta \in$ the set of extended neighborhoods of $(K, K \cap \mathcal{T}) \; \forall \; V_\beta$. ||

The method now to be discussed in relation to generating quasi spaces from others involves the concept of the Cartesian product of a collection of sets.

Definition 5.8. Let σ_1 and σ_2 be collections of sets. The Cartesian product $\sigma_1 \times \sigma_2$ of σ_1 and σ_2 is a collection of sets. $\sigma_1 \times \sigma_2 = \left\{F \mid F = U \times V \text{ where } U \in \sigma_1 \text{ and } V \in \sigma_2\right\}$.

Definition 5.9. Let $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ be quasi spaces. Then $(S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is the quasi topological product of $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$.

A cautionary note is in order. The notation for a topological space, $[S, \sigma]$, describes the space with σ as the topology. However, in speaking with reference to the topological product of two topological spaces $[S_1, \sigma_1]$ and $[S_2, \sigma_2]$, bases for each topology must be used, for $\sigma_1 \times \sigma_2$ is not necessarily a topology on $S_1 \times S_2$. Thus, if $\mathcal{T}_1$ is a basis for σ_1 and $\mathcal{T}_2$ for σ_2, describe the topological product of $[S_1, \sigma_1]$ and $[S_2, \sigma_2]$ as $\langle S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2 \rangle$ where $\mathcal{T}_1 \times \mathcal{T}_2$ is thought of as a basis and not necessarily a topology. "$\langle \; \rangle$" is used here because the usual notation for a topological space uses the topology not just any basis. The following examples illustrates this.

Example 5.10. Let $S = \{a,b\}$ and $\sigma = \{\emptyset, \{a\}, \{a,b\}\}$. Then $[S,\sigma]$ is a topological space and $\sigma \times \sigma = \{\emptyset, \{(a,a)\}, \{(a,b), (a,a)\}, \{(a,a), (b,a)\}, \{(a,a), (b,a), (a,b), (b,b)\}\}$. However, $\{(a,b), (a,a)\} \cup \{(a,a), (b,a)\} \notin \sigma \times \sigma$.

Theorem 5.11. If $(S_1, \mathcal{T}_1)$ and $(S_2, \mathcal{T}_2)$ are quasi spaces, then $(S_1 \times S_2, \mathcal{T}_1 \times \mathcal{T}_2)$ is a quasi space.
Proof. Let $(p,q) \in S_1 \times S_2$. Then, $p \in S_1$ and $q \in S_2$. Then, $\exists\, U \in \mathcal{T}_1$ and $V \in \mathcal{T}_2 \ni p \in U$ and $q \in V$. Thus, $(p,q) \in U \times V$ and $U \times V \in \mathcal{T}_1 \times \mathcal{T}_2$ by the definition of $\mathcal{T}_1 \times \mathcal{T}_2$. If either S_1 or S_2 is empty, then $S_1 \times S_2 = \emptyset$ by definition, $\mathcal{T}_1 \times \mathcal{T}_2 = \{\emptyset\}$, and $(\emptyset, \{\emptyset\})$ is a quasi space. ||

Lemma 5.12. Given $\{A_\alpha\}_{\alpha \in \Lambda}$ and $\{B_\alpha\}_{\alpha \in \Lambda}$, then $\bigcap_{\alpha \in \Lambda} A_\alpha \times \bigcap_{\alpha \in \Lambda} B_\alpha = \bigcap_{\alpha \in \Lambda} (A_\alpha \times B_\alpha)$.
Proof. Let $(p,q) \in \bigcap_{\alpha \in \Lambda} A_\alpha \times \bigcap_{\alpha \in \Lambda} B_\alpha$. Then, $p \in \bigcap_{\alpha \in \Lambda} A_\alpha$ and $q \in \bigcap_{\alpha \in \Lambda} B_\alpha$. Since $p \in A_\alpha \ \forall\, \alpha \in \Lambda$ and $q \in B_\alpha \ \forall\, \alpha \in \Lambda$, $(p,q) \in A_\alpha \times B_\alpha \ \forall\, \alpha \in \Lambda$ and $(p,q) \in \bigcap_{\alpha \in \Lambda} (A_\alpha \times B_\alpha)$. Therefore, $\bigcap_{\alpha \in \Lambda} A_\alpha \times \bigcap_{\alpha \in \Lambda} B_\alpha \subset \bigcap_{\alpha \in \Lambda} (A_\alpha \times B_\alpha)$. Let $(p,q) \in \bigcap_{\alpha \in \Lambda} (A_\alpha \times B_\alpha)$. Then $(p,q) \in A_\alpha \times B_\alpha \ \forall\, \alpha \in \Lambda$. Then, $p \in A_\alpha$ and $q \in B_\alpha \ \forall\, \alpha \in \Lambda$. Thus, $p \in \bigcap_{\alpha \in \Lambda} A_\alpha$ and $q \in \bigcap_{\alpha \in \Lambda} B_\alpha$. Then, $(p,q) \in \bigcap_{\alpha \in \Lambda} A_\alpha \times \bigcap_{\alpha \in \Lambda} B_\alpha$ and $\bigcap_{\alpha \in \Lambda} (A_\alpha \times B_\alpha) \subset \bigcap_{\alpha \in \Lambda} A_\alpha \times \bigcap_{\alpha \in \Lambda} B_\alpha$. Therefore, $\bigcap_{\alpha \in \Lambda} A_\alpha \times \bigcap_{\alpha \in \Lambda} B_\alpha = \bigcap_{\alpha \in \Lambda} (A_\alpha \times B_\alpha)$. ||

It is henceforth assumed that the topological product gives a topological space.

Theorem 5.13. Let $(S,\mathcal{T})$ be a quasi space and let $(S \times S, \mathcal{T} \times \mathcal{T})$ be the quasi topological product of $(S,\mathcal{T})$ with itself. Extend $(S,\mathcal{T})$ to $\langle S,\gamma\rangle$ where $\langle S,\gamma\rangle = [S,\sigma]$ and γ the set of extended neighborhoods. Extend $(S \times S, \mathcal{T} \times \mathcal{T})$ to $\langle S \times S, \beta\rangle$ where β is the set of extended neighborhoods. Then, $\langle S \times S, \beta\rangle = \langle S \times S, \gamma \times \gamma\rangle$.

Proof. It must be shown that β and $\gamma \times \gamma$ are bases for the same collection of sets or that $\beta = \gamma \times \gamma$. The latter course will be taken. Let $W \in \beta$. Then, $W = \bigcap_{i \in I_n} V_i$ where $V_i \in \mathcal{T} \times \mathcal{T}\ \forall\ i \in I_n$. Since $W = \bigcap_{i \in I_n} V_i$, $W = \bigcap_{i \in I_n} (U_i \times T_i)$ and $W = \bigcap_{i \in I_n} U_i \times \bigcap_{i \in I_n} T_i$ by Lemma 5.12 where $U_i \in \mathcal{T}$ and $V_i \in \mathcal{T}\ \forall\ i \in I_n$. Since $U_i \in \mathcal{T}\ \forall\ i \in I_n$ and $T_i \in \mathcal{T}\ \forall\ i \in I_n$, $\bigcap_{i \in I_n} T_i$ and $\bigcap_{i \in I_n} U_i$ are both elements of γ and thus $\bigcap_{i \in I_n} T_i \times \bigcap_{i \in I_n} U_i \in \gamma \times \gamma$. Thus, $\beta \subset \gamma \times \gamma$. Let $W \in \gamma \times \gamma$. Then it is true that, $W = \bigcap_{i \in I_m} N_i \times \bigcap_{j \in I_n} M_j$ where $N_i \in \mathcal{T}\ \forall\ i \in I_m$ and $M_j \in \mathcal{T}\ \forall\ j \in I_n$. If $m \neq n$, then $\lessgtr$ $n < m$. Is $\bigcap_{j \in I_n} M_j = \bigcap_{j \in I_m} Y_j$ where $Y_j = M_j\ \forall\ j = 1,\ldots,n$ and $Y_j = M_n$ $\forall\ j = n+1,\ldots,m$? The answer is yes. Thus, $\bar{W} = \bigcap_{i \in I_m} N_i \times$

$\bigcap_{i \in I_m} Y_i$ and $W = \bigcap_{i \in I_m} (N_i \times Y_i)$. Thus, $W \in \beta$ and $\gamma \times \gamma \subset \beta$. Therefore, $\beta = \gamma \times \gamma$. ||

CHAPTER VI

CONCLUSION

This paper has shown that an elementary structure, a quasi topological space, extends to a mathematical structure less elementary in nature, a topological space. It has shown that most of the rudimentary concepts of a topological space also hold in a quasi topological space. Absent from these concepts was an axiom for a topological space, that the intersection of any finite collection of open sets is open. Using this as an objective, a topological space was reached by extending the collection of neighborhoods of a quasi topological space, where this extension is unique. It was shown under what conditions a quasi topological space is a topological space and how all possible quasi topological spaces that extend to a given topological space can be obtained.

The chapter on classification of spaces showed that Hausdorff, discrete, regular, T_0, T_1, second countable, first countable, and I quasi topological spaces do extend to Hausdorff, discrete, regular, T_0, T_1, second countable, first countable, and I topological spaces respectively. Also evident in Chapter IV was that any quasi topological space that extends to a second countable topological space is second countable. Also any quasi topological space that extends to a T_0 topological space was shown to be T_0.

Chapter V introduced the concept of a subspace and revealed the relationship between a subspace of a quasi topological space and the subspace of the topological space reached by the extension of the quasi topological space, where the same subset is used in defining both subspaces. The relationship is, of course, that the subspace of the quasi topological space extends to the subspace of the topological space. The Cartesian product of the two spaces was introduced in the fifth chapter and the extension of the Cartesian product of a quasi topological space with itself was shown to be the same topological space as the Cartesian topological product of the extension of the original quasi topological space with itself.

Thus, properties, inherent in a topological space were related to the corresponding property in a quasi topological space.

The discussion was halted at this point to leave the reader with some points to consider and possibly verify.

Definition 6.1 A space $(S, \mathcal{T})$ is normal iff given any two disjoint closed sets C_1 and C_2 $\exists$ disjoint open sets U_1 and $U_2 \ni C_1 \subset U_1$ and $C_2 \subset U_2$.

The reader might wish to show that a normal quasi topological space extends to a normal topological space. He might also wish to verify that subspaces of Hausdorff, discrete, regular, T_1, T_0, second countable, first countable, I, and normal quasi topological spaces are respectively Hausdorff, discrete, regular, T_1, T_0, second countable, first countable, I, and normal or that the Cartesian product of Hausdorff, discrete, T_1, T_0, second countable, first countable, and I quasi topological spaces are Hausdorff, discrete, T_1, T_0, second countable, first countable, and I respectively. However, this is not true for regular spaces as the following example illustrates.

Example 6.2. Let $S = \{a,b,c\}$ and $\mathcal{T} = \{\{a\}, \{a,b\}, \{c\}, S\}$.

As yet a type of space has not been found that is not invariant under the extension from a quasi topological space to a topological space. Also it has not been shown, nor a counter example exhibited to the contrary, that the quasi topological product of normal spaces is normal. These problems along with the further development of properties of a topological space originating from a quasi topological space will be investigated by this author, for one.

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INDEX OF NOTATION

NOTATION	MEANING	PAGE
$\{x, y, z\}$	Set that contains x, y, and z	1
$a \in A$	a is an element of A	1
$a > b$	a is greater than b	1
I	Set of integers	1
$\{x \mid x \in I \text{ and } x > 3\}$	Set of all integers such that they are greater than 3	1
$A \subset B$	A is a subset of B	1
$\emptyset$	The empty set	2
$\forall$	For all	2
iff	If and only if	2
$\notin$	Not an element of	2
$-A$	Complement of A	2
$A \cup B$	A union B	2
$A \cap B$	A intersection B	2
(a, b)	The ordered pair of a and b	3
$A \times B$	Cartesian product of A and B	3
$\exists$	There exists	5
$\ni$	Such that	5
$(S, \mathcal{T})$	Notation for a quasi topological space	5
N_p	Neighborhood of p in a quasi topological space	6

NOTATION	MEANING	PAGE
$\{N_p \mid p \in S\}$	Collection of neighborhoods for all $p \in S$ in a quasi topological space	6
S-A	$S \cap -A$	6
$\|\|$	End of proof	6
$\{N_p\}$	Collection of neighborhoods of a particular point p in a quasi topological space	6
N	Set of natural numbers	7
$\{B_\gamma\}_{\gamma \in \beta}$	Collection of sets indexed by β	7
$\bigcup_{\gamma \in \beta} B_\gamma$	Union of $\{B_\gamma\}_{\gamma \in \beta}$	7
$\bigcap_{\gamma \in \beta} B_\gamma$	Intersection of $\{B_\gamma\}_{\gamma \in \beta}$	8
O_p	Open set about p	9
A'	Set of limit points of A	12
$\mathfrak{S}$	Suppose	12
$\bar{A}$	Closure of set A	13
$\{O_x\}$	Collection of open sets about x	14
$f:S \longrightarrow T$	Function from S to T	15
f(x)	Image of x under function f	15
f(S)	Range of S under f	15
$\{O_p \mid p \in S\}$	Collection of all open sets in $(S,\mathcal{T})$	17
gf	Composition of g and f	18

NOTATION	MEANING	PAGE
$f\|A$	Function f restricted to A	18
f^{-1}	Inverse of f	18
$\{a_n\}$	Sequence whose general term is a_n	20
$\bigcup_{i=1}^{n} A_i$	The union of $\{A_i\}_{i \in I_n}$ where I_n denotes the first n natural numbers	20
$\bigcup_{i=1}^{n} f(i)$	The union of $\{\{f(i)\}\}_{i \in I_n}$	20
$a < b$	a is less than b	21
$\nexists$	There does not exist	22
I_n	Set of all natural numbers less than or equal to n	22
$a \leq b$	a is less than or equal to b	23
$(gf\|S): S \longrightarrow L$	Function gf|S from S to L	24
$(gf\|S)(k)$	Image of k under gf|S	24
$[S, \sigma]$	Topological space with S as the set of points and σ the topology (open sets)	25
$\{N'_p \| p \in S\}$	Collection of extended neighborhoods, extended from a quasi topological space	26
$N'(p)$	"Extended" neighborhood of p	26
$a \geq b$	a is greater than or equal to b	28

NOTATION	MEANING	PAGE
$\{N'_p\}$	Extended neighborhoods of a particular point p	28
$\bigcup_{\substack{\alpha \in \lambda \\ \gamma \in \beta}} (A_\alpha \cap B_\gamma)$	The union of all sets of the form $A_\alpha \cap B_\gamma$ where $\alpha \in \lambda$ and $\gamma \in \beta$	34
$\{N_p' \mid p \in S\}$	Same as $\{N'_p \mid p \in S\}$	41
$f: A \xrightarrow{\text{onto}} B$	f is an onto mapping from A to B	45
$\langle S, \tau \rangle$	A topological space where τ is a basis for the topology	56