MATRIX REPRESENTATIONS OF SEMIGROUPS

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Introduction

"A semigroup is a set considered with respect to a binary associative operation defined in it. The concept of a semigroup is so simple and natural that it is hard to say when it first appeared. As Klein, Lectures On The Development of Mathematics in The 19th Century (Part I, Chapter VIII), points out there were doubts, even in the period when the theory of groups was formulated as a separate mathematical discipline, as to whether that which we call a semigroup should be taken as the fundemental concept. However, the problem facing mathematics at that stage of its development made it necessary to choose a more restrictive concept, that of a group."[1]

This thesis is a study of the matrix representations of semigroups. The main problem in this area: "Given an arbitrary semigroup, is it possible to find a matrix representation for that semigroup?"

The matrix representations dealt with are matrix representations over a group. Chapter I is an introduction to semigroups, some of the necessary definitions, and a few preliminary theorems. Chapter II contains the theorems

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[1]Ljapin, Semigroups (American Mathematical Society, Providence, Rhode Island, 1963) p.v. Preface.

and definitions necessary for Chapter III and Chapter IV. In particular, principal ideals of a semigroup and the partitioning of a semigroup by these principal ideals into $\mathcal{D}, \mathcal{H}, \mathcal{L}$, and $\mathcal{R}$-classes have been considered. Probably the most used theorems of the Chapter is Green's Lemma on transformations from $\mathcal{H}$-class to $\mathcal{H}$-class. It is shown in Chapter III that any completely O-simple semigroup is isomorphic to a semigroup of matrices. In Chapter IV an arbitrary semigroup is represented as a semigroup of matrices. In the more general case the mapping may be a homomorphism instead of an isomorphism as in the case for completely O-simple semigroups.

Definition: The ordered pair $(S,*)$ is a semigroup if and only iff (iff)

(i) for all $(\forall)$ $x,y \in S$, there exists $(\exists)$ a unique $s \in S$ such that $(\ni)$ $s=x*y$ and

(ii) $x,y,z \in S$, $x*(y*z)=(x*y)*z$.

If the operation is obvious, the set S will be referred to as the semigroup.

Examples of semigroups are numerous. To list a few:

Any group is a semigroup.

The set M of all square matrices over the complex numbers of order n with respect to ordinary multiplication of matrices is a semigroup. The non-singular matrices in

M form a semigroup with respect to the same operation.

The set $\{1,2,3,\ldots,n\}$ with the operation of finding the greatest common divisor is a semigroup.

Let n be any natural number. Let $M=\{0,1,2,\ldots,n-1\}$ with the operation, $\times$ multiplication modulo n defined on the set M. $(M,\times)$ is a semigroup.

Definition: An element e of a semigroup $(S,*)$ is called a left identity of S iff $\forall\, a\in S,\ e*a=a$. e is called a right identity of S iff $a*e=a$. e is called an identity (two sided) of S iff $e*a=a*e=a$.

Definition: An element 0 of a semigroup $(S,*)$ is called a left zero of S iff $\forall\, a\in S,\ 0*a=0$. 0 is called a right zero of S iff $a*0=0$. An element 0 of S is called a zero of S iff 0 is both a left and right zero of S.

Any binary operation on a semigroup S may be extended to include an identity element by adjoining the element 1 to the set S and defining $1*1=1$ and for all $a\in S$, $1*a=a*1=a$. When this is done, the resulting semigroup $S\cup\{1\}$ will be said to have had an identity element adjoined and will be denoted S^1.

Theorem 1.1: If a semigroup S contains a left identity e_1 and a right identity e_2 then $e_1=e_2$.

Proof: Assume e_1 and e_2 are, respectively, left and right identities of a semigroup $(S,*)$ and suppose that $a \in S$. Then $e_1 * e_2 = e_2$ and $e_1 * e_2 = e_1$ by definition of left and right identities. Therefore $e_1 = e_2$.

Corollary 1.1: A semigroup possesses at most one identity and at most one zero element.

Proof: Assume e_1 and e_2 are identity elements of a semigroup $(S,*)$. If e_1 is an identity of S then $e_1 * e_2 = e_1$. If e_2 is an identity of S then $e_1 * e_2 = e_1$. Hence $e_1 = e_2$. In the same manner, if O_1 and O_2 are zero elements of S then $O_1 * O_2 = O_2$ and $O_1 * O_2 = O_1$. Therefore $O_1 = O_2$.

The image of a mapping, θ, from A into B will be denoted $(a)\theta$, for $a \in A$.

Definition: The mapping θ of the semigroup $(S,\circ)$ into the semigroup $(T,*)$ is said to be a homomorphism, if for any $x,y \in S$, $(x \circ y)\theta = (x)\theta * (y)\theta$.
If θ is a mapping of S onto T, then θ is a homomorphism of $(S,\circ)$ onto $(T,*)$.

Definition: A one-to-one homomorphism is called an isomorphism.

Definition: A mapping of a set A into itself is called a transformation.

The symbol $\mathcal{J}_A$ will represent the set of all transformations of the set A.

Definition: The product of transformations $\theta, \psi \in \mathcal{J}_A$ is defined as the transformation $\theta \circ \psi \in \mathcal{J}_A \ni \forall a \in A \quad (a)(\theta \circ \psi) = [(a)\theta]\psi$.

Theorem 1.2: The set $\mathcal{J}_A$ of all transformations of an arbitrary set A is a semigroup with respect to the operation of forming the product of the transformations.

Proof: Let A be a set and $\mathcal{J}_A$ the set of transformations of A. Suppose $\theta, \psi, \mu \in \mathcal{J}_A$.

$\forall a \in A$; $(a)(\theta \circ \psi) = [(a)\theta]\psi$. But by definition of transformation $(a)\theta \in A$, $[(a)\theta]\psi \in A$ and consequently so is $(a)(\theta \circ \psi) \in A$. Therefore $(\mathcal{J}_A, \circ)$ is closed under the operation.

For all $a \in A$, $(a)\{(\theta \circ \psi) \circ \mu\} = \{(a)(\theta \circ \psi)\}\mu = \{[(a)\theta]\psi\}\mu$ by definition of product of transformations. Also $(a)\{\theta \circ (\psi \circ \mu)\}$ $= [(a)\theta](\psi \circ \mu) = \{[(a)\theta]\psi\}\mu$. Therefore $(a)\{(\theta \circ \psi) \circ \mu\} = (a)\{\theta \circ (\psi \circ \mu)\}$ So $(\theta \circ \psi) \circ \mu = \theta \circ (\psi \circ \mu)$, giving that $(\mathcal{J}_A, \bullet)$ is associative. Satisfying the necessary conditions, $(\mathcal{J}_A, \circ)$ is a semigroup.

Definition: A one-to-one mapping of a set A onto itself, is called a permutation. The symbol $\mathfrak{G}_A$ will denote the set of permutations of the set A.

Theorem 1.3: $(\mathcal{G}_A, \circ)$ is a semigroup.

Proof: Let $\gamma, \rho \in \mathcal{G}_A$. Suppose $\gamma \circ \rho$ is not a permutation. Then $\gamma \circ \rho$ is either not one-to-one or else $\gamma \circ \rho$ is not onto.

Suppose $\gamma \circ \rho$ is not one-to-one. Then $\exists\ a, b \in A \ni (a)(\gamma \circ \rho) = (b)(\gamma \circ \rho)$ but $a \neq b$. $(a\gamma)\rho = a(\gamma \circ \rho) = b(\gamma \circ \rho) = (b\gamma)\rho$. But ρ is one-to-one and therefore $(a)\gamma = (b)\gamma$. Also γ is one-to-one and so $a = b$. Hence $\gamma \circ \rho$ is one-to-one.

Suppose $\gamma \circ \rho$ is not onto. Then $(A)(\gamma \circ \rho) \subseteq [(A)\gamma]\rho$. (Where $(A)(\gamma \circ \rho)$ denotes the image set of $\gamma \circ \rho$ and similarly for $(A)\gamma$).

Suppose $b \in [(A)\gamma]\rho$. Then $\exists\ c \in (A)\gamma \ni (c)\rho = b$ because ρ is onto. Hence $\exists\ a \in A \ni (a)\gamma = c$ because γ is onto.

Therefore $[(a)\gamma]\rho = b$ and so $(A)(\gamma \circ \rho) = [(A)\gamma]\rho$. Therefore $\gamma \circ \rho$ is onto. Hence $\gamma \circ \rho$ is a permutation. Associativity holds for general transformations and in particular for permutations.

Definition: The image of a mapping θ of a semigroup S into a semigroup S^1 is called a homomorphic representation of S iff θ is a homomorphism. If θ is an isomorphism then θ is called an isomorphic representation of S.

Isomorphic representations are also referred to as true or faithful representations.

Theorem 1.4: Any semigroup with identity, S^1, is isomorphic to a semigroup of transformations. Namely $(\mathcal{G}_{S^1}, \circ)$.

Proof: Suppose $(S^1, *)$ is a semigroup and $\mathcal{G}_{S^1}$ is the set of permutations of S^1. By theorem 1.3, $(\mathcal{G}_{S^1}, \circ)$ is a semigroup. For $a \in S^1$ let ρ_a be the permutation defined by $(s)\rho_x = s * x$, $s \in S^1$. Let θ be a mapping from S^1 into $\mathcal{G}_{S^1}$ defined by $(s)\theta = \rho_s$.

For each $x \in S^1$ $\exists$ one and only one $(x)\theta = \rho_x \in \mathcal{G}_{S^1}$ because $s * x = s * x \, \forall s \in S$.

For each $\rho_x \in \mathcal{G}_{S^1} \exists$ one and only one $x \in S^1$. To see this suppose $x \neq y$ and $(x)\theta = (y)\theta$. Then $\rho_x = \rho_y$ which implies $s * x = s * y \; \forall \; s \in S^1$ which is impossible because then $1 * x = 1 * y$ or $x = y$. Therefore θ is one-to-one.

Suppose $\exists$ a $\rho_x \in \mathcal{G}_{S^1} \ni$ there does not exist an $x \in S^1 \ni (x)\theta = \rho_x$. This is impossible by definition of ρ_x and $(x)\theta$. Let x and $y \in S^1$ and $\rho_x, \rho_y \in \mathcal{G}_{S^1}$. Then $(x * y)\theta = \rho_{x*y} = s * (x * y) \; \forall \; s \in S^1 = (s * x) * y \; \forall \; s \in S^1$. But $s * x$ is $(s)\rho_x$ and $((s)\rho_x) * y = ((s)\rho_x)\rho_y$. And by definition $((s)\rho_x)\rho_y = (s)(\rho_x \circ \rho_y) \, \forall \, s \in S^1$.

Therefore $(x * y)\theta = \rho_x \circ \rho_y = (x)\theta \circ (y)\theta$. Hence θ is an isomorphism.

Let G be a group and $G^0=G\cup\{0\}$ the group with a zero element adjoined.

Definition: Let X and Y be index sets. A mapping θ of $X\times Y$ into G^0 is called a $X\times Y$ matrix over G^0.

Definition: Let X be an index set with S and T subsets of X. If a is an element of the group $(G^0,+)$,[2] $i\to a_i$ will be used to represent the mapping of X into G^0.

$\sum_{i\in S} a_i=0$ if $a_i=0\ \forall\ i\in S$;

$\sum_{i\in T} a_i=a$ if $\exists\ j\in T\ni a_j\neq 0$ and $a_i=0\ \forall\ i\neq j$

$\sum_{i\in T} a_i$ is undefined if $\exists\ j,k\in T\ni a_j\neq 0,\ a_k\neq 0,\ j\neq k$

If $(i,j)\in X\times Y$ then for some $a_{ij}\in G^0$, a_{ij} will denote $(i,j)\theta$ and will be called the element in the ith row and the jth column of the matrix denoted (a_{ij}).

In keeping with standard notations a matrix will often be denoted with a capital letter.

Definition: Let X,Y,Z be index sets. Let $A=(a_{ij})$ be an $X\times Y$ matrix over G^0 and $B=(b_{jk})$ be a $Y\times Z$ matrix over G^0. If for every pair $(i,k)\in X\times Z\ni c_{ik}=\sum_{j\in X} a_{ij}b_{jk}$ is

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[2]If the group operation + is to be used more than once the suggestive notation Σ will be used to indicate this.

defined then the matrix product AB is the XxZ matrix $C=(c_{ik})$ over G^0.

Definition: An XxY matrix A over G^0 is called row monomial iff each row of A contains at most one non-zero element. A is called column monomial iff each column of A contains at most one non-zero element.

Definition: If I and Λ are index sets, an $I\times\Lambda$ matrix A over G^0 is called a Rees $I\times\Lambda$ matrix over G^0 iff $\exists$ one and only one $a_{ij} \in A \ni a_{ij} \neq 0$.

Definition: Let $A=(a_{i\lambda})$ and $B=(b_{j\mu})$ be Rees $I\times\Lambda$ matrices over G^0. Let $P=(p_{\lambda j})$ be a fixed $\Lambda\times I$ matrix over G^0. The Rees matrix product $A\circ B$ of A and B is $A\circ B=APB$, where the products APB are the matrix product defined above.

P is called a sandwich matrix.

Because there is only one non-zero element in a Rees matrix, the Rees matrix $(a_{i\lambda})$ will often be denoted $(a)_{i\lambda}$ to distinguish it from the sandwich matrix.

Chapter II

Fundamental Theorems

In this chapter the semigroup is partitioned into ideals by the relations $\mathcal{D}, \mathcal{L}, \mathcal{R}, \mathcal{H}$ and $\mathcal{J}$. Green's theorem will provide a mapping from $\mathcal{H}$-class to $\mathcal{H}$-class that will be important in the determination of the non-zero elements in the matrix. Theorem 2.11 provides a possible group, contained in the semigroup, from which elements for the matrix may be selected.

Definition: A non-empty subset A of a semigroup S is a right ideal of S iff AS A. A is a left ideal of S iff $SA \subseteq A$. A is a two sided ideal of S iff A is a right ideal and A is a left ideal of S.

Consider the non-empty subset B of the semigroup S. Let $\{A_1, A_2, A_3, \ldots\}$ be the set of left ideals of S $\ni$ for each A_i in the set, $B \subseteq A_i$. Suppose $s \in S$ and $a \in \cap A_i$. Then $a \in A_i$ for each $i=1,2,3,\ldots$ Therefore $sa \in A_i$ for each $i=1,2,3,\ldots$ So $sa \in \cap A_i$ for each $i=1,2,3,\ldots$ Then $S(\cap A_i) \subseteq \cap A_i$ and $\cap A_i$ is a left ideal of S.

Definition: Let B be a non-empty subset of a semigroup S and $\{A_1, A_2, A_3, \ldots\}$ be the set of left ideals of S $\ni$ $B \subseteq A_i$. Then $\cap A_i$ is called the left ideal of S generated by B. If the subset B has as its only element the element b then $\cap A_i$ is called a principal left ideal of S.

In the following discussion and definition the left ideals of S were obtained by multiplying on the left by S. If the same procedure were followed using multiplication on the right by S the same ideas would follow for right ideals. For each definition about left ideals it can be seen that commuting the multiplication provides a corresponding definition about right ideals. These definitions will be referred to as duals of one another.

The following example might help to illustrate the preceding discussion and definition.

Consider the semigroup (I_6,X) where $I_6=\{0,1,2,3,4,5\}$ and the operation is multiplication mod 6. Let B in the definition above be $\{0,2\}$. The left ideals that contain B are $A_1=I_6, A_2=\{0,2,4\}$. $A_1\cap A_2=\{0,2,4\}=A_2$. A_2 is the left ideal of S generated by $B=\{0,2\}$. If $B=\{3\}$ then the left ideals $A_1=I_6, A_2=\{0,3\}$ are the left ideals containing B. Therefore $A_1\cap A_2=\{0,3\}=A_2$. In this case A_2 is a principal ideal.

The left principal ideals of S generated by a will be denoted by S^1a. The symbol S^1 will be used to denote S if S has an identity element and $S\cup\{1\}$ if S does not have an identity element.

Definition: Let SxS be the cross product of the semigroup S and (a,b) SxS.

$\mathcal{L}= \{(a,b) | S^1 a = S^1 b\}$ will be called an $\mathcal{L}$-relation,

$\mathcal{R}= \{(a,b) | aS^1 = bS^1\}$ will be called an $\mathcal{R}$-relation and

$\mathcal{J}= \{(a,b) | S^1 aS^1 = S^1 bS^1\}$ will be called an $\mathcal{J}$-relation.

Let $(\mathcal{J}_2, \circ)$ be the semigroup of transformations of the set $\{1,2\}$ and the operation is the product of transformations.

∘	(11)	(12)	(21)	(22)
(11)	(11)	(11)	(22)	(22)
(12)	(11)	(12)	(21)	(22)
(21)	(11)	(21)	(12)	(22)
(22)	(11)	(22)	(11)	(22)

FIGURE 1

SEMIGROUP OF TRANSFORMATIONS

It is seen from figure 1 that $(11)\mathcal{J}_2^1 = (22)\mathcal{J}_2^1$ are right principal ideals of $(\mathcal{J}_2, \circ)$. Then $\mathcal{R} = \{((11),(11)),\ ((11),(22)),\ ((22),(11)),\ ((22(,(22))\}$ is an $\mathcal{R}$-relation of $(\mathcal{J}_2, \circ)$. $(22)\mathcal{J}_2^1 = \{(22)\}$. Therefore $\mathcal{L} = \{((22),(22))\}$.

If $a,b \in S$ are in the relation $\mathcal{L}$, $a\mathcal{L}b$ will often be used to denote this fact. Similarly for $\mathcal{R},\mathcal{J}$.

In what follows, it is important to consider sets of order pairs formed by taking the composition of the relations $\mathcal{R}$ and $\mathcal{L}$ and intersection of $\mathcal{R}$ and $\mathcal{L}$. The largest of these sets is obtained by taking the composition of the relations $\mathcal{R}$ and $\mathcal{L}$.

Definition: $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \{ (a,c) \mid \exists\, b \in S \ni (a,b) \in \mathcal{L}, (b,c) \in \mathcal{R} \}$.
The smallest of these will be obtained from the intersection of the relations $\mathcal{R}$ and $\mathcal{L}$.

Definition: $\mathcal{L}$ and $\mathcal{R}$ are relations. $\mathcal{H}$ is the relation $\mathcal{L} \cap \mathcal{R}$.

From the previous example: $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \{((22)(11))((22)(22))\}$ and $\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \{((22)(22))\}$.

Given a semigroup S, $\forall$ $a \in S$ $S^1 a = S^1 a$ and so $a \mathcal{L} a$. $\forall$ $a,b \in S \ni a \mathcal{L} b$ then $S^1 a = S^1 b$ or $S^1 b = S^1 a$ and so $b \mathcal{L} a$. If $a \mathcal{L} b$ and $b \mathcal{L} c$ then $S^1 a = S^1 b = S^1 c$ or $a \mathcal{L} c$. Therefore $\mathcal{L}$ is an equivalence relation. In like manner $\mathcal{R}, \mathcal{J}$ and consequently $\mathcal{D}$ and $\mathcal{H}$ can be shown to be equivalence relations.

The set of all elements of the semigroups S that are $\mathcal{L}$ related to a will be denoted by L_a and called the $\mathcal{L}$-class containing a. Similarly the elements that are in the same $\mathcal{R}$-class, $\mathcal{J}$-class, $\mathcal{H}$-class and $\mathcal{D}$-class as a will be denoted R_a, J_a, H_a, D_a.

If $\mathcal{R} = \{((11),(11)),((11),(22)),((22),(11)),((22),(22))\}$ and $\mathcal{L} = \{((22),(22))\}$ as in the previous example, then $R_{(11)} = R_{(22)} = \{(11),(22)\}$ and $L_{(22)} = \{(22)\}$.

Definition: An element a of a semigroup S is called regular iff $\exists\ x\in S \ni axa=a$. A semigroup S is called regular iff every element of S is regular.

An idea closely related to that of regularity is given in the next definition.

Definition: a and b elements of a semigroup S are inverses of each other iff aba=a and bab=b.

Because $\mathcal{R},\mathcal{L},\mathcal{D},\mathcal{H}$ are equivalence relations it is known that the corresponding equivalence classes for $\mathcal{R},\mathcal{L},\mathcal{D}$, and $\mathcal{H}$ are disjoint. Using this fact, a helpful way of representing the $\mathcal{R},\mathcal{L},\mathcal{D}$, and $\mathcal{H}$-classes in a relatively sinple way is given in Clifford and Preston and is called by them an egg-box picture.

D_1	R_a	R_b	R_c
L_a	H_{aa}	H_{ab}	H_{ac}
L_b	H_{ba}	H_{bb}	H_{bc}

D_2	R_d	R_e	R_f	R_g	R_h
L_c	H_{cd}	H_{ce}	H_{cf}	H_{cg}	H_{ch}
L_d	H_{dd}	H_{de}	H_{df}	H_{dg}	H_{dh}
L_e	H_{ed}	H_{ee}	H_{ef}	H_{eg}	H_{eh}
L_f	H_{fd}	H_{fe}	H_{ff}	H_{fg}	H_{fh}
L_g	H_{gd}	H_{ge}	H_{gf}	H_{gg}	H_{gh}

D_3	R_i	R_j	R_k	R_l	R_m
L_h	H_{hi}	H_{hj}	H_{hk}	H_{hl}	H_{hm}
L_i	H_{ii}	H_{ij}	H_{ik}	H_{il}	H_{im}
L_j	H_{ji}	H_{jj}	H_{jk}	H_{jl}	H_{jm}
L_k	H_{ki}	H_{kj}	H_{kk}	H_{kl}	H_{km}
L_l	H_{li}	H_{lj}	H_{lk}	H_{ll}	H_{lm}

FIGURE 2

POSSIBLE EGG-BOX PICTURE

Where each large square is a $\mathcal{D}$-class, each row an $\mathcal{L}$-class, each column an $\mathcal{R}$-class and each square an $\mathcal{H}$-class.

Lemma 2.1(Green): Let a and b be $\mathcal{R}$-equivalent elements of a semigroup S, and let s and s' be elements of S^1 such that $as=b$ and $bs'=a$. Then the mappings $x \to xs\,(x \in L_a)$ and $y \to ys'\,(y \in L_b)$ are mutually inverse, $\mathcal{R}$-class preserving, one-to-one mappings of L_a onto L_b, and of L_b onto L_a, respectively.

Proof: Denote the two mappings by θ and θ'. It is noted that θ is the inner right translation ρ_s. Restricted to L_a and θ' is the inner right translation $\rho_{s'}$ restricted to L_b.

Suppose $a\mathcal{R}b$. Then by definition $aS^1=bS^1$. But this implies that $\exists\ s \in S^1 \ni as=b$ and $s' \in S^1 \ni a=bs'$. Let $x \in S^1 \ni x \in L_a$. $S^1x=S^1a$ by definition and so $(S^1x)s=(S^1a)s$. But because of associativity $S^1(xs)=S^1(as)$ and hence $xs\mathcal{L}as$ or, because $as=b$, $xs\mathcal{L}b$. Therefore $xs \in L_b$. And θ maps L_a into L_b.

Again let $x \in L_a$. Then $S^1x=S^1a$ and so there exists $t \in S^1$ such that $x=1x=ta$, and $x\theta\theta'=xss'=tass'=tbs'=ta=x$.

Thus $\theta\theta'$ is the identity transformation on L_a. Similarly, $\theta\theta'$ is the identity transformation on L_b, and θ and θ' are mutually inverse, one-to-one mappings of L_a and L_b onto each other.

To see that θ is $\mathcal{R}$-class preserving, it is noted that if $x \in L_a$ and $y=x\theta=xs$, then $ys'=x$, so that $y\mathcal{R}x$. Similarly, θ' is also $\mathcal{R}$-class preserving.

As a consequence of Green's theorem the following theorem follows.

Theorem 2.2: Let a and c be $\mathcal{D}$-equivalent elements of a semigroup S. Then there exists b in S such that $a\mathcal{R}b$ and $b\mathcal{L}c$, and hence $as=b$, $bs'=a$, $tb=c$, $t'c=b$, for some $s, s', t, t' \in S$. The mapping $x \rightarrow txs \, (x \in H_a)$ and $z \rightarrow t'z's'$ $(z \in H_c)$ are mutually inverse, one-to-one mappings of H_a and H_c onto each other. Any two $\mathcal{H}$-classes contained in the same $\mathcal{D}$-class have the same cardinal number.

Proof: By the dual of Green's Lemma, the mappings $\tau: y \rightarrow ty \, (y \in R_b)$ and $\tau': z \rightarrow tz \, (z \in R_c)$ are mutually inverse, $\mathcal{L}$-class preserving, one-to-one mappings of R_b onto R_c and R_c onto R_b. Let θ and θ' be as in Green's Lemma, but restricted to H_a and H_b respectively. (Since the unrestricted θ and θ are $\mathcal{R}$-class preserving, they map H_a and H_b upon each other in a one-to-one fashion.)

Similarly, let τ and τ' be restricted to H_b and H_c, respectively. Then $\theta\tau$ and $\tau'\theta'$ are mutually inverse, one-to-one mappings of H_a and H_c upon each other, But these are the mappings defined in the theorem.

One of the $\mathcal{D}$-classes, D, for the semigroup $(\mathcal{T}_4, \circ)$ is composed, in part, as shown in figure 3.

$L_1 = \{(1222), (2111), (1211), (2122), (1121), (2212), (1112), (2221), (1122), (2211), (1212), (2121), (1221), (2112)\}$

$L_2 = \{(1333), (3111), (1311), (3133), (1131), (3313), (1113), (3331), (1133), (3311), (1313), (3131), (1331), (3113)\}$

$R_1 = \{(1222), (2111), (1333), (3111), (1444), (4111), (2333), (3222), (2444), (4222), (3444), (4333)\}$

$R_2 = \{(1211), (2122), (1311), (3133), (1411), (4144), (2322), (3233), (2422), (4244), (3433), (4344)\}$

FIGURE 3

ONE OF THE $\mathcal{D}$-CLASSES, D, FOR THE SEMIGROUP $(\mathcal{J}_4, \circ)$, IN PART

According then to Lemma 2.1, pick out two $\mathcal{R}$-equivalent elements of, say, R_1. $(1222)\mathcal{R}(3111)$. Then $(1222)\circ(3122)=(3111)$ and $(3111)\circ(2211)=(1222)$. So in the Lemma 2.1 choose $s=(3122)$ and $s'=(2211)$.

$x \longrightarrow x(3122)$	and	$y \longrightarrow y(2211)$
$(1222) \longrightarrow (3111)$		$(1333) \longrightarrow (2111)$
$(2111) \longrightarrow (1333)$		$(3111) \longrightarrow (1222)$
$(1211) \longrightarrow (3133)$		$(1311) \longrightarrow (2122)$
$(2122) \longrightarrow (1311)$		$(3133) \longrightarrow (1211)$
$(1121) \longrightarrow (3313)$		$(1131) \longrightarrow (2212)$
$(2212) \longrightarrow (1131)$		$(3313) \longrightarrow (1121)$
$(1112) \longrightarrow (3331)$		$(1113) \longrightarrow (2221)$
$(2221) \longrightarrow (1113)$		$(3331) \longrightarrow (1112)$
$(1122) \longrightarrow (3311)$		$(1133) \longrightarrow (2211)$
$(2211) \longrightarrow (1133)$		$(3311) \longrightarrow (1122)$
$(1212) \longrightarrow (3131)$		$(1313) \longrightarrow (2121)$
$(2121) \longrightarrow (1313)$		$(3131) \longrightarrow (1212)$
$(1221) \longrightarrow (3113)$		$(1331) \longrightarrow (2112)$
$(2112) \longrightarrow (1331)$		$(3113) \longrightarrow (1221)$

Notice also that:

$x \longrightarrow x(3122)$	and	$y \longrightarrow y(2211)$
$(1222) \longrightarrow (3111)$		$(1333) \longrightarrow (2111)$
$(2111) \longrightarrow (1333)$		$(3111) \longrightarrow (1222)$
$(1211) \longrightarrow (3133)$		$(1311) \longrightarrow (2122)$
$(2122) \longrightarrow (1311)$		$(3133) \longrightarrow (1211)$

FIGURE 4

MUTUALLY INVERSE ONE-TO-ONE MAPPINGS

OF L_1 ONTO L_2

It is seen that under the mappings in figure 4 $\mathcal{R}$-classes are preserved.

In the same manner, the mappings may be defined so that they are mutually inverse $\mathcal{L}$-class preserving, one-to-one mappings of R_a onto R_b.

To illustrate theorem 2.2 the computation in figure 4 is carried out for $\mathcal{L}$-classes and $\mathcal{R}$-classes. The needed $\mathcal{H}$-classes would be :

$H_1 = \{(1222), (2111)\}$; $H_2 = \{(1333), (3111)\}$;
$H_3 = \{(1211), (2122)\}$ and $H_4 = \{(1311), (3133)\}$.[1]

Theorem 2.3: The set product LR of any $\mathcal{L}$-class L and any $\mathcal{R}$-class R of a semigroup S is always contained in a single $\mathcal{D}$-class of S.

Proof: Let $a, a', b, b' \in S \ni a \mathcal{L} a'$ and $b \mathcal{R} b'$. But $a \mathcal{L} a'$ implies $ab \mathcal{L} a'b$ because $S^1 a = S^1 a'$ implies $S^1(ab) = (S^1 a)b = (S^1 a')b = S^1(a'b)$. In the same manner, $b \mathcal{R} b'$ implies $a'b \mathcal{R} a'b'$. But applying the definition of $\mathcal{L} \circ \mathcal{R}$ it is seen that $(ab, a'b') \in \mathcal{L} \circ \mathcal{R}$ or that $ab \mathcal{D} a'b'$.

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[1]For the remainder of the D structure of this semigroup see A.H.Clifford and G.B.Preston, The Algebraic Theory Of Semigroups (American Mathematical Society, Providence, Rhode Island, 1961) Volume I, p. 55

Lemma 2.4: An element a of a semigroup S is regular iff R_a contains an idempotent.

Proof: Suppose $a \in S$ and a is regular. Then $axa=a$ for some $x \in S$. But then $aS^1=axS^1$ and so $ax \in R_a$ and $(xa)(xa)=(x)(axa)=xa$. Therefore ax is idempotent.

Suppose that R_a contains an idempotent element e. Then $aS^1=eS^1$. Hence $\exists\, x \in S^1 \ni a=a1=ex$. But $ea=e(ex)=e^2x=ex=a$. Also $\exists\, y \in S^1 \ni e=ay$. Then $a=ea=aya$. Therefore a is regular. The dual of this Lemma is also true.

Theorem 2.5: (i) If a $\mathcal{D}$-class D of a semigroup S contains a regular element, then every element of D is regular,

(ii) If D is regular, then every $\mathcal{L}$-class and every $\mathcal{R}$-class contained in D contains an idempotent.

Proof: (i) Let a be a regular element of a $\mathcal{D}$-class $D \ni axa=a$. Then the $\mathcal{R}$-class R containing $e=ax$ contains an idempotent element, namely e. Then every element of R is regular by Lemma 2.4. But, every $\mathcal{L}$-class of D contains an element of R and every $\mathcal{R}$-class of D contains an element of each $\mathcal{L}$-class. Therefore every $\mathcal{R}$-class and $\mathcal{L}$-class of D contains regular elements and hence idempotent elements. Therefore every element of D is regular.

(ii) If D is regular then $axa=a$ for some $x \in S$. But ax is an element of some $\mathcal{R}$-class and xa is an element of some $\mathcal{L}$-class. $(ax)(ax)=(axa)x=ax$. Also $(xa)(xa)=(x)(axa)=xa$. Therefore, ax is an idempotent in a $\mathcal{R}$-class and xa is an idempotent in an $\mathcal{L}$-class.

Lemma 2.6: If a and a' are inverse elements of a semigroup S then $e=aa'$ and $f=a'a$ are idempotents $\ni$ $ea=af=a$ and $a'e=fa'=a'$. Hence $e\in R_a \cap L_{a'}$ and $f\in R_{a'} \cap L_a$. The elements a,a',e,f all belong to the same $\mathcal{D}$-class of S.

Proof: Suppose $e=aa'$ and $f=a'a$ where a and a' are inverse elements of S. $e^2=(aa')(aa')=a(a'aa')=aa'=e$

$$f^2=(a'a)(a'a)=(a'aa')a=a'a=f$$

Therefore e and f are idempotent elements of S. If $e=aa'$ then $e\in R_a$ and $e\in L_{a'}$. Therefore $e\in R_a\cap L_{a'}$. Likewise, if $f=a'a$ then $f\in R_{a'}$ and $f\in L_{a'}$. Therefore $f\in R_{a'}\cap L_a$. $e\mathcal{R}a$ and $a\mathcal{L}f$ implies $e\mathcal{D}f$. $a\mathcal{R}e$ and $e\mathcal{L}a$ implies $a\mathcal{D}a$. Also e and a in the same $\mathcal{R}$-class implies $e\mathcal{D}a$. Therefore e,f,a,a' are all elements of the same $\mathcal{D}$-class.

Lemma 2.7: If a is a regular element of a semigroup S, then $aS^1=aS$ and $S^1a=Sa$.

Proof: Obviously $a\in aS^1$ because $a=a\cdot 1$. It is necessary then to show that $a\in aS$. But a is regular and so $axa=a$. Let $f=xa$ which has been shown previously to be an idempotent.

Then af=a implies that $a \in aS$. In like manner $S^1a=Sa$.

Lemma 2.8: If a and b are regular elements of S, then $a \mathcal{L} b$ iff $Sa=Sb$.

Proof: Suppose a and b are regular elements of S and $a \mathcal{L} b$. Then $S^1a=S^1a$. But by Lemma 2.7, $S^1a=Sa$ and $S^1b=Sb$. Therefore $Sa=Sb$.

Suppose a and b are regular and $Sa=Sb$. Then by Lemma 2.7, $Sa=S^1a$ and $Sb=S^1b$. Therefore $S^1a=S^1b$ and hence $a \mathcal{L} b$. Similarly for $a \mathcal{R} b$.

Lemma 2.9: Any idempotent element e of a semigroup S is a right identity element of L_e, a left identity element of R_e, and a two sided identity element of H_e.

Proof: If $a \in L_e$ then $a \in S^1e$ and hence $\exists\ x \in S^1 \ni a=xe$. Therefore $ae=xee=xe=a$. So e is a right identity of L_e. In the same manner e is a left identity of R_e. If $a \in H_e=R_e \cap L_e$, then $ea=ae=a$.

Definition: A subset T of a semigroup S is a subgroup of S iff $aT=T\ a=T\ \forall a \in T$.

Lemma 2.10: If a and $ab \in H$ then $Hb=H$.

Proof: Suppose a and $ab \in H$. Then $a \mathcal{R} ab$ and $\exists$ $s \in S^1 \ni as = ab$. Let $s=b$. Then by Lemma 2.1, $x \to xb$ is a one-to-one mapping of H onto itself. Therefore $Hb=H$.

Dually if b and $ab \in H$ then $aH=H$.

Theorem 2.11 (Green): If a, b and ab all belong to the same $\mathcal{H}$-class H of a semigroup S, then H is a subgroup of S. In particular, any $\mathcal{H}$-class containing an idempotent is a subgroup of S.

Proof: Suppose $a, b, ab \in H$. Then by Lemma 2.10 $aH=Hb=H$.

Let x be an arbitrary element of H. Then $x, ax \in H$ and $x, xb \in H$. From Lemma 2.10 it follows that $Hx=H$ and $xH=H$. Therefore $Hx=xH=H$ $\forall$ $x \in H$. Hence H is a subgroup.

Theorem 2.12: If a and b are elements of a semigroup S, then $ab \in R_a \cap L_b$ iff $R_b \cap L_a$ contains an idempotent. If this is the case, then $aH_b = H_a b = H_a H_b = R_a \cap L_b$.

Proof: Assume first that $ab \in R_a \cap L_b$. From $ab \in R_a$ there exists $b' \in S$ such that $(ab)b' = a$. By Green's Lemma, the mappings $\theta: x \longrightarrow xb$ $(x \in L_a)$ and $\theta': y \longrightarrow yb'$ $(y \in L_{ab})$ are mutually inverse, $\mathcal{R}$-class preserving, one-to-one mappings of L_a onto L_{ab} and of L_{ab} onto L_a, respectively. But $ab \in L_b$, and so $L_{ab} = L_b$. Thus θ' maps the element b of

L_b upon the element bb' of L_a, and moreover $bb' \in R_b$ since θ' is $\mathcal{R}$-class preserving. Hence $bb' \in R_b \cap L_a$. If $x \in L_a$, when $xbb' = x\theta\theta' = x$; putting $x = bb'$, therefore bb' is idempotent.

Conversely, assume that $R_b \cap L_a$ contains an idempotent e. Then $eb = b$ by Lemma 2.9. Since $e\mathcal{R}b$, it follows from Green's Lemma that $\theta : x \longrightarrow xb$ $(x \in L_e)$ is an $\mathcal{R}$-class preserving, one-to-one mapping of L_e onto L_b. Since $a \in L_e$, $ab \in L_b$; moreover, $ab \in R_a$ since θ is $\mathcal{R}$-class preserving. Hence $ab \in R_a \cap L_b$.

Continuing with the hypothesis that $R_b \cap L_a$ contains an idempotent e, let $x \in H_a$ and $y \in H_b$. Then $e = R_y \cap L_x$, and hence $xy \in R_x \cap L_y = R_a \cap L_b$. Hence $H_a H_b \subseteq R_a \cap L_b$. Since $L_e = L_a$ and $L_b = L_{ab}$, $\theta : x \longrightarrow xb$ maps L_a upon L_{ab}. Since θ is $\mathcal{R}$-class preserving, it maps H_a upon H_{ab}, and so $H_a = H_{ab}$. Hence, $H_{ab} \subseteq H_a H_b \subseteq R_a \cap L_b = H_{ab} = H_a$, and equalities hold all down the line. Dually $aH_b = H_{ab}$.

Theorem 2.13: Let a be a regular element of a semigroup S.

(i) Every inverse of a lies in Da.

(ii) An $\mathcal{H}$-class H contains an inverse of a if and only if both of the $\mathcal{H}$-classes $R_a \cap L_b$ and $R_b \cap L_a$, contain idempotents.

(iii) No $\mathcal{H}$-class contains more than one inverse of a.

Proof: (i) If a and a' are inverse elements of each other they all belong to the same $\mathcal{D}$-class by Lemma 2.6.

(ii) Suppose H_b contains an inverse a' of a. By Lemma 2.6 the $\mathcal{H}$-classes $R_a \cap L_b = R_a \cap L_{a'}$ and $R_b \cap L_a = R_{a'} \cap L_a$ contain the idempotents aa' and $a'a$ respectively.

Conversely, suppose e is an idempotent in $R_a \cap L_b$, and that f is an idempotent in $R_b \cap L_a$. From $a\mathcal{R}e$ and $a\mathcal{L}f$, $ea=a=af$, by Lemma 2.9 and $e=ax$, $f=ya$, for some $x,y \in S$, by Lemma 2.8. Let $a'=fxe$.

Then $fa'=a'e=a'$,

$$aa'=afxe=axe=e^2=e$$

$$a'a=fa'a=yaa'a=yea=ya=f$$

Since $aa'a=ea=a$ and $a'aa'=a'e=a'$, a and a' are mutually inverse. From $fa'=a'$ and $a'a=f$, $a'\mathcal{R}f$. From $a'e=a'$ and $aa'=e$. $a'\mathcal{L}e$. Hence $a' \in R_f \cap L_e = R_b \cap L_b = H_b$.

(iii) Let b and c be $\mathcal{H}$-equivalent inverse elements of a. By Lemma 2.6, ab is an idempotent element in $R_a \cap L_b$, and ac is an idempotent in $R_a \cap L_c$. If ab and ac are idempotents such that $H_{ab}=H_{ac}$ then by Lemma 2.9, each is a two sided identity of the other and hence $L_b=L_c$ and $ab=ac$.

Similarly from $R_b=R_c$, $ba=ca$.

Therefore $b=bab=cab=cac=c$.

Chapter III

Representations of Completely O-simple Semigroups

A special class of semigroups known as completely O-simple semigroups can be represented isomorphically by a semigroup of matrices. It will be found that theorem 2.11 provides the group onto which the matrices will be defined. Theorem 3.14 will then provide the necessary isomorphism.

Definition: A semigroup S is said to be simple iff S does not contain any proper two sided ideals.

Definition: A semigroup S with zero O is called O-simple iff (i) $S^2 = SS \neq O$,

(ii) O is the only proper two sided ideal of S.

Lemma 3.1: Let S be a semigroup with zero O, and such that $S \neq O$. Then S is O-simple iff $SaS=S$ for every $a \neq O$ of S.

Proof: Suppose S is O-simple. Let $B=\{b \mid SbS=O\}$. From the expression $SBS=O \subseteq B$ it is seen that B is an ideal of S. But because S is O-simple $B=O$ or $B=S$. If $B=S$ then $S^3=O$. But since $S^2=S$, $O=S^3=S^2=S$ which is impossible because $S^2 \neq O$ if S is O-simple. Therefore $B=O$ and $SaS \neq O$ for every

$a \neq 0$. But SaS is an ideal $\forall\, a \neq 0$ and $SaS \neq 0$. Therefore $SaS = S$.

Suppose $SaS = S\ \forall\, a \neq 0$. Let $A \neq 0$ be an ideal of S and $a \neq 0 \ni a \in A$. Then $S = SaS \subseteq SAS \subseteq A$ so that $S = A$. Hence S contains no proper ideal $\neq 0$. But $S \neq 0$ therefore S contains an element $a \neq 0$. From $S = SaS \subseteq S^2$, it is seen that $S^2 \neq 0$. Therefore S is 0-simple.

It should be noted from this Lemma that neither $Sa = 0$ or $aS = 0$ if S is to be 0-simple.

Definition: An ideal M of a semigroup S is called minimal iff there does not exist an ideal $N \ni N \subset M$.

Definition: An ideal M of a semigroup S is called 0-minimal iff (i) $M \neq 0$

(ii) 0 is the only proper ideal of S contained in M.

Comparing this definition with the definition for 0-simple it is seen that any 0-simple semigroup is a 0-minimal ideal of itself.

It will be necessary to indicate whether the zero element is to be omitted when discussing semigroups with zero elements. If L is the subset under consideration and S^0 is the semigroup with zero element, then $L \backslash 0$ will indicate the set L without the zero element.

The word ideal in the preceeding two definitions is intended to mean two sided ideals. It should be noticed that the definitions apply equally well for left or right ideals.

Lemma 3.2: Let L be a O-minimal left ideal of a semigroup S with O, and $c \in S$. Then Lc is either O or a O-minimal left ideal of S.

Proof: Suppose $Lc \neq O$. Lc is a left ideal of S generated by c. Let A be a left ideal of S contained in Lc. Let $B = \{b \mid b \in L \text{ and } bc \in A\}$, then $Bc \subseteq A$. If $x \in A \ni x = yc$ then $yc \in A$ and by definition of B, $yc \in Bc$. Therefore $A \subseteq Bc$ and $A = Bc$. If $b \in B$ and $s \in S$, then $sbc \in sA \subseteq A$, and $b \in sL \subseteq L$. Hence $sb \in B$, and so B is a left ideal of S. From the O-minimality of L, either B=O or B=L, and therefore A=O or A=Lc.

Theorem 3.3: Let S be a semigroup with O. Let M be a O-minimal ideal of S containing at least one O-minimal left ideal of S. Then M is the union of all the O-minimal left ideals of S contained in M.

Proof: Let $A = \cup B_i$ $(i = 1, 2, 3, \ldots)$ where B is a O-minimal left ideal of S contained in M. A is a left ideal of S. Let $a \in A$ and $c \in S$. Then $\exists\ B_j \subseteq \cup B_i \ni a \in B_j$. By Lemma 3.2, $B_j c = O$ or $B_j c$ is a minimal left ideal of S. Also $B_j c \subseteq Mc \subset M$, and $B_j c \subseteq A$. Hence $ac \in A$ and thus A is a right ideal of S. $A \neq 0$ because it contains at least one 0-minimal left ideal of S. Therefore A is a non-zero ideal of S contained in M and hence A=M because M is 0-minimal.

Lemma 3.4: If L is a 0-minimal left ideal of a semigroup S with zero 0, then $L \setminus 0$ is an $\mathcal{L}$-class of S.

Proof: Let $a \in L \setminus 0$. Then either $Sa=0$ or $Sa=L$.

Suppose $Sa=L \, \forall a \in L \setminus 0$. Then $S^1 a=S^1 b \; \forall a,\, b \in L \setminus 0$. Hence $L \setminus 0 \subseteq L_a$. If $c \in L_a$ then $c \in S^1 a=L$ and hence $L_a \subseteq L \setminus 0$. Therefore $L \setminus 0=L_a$.

Suppose $Sa=0$ for some $a \in L \setminus 0$. Then $L=\{0,a\}$ is a non-zero left ideal of S contained in L. Then if $S^2 a=L$ and $\forall$ $x \in L \setminus 0$, $S^1 x=S^1 a$ it can be concluded $x=a$. Therefore $L \setminus 0= \{a\} =L_a$.

Lemma 3.5: Let L be a 0-minimal left ideal of a 0-simple semigroup S, and let $a \in L \setminus 0$. Then $Sa=L$.

Proof: Since S_a is a left ideal of S contained in L, it follows that $Sa=0$ or $Sa=L$. The case $Sa=0$ is ruled out by Lemma 3.1.

Definition: Let e,f be idempotents of a semigroup S. $e \leq f$ iff $ef=fe=e$.

Definition: An idempotent element f of a semigroup S is called primitive iff $f \neq 0$ and if $e \leq f$ then $e=0$ or $e=f$ for all idempotents $e \in S$.

Definition: A semigroup S is said to be completely 0-simple iff S is a 0-simple semigroup containing a primitive idempotent.

Example: Any group G is a simple semigroup because for any $A \subseteq G$, $GAG=G$ and G is not a proper ideal of G. $G \cup \{0\}$ is completely 0-simple and the primitive idempotent is the identity of the group.

Let I be a set. Let $S = (I \times I) \cup \{0\}$. For $i,j,k,l \in I$ define

$$(i,j) \circ (k,l) = \begin{cases} (i,l) & \text{if } j=k, \\ 0 & \text{if } j \neq k; \end{cases}$$

$$0 \circ (i,j) = (i,j) \circ 0 = 0 \circ 0 = 0$$

Then $(S,\circ)$ is a completely 0-simple semigroup.

Theorem 3.6: Let S be a 0-simple semigroup. Then S is completely 0-simple iff it contains at least one 0-minimal left ideal and at least one 0-minimal right ideal.[1]

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[1] There are 11 Lemmas needed to prove this theorem. They are not used elsewhere in the paper. See A.I.Clifford and G.B.Preston, The Algebraic Theory Of Semigroups (American Mathematical Society, Providence, Rhode Island, 1961) Volume I, p.78

Theorem 3.7: A completely 0-simple semigroup is the union of its 0-minimal left (right) ideals.

Proof: Let S be a completely 0-simple semigroup. By definition S is 0-simple. Therefore $S^2 \neq 0$ and 0 is the only proper ideal of S. So S is 0-minimal. By theorem 3.6, S contains at least one 0-minimal left (right) ideal. By theorem 3.3, S is the union of all the 0-minimal left (right) ideals of S.

Theorem 3.8: If S is a completely 0-simple semigroup then $S \setminus 0$ is a $\mathcal{D}$-class of S.

Proof: Let S be a completely 0-simple semigroup. Let a and b be non-zero elements of S. By Corollary 3.1, a belongs to some 0-minimal right ideal L of S, and b belongs to some 0-minimal right ideal R of S. By Lemma 3.5, $L=Sa$ and $R=bS$. By Lemma 3.4 and its dual, $L_a = L \setminus 0$ and $R_b = R \setminus 0$. Since $a \in L$ and $b \in R$, $bSa \subseteq R \cap L$. Since S is 0-simple, and $a \neq 0$, $b \neq 0$, $SaS=S$ and $SbS=S$. Hence $S=S^2=SbSSaS \subseteq (bSa)S \neq 0$, so that $bSa \neq 0$. Since $R_b \cap L_a$ contains the non-empty set $bSa \setminus 0$, it follows that $a \mathcal{D} b$.

Corollary 3.1: If S is a completely 0-simple semigroup then S is regular.

Proof: By definition of complete 0-simplicity, the $\mathcal{D}$-class $S\backslash 0$ contains an idempotent. By theorem 2.5 (i), every element of $S\backslash 0$ is regular. Since 0 is regular, S is regular.

Terminology: A completely 0-simple semigroup S is said to be 0-bisimple iff $S\backslash 0$ is a $\mathcal{D}$-class of S.

Theorem 3.9: Let S be a completely 0-simple semigroup.

(i) If $a\in S$ and $a^2\neq 0$, then $a^2\in H_a$, and H_a is a group.

(ii) If $a,b\in S$ and $ab\neq 0$, then $ab\in R_a\cap L_b$.

Proof: (i) By Corollary 3.1, a belongs to some 0-minimal left ideal L of S. Then $a^2\in L$. By Lemma 3.4, $L\backslash 0$ is an $\mathcal{L}$-class of S. Since $a^2\neq 0$ by hypothesis, and $a\neq 0$, both a and a^2 belong to $L\backslash 0$, so that $a\mathcal{L}a^2$. Dually $a\mathcal{R}a^2$. Therefore $a\mathcal{H}a^2$ and hence by theorem 2.11, H_a is a group.

(ii) If $ab\neq 0$ then $a\neq 0$ and $b\neq 0$. By theorem 3.8, $a\mathcal{D}b$, and hence $R_b\cap L_a\neq 0$. Suppose $c\in R_b\cap L_a$. Then $c^2\in L_aR_b$. By theorem 2.3, either $L_aR_b=0$ or $L_aR_b\subseteq S\backslash 0$. The former is excluded by $ab\neq 0$. So $c^2\neq 0$. By (i) $H_c=R_b\cap L_a$ is a group. By theorem 2.12, $ab\in R_a\cap L_b$.

To illustrate theorem 3.9 consider figure 5.

t_3	e	1	2	0
e	e	1	2	0
1	1	2	e	0
2	2	e	1	0
0	0	0	0	0

$\mathcal{L} = \{(ee),(e1),(e2),(1e),$
$(11),(12),(2e),(21),(22)\} = \mathcal{R}.$

$\mathcal{H} = \mathcal{L} \cap \mathcal{R} = \mathcal{L}$

FIGURE 5

ILLUSTRATION OF THEOREM 3.9

Therefore $H_e = \{e,1,2\}$.

Obviously H_e is the group of Integers mod. 3.

The Rees $I \times \Lambda$ matrix semigroup over the group G^0 with sandwich matrix P will be denoted by $\mathcal{M}^0(G;I;\Lambda;P)$ or $\mathcal{M}^0$. G will be referred to as the structure group of $\mathcal{M}^0$. Denote the elements of $\mathcal{M}^0$ by $(a)_{i\lambda}$ with $a \in G^0$, $i \in I$, and $\lambda \in \Lambda$.

Let $R_i = \{(a)_{i\lambda} \mid a \in G, \lambda \in \Lambda\}$ and $R_i^0 = R_i \cup 0$;

$L_\lambda = \{(a)_{i\lambda} \mid a \in G,\ i \in I\}$ and $L_\lambda^0 = L_\lambda \cup 0$;

$H_{i\lambda} = R_i \cap L_\lambda = \{(a)_{i\lambda} \mid a \in G\}$.

Lemma 3.10: The Rees $I \times \Lambda$ matrix semigroup $\mathcal{M}^0(G;I;\Lambda;P)$ over a group with zero G^0, and with sandwich matrix P is regular iff each row and each column of P contains a non-zero entry.

Proof: Let $P=(p_{\lambda i})$. Let $a,b\in G; i,j\in I; \lambda,\mu\in\Lambda$. Then $(a)_{i\lambda}\circ(b)_{j\mu}\circ(a)_{i\lambda}=(ap_{\lambda j}\,bp_{\mu i}\,a)_{i\lambda}$.
This is equal to $(a)_{i\lambda}$ iff $p_{\lambda j}\,bp_{\mu i}=a^{-1}$ (a^{-1} is the inverse of a). With $(a)_{i\lambda}$ given, $\exists$ such an element $(b)_{j\mu}$ in $\mathcal{M}^0$ iff $p_{\lambda j}\neq 0$ and $p_{\mu i}\neq 0$ for some $j\in I$ and $\mu\in\Lambda$, that is, iff the λth row and the ith column of P each contains a non-zero element of G .

Theorem 3.11: The set of all Rees $I\times\Lambda$ matrices over the group G^0 with sandwich matrix P is a semigroup with respect to matrix multiplication.

Proof: Let $(a)_{i\lambda}$, $(b)_{j\mu}\in\mathcal{M}^0$ with $(p_{\lambda j})\in P$. The $(a)_{i\lambda}\circ(b)_{j\mu}=(ap_{\lambda j}b)_{i\mu}\in\mathcal{M}^0$. Also for $(a)_{i\lambda}$, $(b)_{j\mu}$, $(c)_{k\gamma}\in\mathcal{M}^0$; $[(a)_{i\lambda}\circ(b)_{j\mu}]\circ(c)_{k\gamma}=[(ap_{\lambda j}\,b)_{i\mu}]\circ(c)_{k\gamma}=([ap_{\lambda j}\,b]\,p_{\mu k}\,c)_{i\gamma}=(ap_{\lambda j}\,[bp_{\mu k}\,c]_{i\gamma})=(a)_{i\lambda}\circ(bp_{\mu k}\,c)_{j\gamma}=(a)_{i\lambda}\circ[(b)_{j\mu}\circ(c)_{k\gamma}]$. Hence $\mathcal{M}^0$ is a semigroup.

Lemma 3.12: (i) For each i in I, R_i^0 is a right ideal of $\mathcal{M}^0$; any two R-equivalent elements of $\mathcal{M}\setminus 0$ must belong to the same R_i, for some i in I.
(ii) If P is regular, then, for each $i\in I$, R_i^0 is a 0-minimal right ideal of $\mathcal{M}^0$, and R_i is an $\mathcal{R}$-class.
(iii) If, for some i in I, $p_{\lambda i}=0$ for every λ in Λ, then R_i^0 is a two-sided ideal of $\mathcal{M}^0$ such that $\mathcal{M}^0R_i^0=0$; in particular, $(R_i^0)^2=0$.

(iv) The set $H_{i\lambda}$ $(i\in I, \lambda\in\Lambda)$ contains an idempotent element iff $p_{\lambda i}\neq 0$. If $p_{\lambda i}\neq 0$, then $H_{i\lambda}$ is an $\mathcal{H}$-class of $\mathcal{M}^0$, and is a subgroup of $\mathcal{M}^0$ with identity element $e_{i\lambda}=(p_{\lambda i}^{-1})_{i\lambda}$. The mapping $a\to(ap_{\lambda i}^{-1})_{i\lambda}$ is an isomorphism of the group G onto $H_{i\lambda}$.

(v) For every $i,j\in I$ and λ,μ in Λ,

$$H_{i\lambda}\circ H_{j\mu} = \begin{cases} H_{i\mu} & \text{if } P_{\lambda j} \neq 0, \\ 0 & \text{if } P_{\lambda j} = 0 \end{cases}$$

Proof: Lemma 3.12 (i) Let $(a)_{i\lambda}\in R_i^0$ and $(b)_{j\mu}\in\mathcal{M}^0$. $(a)_{i\lambda}\circ(b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}\in R_i^0$. Therefore R_i^0 is a right ideal of $\mathcal{M}^0$. Let $(a)_{i\lambda}$ and $(b)_{j\mu}$ be non-zero, $\mathcal{R}$-equivalent elements of $\mathcal{M}^0$. Then there exists $(c)_{k\nu}$ in $\mathcal{M}^0$ such that $(a)_{i\lambda}\circ(c)_{k\nu}=(b)_{j\mu}$, hence $(ap_{\lambda k}c)_{i\nu}=(b)_{j\mu}$. Since $b\neq 0$, this requires $j=i$.

(ii) Assume that P is regular, and let $(a)_{i\lambda}$ and $(b)_{i\mu}$ be (non-zero) elements of R_i. Since P is regular, there exists k in I such that $p_{\lambda k}\neq 0$. Then $(a)_{i\lambda}\circ(c)_{k\mu} = (b)_{i\mu}$ if $c=p_{\lambda k}^{-1}a^{-1}b\neq 0$. $\therefore R_i^0\neq 0$. This shows that R_i^0 is a 0-minimal right ideal of $\mathcal{M}^0$, and that any two elements of R_i are R-equivalent. That R_i is an $\mathcal{R}$-class then follows from (i).

(iii) Suppose $p_{\lambda i}=0$ for every $\lambda\in\Lambda$. If $(a)_{i\lambda}\in R_i^0$ and $(b)_{j\mu}\in\mathcal{M}^0$, then $(b)_{j\mu}\circ(a)_{i\lambda}=(bp_{\mu i}a)_{j\lambda}=0$. Hence $\mathcal{M}^0\circ R_i^0=0\subseteq R_i^0$. From this and (i) it follows that R_i^0 is a two sided ideal.

(iv) Let $(a)_{i\lambda} \in H_{i\lambda}$. From $(a)_{i\lambda} \circ (a)_{i\lambda} = (ap_{\lambda i}a)_{i\lambda}$ it is seen that $(a)_{i\lambda}$ is idempotent iff $p_{\lambda i} \neq 0$ and $a = p_{\lambda i}$. Assume $p_{\lambda i} \neq 0$. Then $H_{i\lambda}$ contains the idempotent $e_{i\lambda} = (p_{\lambda i}^{-1})_{i\lambda}$. For a in G, let $a\theta = (ap_{\lambda i}^{-1})_{i\lambda}$. Then for a and $b \in G$, $(a\theta)\circ(b\theta) = (ap_{\lambda i}^{-1}p_{\lambda i}b_{\lambda i}^{-1})_{i\lambda} = (ab)\theta$, and, since for each a $\exists$ one and only one $(a)\theta$ and for each $(a)\theta$ $\exists$ one and only one a, θ is a one-to-one mapping of G onto $H_{i\lambda}$, it follows that θ is an isomorphism of G upon $H_{i\lambda}$. Hence $H_{i\lambda}$ is a subgroup of M^0 with identity $e_{i\lambda}$.

Let H be the $\mathcal{H}$-class of M^0 containing $e_{i\lambda}$. Evidently $H_{i\lambda} \subseteq H$ since $H_{i\lambda}$ is a group. But by (i) and its dual, $H \subseteq R_i \cap L_\lambda = H_{i\lambda}$, and hence $H = H_{i\lambda}$.

(v) Let $(a)_{i\lambda} \in H_{i\lambda}$ and $(b)_{j\mu} \in H_{j\mu}$. We have $(a)_{i\lambda} \circ (b)_{j\mu} = (ap_{\lambda j}b)_{i\mu}$. If $p_{\lambda j} = 0$, this is 0. If $p_{\lambda j} \neq 0$, it belongs to $H_{i\mu}$. In the latter event, any element $(c)_{i\mu}$ of $H_{j\mu}$ may be obtained as such a product by taking $a = p_{\lambda j}^{-1}$ and $b = c$.

Theorem 3.13: A Rees matrix semigroup is 0-simple iff it is regular, and if so it is completely 0-simple.

Proof: Let $M^0(G,I,\Lambda,P)$ be a Rees matrix semigroup. Suppose first that M^0 is not regular. By Lemma 3.10, there is a row or column of P which consists of zeros, say the <u>ith</u> column: $p_{\lambda i} = 0$ for all λ in Λ. By Lemma 3.12 (iii), R_i^0 is a non-zero nilpotent ideal of M^0, and so can not be 0-simple.

Assume conversely that $\mathcal{M}^0$ is regular. Let $(a)_{i\lambda}$ and $(b)_{j\mu}$ be any two elements of $\mathcal{M}^0$ with $a\neq 0$. By Lemma 3.10, there exists ν in Λ and k in I such that $p_{\nu i}\neq 0$ and $p_{\lambda k}\neq 0$. Let $c=b(p_{\nu i}ap_{\lambda k})^{-1}$, and let e be the identity element of G. Then $(c)_{j\nu}\circ(a)_{i\lambda}\circ(e)_{k\mu}=(b)_{j\mu}$, and it follows from Lemma 3.1 that $\mathcal{M}^0$ is 0-simple.

By Lemma 3.12 (iv), the non-zero idempotents of $\mathcal{M}^0$ are the elements $e_{i\lambda}=(p_{\lambda i}^{-1})_{i\lambda}$. There is one such for each pair, $i\lambda$ $(i\in I, \lambda\in\Lambda)$ such that $p_{\lambda i}\neq 0$. If $e_{i\lambda}\circ e_{j\mu}=e_{j\mu}\circ e_{i\lambda}=e_{j\mu}$, then $i=j$ and $\lambda=\mu$, so that $e_{i\lambda}=e_{j\mu}$. Thus every non-zero idempotent of $\mathcal{M}^0$ is primitive, and so $\mathcal{M}^0$ is completely 0-simple.

Let D be a $\mathcal{D}$-class of a semigroup S. Let $\{R_i \mid i\in I\}$ and $\{L_\lambda \mid \lambda\in\Lambda\}$ be the sets of $\mathcal{R}$ and $\mathcal{L}$-classes of S contained in D. Then the set of $\mathcal{H}$-classes contained in D is $\{H_{i\lambda} \mid i\in I, \lambda\in\Lambda\}$ and $H_{i\lambda}=R_i\cap L_\lambda$. Choose an $\mathcal{H}$-class of D that contains an idempotent element and call it H_{11}. For each $i\in I$ select and fix an element r_i of H_{i1}. For each $\lambda\in\Lambda$ select and fix an element q_λ of $H_{1\lambda}$.

The $\Lambda\times I$ matrix $P=(p_{\lambda i})$ over H_{11} is then

$$p_{\lambda i} = \begin{cases} q_\lambda r_i & \text{if } q_\lambda r_i\in H_{11} \\ 0 & \text{otherwise} \end{cases}$$

Definition: Let D be a $\mathcal{D}$-class of a semigroup $(S,\circ)$. Let $T=D\cup 0$. T will be called the trace of D.

If a and b are any elements of D then a product $(*)$ may be defined in D.

$$a * b = \begin{cases} ab \text{ if } ab \in R_a \cap L_b, \\ 0 \text{ otherwise} \end{cases}$$

$$a * 0 = 0 * a = 0$$

Theorem 3.14: Every element of D is uniquely representable in the form $r_i a q_\lambda$ with a in H_{11}, in I, and λ in Λ. The one-to-one mapping $\emptyset$ of $\mathcal{M}^0$ upon T defined by

$$(a)_{i\lambda}\emptyset = \begin{cases} r_i a q_\lambda \text{ if } a \neq 0 \\ 0 \text{ if } a=0, \end{cases}$$

is an isomorphism.

Proof: For each $\lambda \in \Lambda$, let e be an idempotent in L_λ. L_λ exists by theorem 2.5. By theorem 2.13, q_λ has a unique inverse q_λ' in $R_{e_\lambda} \cap L$. Then $eq_\lambda = q_\lambda$ and $q_\lambda q_\lambda' = e$, where e is the idempotent in H_{11}. By Green's Lemma 2.1, the mappings $x \to xq_\lambda (x\ L)$ and $y \to yq_\lambda'$ $(y \in L_\lambda)$ are mutually inverse, $\mathcal{R}$-class preserving, one-to-one mappings of L_1 onto L_λ.

Dually, for each $i \in I$, $\exists$ an inverse r_i' of $r_i \in R_1$, and the mappings $x \to r_i x (x \in R_1)$ and $y \to r_i' y (y \in R_i)$ are mutually inverse, $\mathcal{L}$-class preserving, one-to-one mappings of R_1 onto R_i'. Combining the two mappings as in theorem 2.2,

$x \to r_i x q_\lambda$ $(x \in H_{11})$ and $y \to r_i' y q$ $(y \in H_{i\lambda})$ are mutually inverse one-to-one mappings of H_{11} onto $H_{i\lambda}$.
Since every element of D belongs to exactly one $H_{i\lambda}$, it follows that the mapping $\emptyset$ defined in the theorem is a one-to-one mapping of $\mathcal{M}^0$ onto $D \cup 0 = T$.

To show $\emptyset$ is an isomorphism suppose $q_\lambda r_i \in H_{11}$. Then by theorem 2.12, $(r_i a q_\lambda)(r_j b q_\mu) \in H_{i\mu} = R_i \cap L_\mu$.
Therefore $r_i a q_\lambda \in R_i$, $r_j b q_\mu \in L_\mu$.

From the definition of the trace product $(*)$ $(r_i a q_\lambda) * (r_j b q_\mu)$ is the product in D. From the definition of $p_{\lambda i}$ $[(a)_{i\lambda} \emptyset] * [(b)_{j\mu} \emptyset] = (r_i a q_\lambda) * (r_j b q_\mu) = r_i a q_\lambda r_j b q_\mu = r_i a p_{\lambda j} b q_\mu = (a p_{\lambda j} b)_{j\mu} \emptyset = [(a)_{i\lambda} \circ (b)_{j\mu}] \emptyset$. Now suppose $q_\lambda r_j \notin H_{11}$. Then $p_{\lambda j} = 0$, $H_{j\lambda}$ does not contain an idempotent, and $(r_i a q_\lambda) * (r_j b q_\mu) \notin H_{i\mu}$.
Again by the definition of $p_{\lambda i}$ $(r_i a q_\lambda) * (r_j b q_\mu) = 0 \in T$.
Therefore $\emptyset = 0$ which is true by definition of $\emptyset$.

Theorem 3.15: (Rees) A semigroup is completely 0-simple iff it is isomorphic with a regular Rees matrix semigroup over a group with zero.

Proof: If a semigroup is isomorphic with a regular Rees matrix semigroup with zero, then it is completely 0-simple by theorem 3.13.

Conversely, let S be a completely 0-simple semigroup.

By theorem 3.8, S is O-bisimple, so that $D=S\setminus 0$ is a $\mathcal{D}$-class of S. Construct $\mathcal{M}^0(H_{11};I,\Lambda:P)$ for D as in theorem 3.14. Now theorem 3.9 (ii) shows that S is isomorphic with the trace $T=D\cup 0$ of D, product ($*$) therein being defined by (2). By theorem 3.14, T and $\mathcal{M}^0$ are isomorphic. Hence, S and $\mathcal{M}^0$ are isomorphic.

Consider the following example of a semigroup $(S,*)$ where $*$ is defined by the table in figure 6.

$*$	a	b	c	d	0
a	a	b	0	0	0
b	0	0	a	b	0
c	c	d	0	0	0
d	0	0	c	d	0
0	0	0	0	0	0

FIGURE 6

EXAMPLE OF A SEMIGROUP

$(S,*)$

S is O-simple because $S^2\neq 0$ and 0 is the only proper two sided ideal. Also a and d are primitive idempotents.

The $\mathcal{R}$-classes of S are $R_1=\{a,b\}$ $R_2=\{c,d\}$. The $\mathcal{L}$-classes of S are $L_1=\{a,c\}$ $L_2=\{b,d\}$. The $\mathcal{H}$-classes of S are $H_{11}=R_1\cap L_1=\{a\}$ $H_{12}=R_1\cap L_2=\{b\}$ $H_{21}=R_2\cap L_1=\{c\}$ $H_{22}=R_2\cap L_2=\{d\}$.

Let $r_1 = a$ and $q_1 = a$

$r_2 = c$ $\qquad q_2 = b$

$p_{11} = q_1 r_1 = aa = a \qquad p_{12} = q_1 r_2 = ac = 0$

$p_{21} = q_2 r_1 = ba = 0 \qquad p_{22} = q_2 r_2 = bc = a$

$$p = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

$(a)_{11}\ \emptyset \longleftrightarrow r_1 a q_1 = a \qquad (a)_{12}\ \emptyset \longleftrightarrow r_1 a q_2 = b$

$(a)_{21}\ \emptyset \longleftrightarrow r_2 a q_1 = c \qquad (a)_{22}\ \emptyset \longleftrightarrow r_2 a q_2 = d$

The matrix representation of S is then:

$$a \rightarrow \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} b \rightarrow \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} c \rightarrow \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} d \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} 0 \rightarrow \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Chapter IV

Schutzenberger Representations

In order to represent a semigroup as a semigroup of matrices over a group it is first necessary to construct the structure group.

Definition: Let S^1 be a semigroup and H an $\mathcal{H}$-class of S^1. $T(H) = \{t \mid t \in S^1 \text{ and } Ht \subseteq H\}$. Notice that for each $t \in T(H)$, the image of H under the transformation $S^1 t$ is a subset of H. The symbol ρ_t will denote the transformation of S^1 by an element of T(H).

Definition: $\Gamma(H) = \{\gamma_t \mid t \in T(H) \text{ and } \gamma_t = \rho_t|H\}$ (Where $\rho_t|H$ is ρ_t restricted to H)

In other words, each γ_t is, in effect, a transformation of the $\mathcal{H}$-class into itself and $\Gamma(H)$ is the set of all such transformations.

Lemma 4.1: Let H be an $\mathcal{H}$-class of a semigroup S. Let $h_0 \in H$, and let $t \in S^1$ such that $h_1 = h_0 t \in H$. Then $h_0 = h_1 t'$ for some t' in S^1, and the mappings $\gamma_t; x \rightarrow xt$ and $\gamma_{t'}: x \rightarrow xt$ are mutually inverse permutations of H. Thus t and t' belongs to T(H), and $\gamma_t \gamma_{t'} = \gamma_{t'} \gamma_t = \gamma_1$.

If L is the $\mathcal{L}$-class containing H, then the mappings $x \rightarrow xt$ and $x \rightarrow xt'$ are mutually inverse, one-to-one, $\mathcal{R}$-class preserving mappings of L upon itself.

Proof: Both h_1 and $h_0 \in H$ and t and $t' \in T(H)$. By Green's Lemma the mappings $\gamma_t : x \to xt\,(x \in L_{h_0})$ and $\gamma_{t'} : x \to xt'\,(x \in L_{h_1})$, are mutually inverse, $\mathcal{R}$-class preserving, one-to-one mappings of L_{h_0} onto L_{h_1} and of L_{h_1} onto L_{h_0}. Because in this case $L_{h_0} = L_{h_1}$ and the mappings are $\mathcal{R}$-class preserving the mappings γ_t and $\gamma_{t'}$ are mutually inverse one-to-one mappings of H onto itself.

The second part of the theorem is the same direct application of Green's Lemma.

Definition: A set $\Gamma(H)$ of transformations of H is said to be simply transitive iff given any two elements x and y of H $\ni$ one and only one element of $\Gamma(H)$ mapping x onto y.

Theorem 4.2: Let H be an $\mathcal{H}$-class of a semigroup S. Then the semigroup $\Gamma(H)$ of transformations of H induced by the inner right translations of S^1 is a simply transitive group of permutations of H. It follows that $|\Gamma(H)| = |H|$ ($|H|$ denotes the order of H). If H is itself a subgroup of S, then $\Gamma(H) \cong H$. ($\Gamma(H)$ is isomorphic to H).

Proof: Let $\gamma_t \in \Gamma(H)$ with t in T(H). If $h_0 \in H$, then $h_1 = h_0 t \in H$, and, by Lemma 4.1, γ_t has a group inverse $\gamma_{t'}$ in $\Gamma(H)$. Hence $\Gamma(H)$ is a group.

To show $\Gamma(H)$ is simply transitive, let h_0 and h_1 be any two elements of H. From $h_0 \mathcal{R} h_1$ it is true $h_0 t=h_1$ for some $t\in S^1$; by Lemma 4.1, $t\in T(H)$ and $h_0\gamma_t=h_1$.

To show that γ_t is the only element of $\Gamma(H)$ mapping h_0 onto h_1, suppose $h_0\gamma_s=h_1$ for some $s\in T(H)$. Let x be an arbitrary element of H. From $x\mathcal{L}h_0$, $x=yh_0$ for some $y\in S^1$, and so $x\gamma_t=xt=yh_0t=yh_1=yh_0s=xs=x\gamma_s$. Hence $\gamma_s=\gamma_t$.

Suppose H is a group. Let e be the identity of H, and let h be an arbitrary element of H. From what has been said there is exactly one element of $\Gamma(H)$ mapping e upon h. But γ_h maps e upon h. Thus $\Gamma(H)=\{\gamma_h | h\in H\}$.

Lemma 4.3: Let H be an $\mathcal{H}$-class of a semigroup S, and let R and L be the $\mathcal{R}$ and $\mathcal{L}$-classes of S containing H.

(i) For every $s\in S$, either $Hs\cap R=\emptyset$ or else Hs is an $\mathcal{H}$-class contained in R, and Ls is the $\mathcal{L}$-class of S containing Hs.

(ii) If $Hs\cap R=\emptyset$, then $Hst\cap R=\emptyset$ for every t in S.

Proof: (i) Suppose $Hs\cap R\neq\emptyset$. Let $b\in Hs\cap R$. Then $b=as$ for some $a\in H$. Since $a\mathcal{R}b$, $a=bs'$ for some $s'\in S$. By Green's Lemma 2.1, $x\to xs$ is an $\mathcal{R}$-class preserving, one-to-one mapping of L_a onto L_b. Hence $Hs=H_b\subseteq R$, and $L_s=L_b\supseteq H$.

(ii) If $Hst\cap R\neq\emptyset$, and $b\in Hst\cap R$, then $b=ast$ for some $a\in H$, and $b\mathcal{R}a$, so that $a=bt'$ for some $t\in S$.

But the equations $b=(as)t$ and $as=b(t's)$ imply that $b\mathcal{R}as$, so $as\in Hs\cap R$, contrary to the hypothesis that $Hs\cap R=\emptyset$.

As has been necessary so often, Green's theorem will again be resorted to, to "move" from $\mathcal{L}$-class to $\mathcal{L}$-class and from $\mathcal{R}$-class to $\mathcal{R}$-class or combining the two ideas from $\mathcal{H}$-class to $\mathcal{H}$-class. The $\mathcal{R}$-classes will be $\{R_i \mid i\in I\}$. The $\mathcal{L}$-classes will be $\{L_\lambda \mid \lambda\in\Lambda\}$. They will be contained in the $\mathcal{D}$-class D. The $\mathcal{H}$-classes will be denoted as $H_{i\lambda}=R_i\cap L_\lambda$ and it will be assumed I and Λ have an element in common. Also, $H_{11}=H$.

For each $\lambda\in\Lambda$, pick an element $h_\lambda\in H_{1\lambda}$. Since $h_\lambda\mathcal{R}h_1$, $\exists$ elements $q_\lambda, q'_\lambda\in S^1 \ni h_\lambda=h_1q_\lambda$ and $h_1=h_\lambda q'_\lambda$. By Lemma 2.1, the mappings $x\to xq_\lambda$ and $y\to yq'_\lambda$ are mutually inverse, one-to-one, $\mathcal{R}$-class preserving mappings of L_1 and L_λ onto each other. For each $\lambda\in\Lambda$, make and fix a selection of such elements $q_\lambda, q'_\lambda\in S^1$.

Definition: Let $s\in S$. $M_D(s)=(m_{\lambda\mu}(s))$ is a $\Lambda x\Lambda$ matrix over $\Gamma(H)^0$ defined as

$$m_{\lambda\mu}(s) = \begin{cases}\gamma(q_\lambda s q'_\mu) & \text{if } H_{1\lambda}s=H_{1\mu}\\ 0 & \text{otherwise}\end{cases}$$

Theorem 4.4: The mappings $s\to M_D(s)$ is a representation of S by row monomial $\Lambda x\Lambda$ matrices over $\Gamma(H)^0$. Given $\lambda,\mu\in\Lambda$ and $\gamma(t)$ in $\Gamma(H)$ $(t\in T(H))$, $\exists$ an $s\in S\ni m_{\lambda\mu}(s)=\gamma t$.

Proof: First it is shown that each $M_D(s)$ is row monomial. Let $s\in S$ and $\lambda\in\Lambda$. $H_{1\lambda}=R_1\cap L_\lambda$ implies that $H_{1\lambda}\subseteq R_1$. Applying Lemma 4.3 (i) to $H_{1\lambda}$ either $H_{1\lambda}s\cap R=\emptyset$ or else $H_{1\lambda}s=H_{1\mu}$, $\mu\in\Lambda$. In the first case, the condition of the definition has not been satisfied and therefore the λth row is 0. In the second case, it has $\gamma(q_\lambda s q'_\mu)$ in the μ-column and zeros elsewhere.

To show that $m_{\lambda\mu}(s)=\gamma(t)$ take $s=q'_\lambda\ tq_\mu$. Then $H_{1\lambda}s=H_{1\lambda}q'_\lambda\ tq_\mu=Htq_\mu=Hq_\mu=H_{1\mu}$ and then $m_{\lambda\mu}(s)=\gamma(q_\lambda sq'_\mu)$. If h is any element of H, then $hq_\lambda sq'_\mu=hq_\lambda q'_\lambda\ tq_\mu q'_\mu=ht$ because $xq_\lambda q'_\lambda=x$ and $xq_\mu q'_\mu=x\ \forall x\in H$. Hence $\gamma(q_\lambda sq'_\lambda)=\gamma(t)$.

To show $M_D(s)M_D(t)=M_D(st)\ \forall s,t\in S$ it must be shown that

(1) $\sum_{\mu\in\Lambda} m_{\lambda\mu}(s)m_{\mu\nu}(t)=m_{\lambda\nu}(st)\ \forall\lambda,\nu\in\Lambda$. Suppose first that $m_{\lambda\nu}(st)\neq 0$. Then $H_{1\lambda}st=H_{1\nu}$ and

(2) $m_{\lambda\nu}(st)=\gamma(q_\lambda stq'_\lambda)$.

By Lemma 4.3 (ii), $H_{1\lambda}s\cap R_1\neq 0$, since otherwise $H_{1\nu}=H_{1\lambda}st\cap R_1=0$. By Lemma 4.3 (i), $H_{1\lambda}s=H_{1\kappa}$ for some $\kappa\in\Lambda$, and so $m_{\lambda\kappa}(s)=(q_\lambda sq'_\kappa)$, $m_{\lambda\mu}(s)=0$ for $\mu\neq\kappa$. Since $H_{1\nu}=H_{1\lambda}t$, it follows that $m_{\kappa\nu}t=\gamma(q_\kappa tq'_\nu)$. Hence the left side of (1) is (3) $\gamma(q_\lambda sq'_\kappa)\gamma(q_\kappa tq'_\nu)=\gamma(q_\lambda sq'_\kappa\ q_\kappa tq'_\nu)$. To show (2) and (3) equal it is sufficient to show $hq_\lambda sq'_\kappa q_\kappa tq'_\nu=hq_\lambda stq'_\nu\ \forall h\in H$. But this is so because $hq_\lambda s\in H_{1\lambda}s=H_{1\kappa}$, and $x\longrightarrow xq'_\kappa q_\kappa$ is the identity mapping of $H_{1\kappa}$ onto itself.

Now suppose that $m_{\lambda\nu}(st)=0$, so that $H_{1\lambda}st\neq H_{1\nu}$. It is necessary to show the left side of (1) is also 0. If

$H_{1\lambda} s \cap R = \emptyset$ then $m_{\lambda\mu}(s) = 0 \, \forall \mu \in \Lambda$, and (1) is 0 in this case. If $H_{1\lambda} s \cap R \neq 0$, then by Lemma 4.3 (i), $H_{1\lambda} s = H_{1\kappa} s = H_{1\kappa}$ for some $\kappa \in \Lambda$. Since $m_{\lambda\mu}(s) = 0 \, \forall \mu \neq \kappa$, it is sufficient to show that $m_{\kappa\nu}(t) = 0$. But $H_{1\kappa} t = H_{1\lambda} st \neq H_{1\nu}$. Therefore $m_{\kappa\nu}(t) = 0$.

As an example of Schutzenberger representations consider the following set of transformations and the operation defined by the table in figure 7.

*	(11)	(12)	(21)	(22)
(11)	(11)	(11)	(22)	(22)
(12)	(11)	(12)	(21)	(22)
(21)	(11)	(21)	(12)	(22)
(22)	(11)	(22)	(11)	(22)

FIGURE 7

SET OF TRANSFORMATIONS

$$R_1 = \{(11)\ (22)\} \qquad L_1 = \{(11)\}$$

$$L_2 = \{(22)\}$$

$$H_{11} = R_1 \cap L_1 = \{(11)\} \qquad H_{12} = R_1 \cap L_2 = \{(22)\}$$

$T(H) = \{(11), (12)\}$ $\gamma_{(11)} = (11)(11) = (11)$ $\gamma_{(12)} = (11)(12) = (11)$

$\Gamma(H)$ is the group consisting of the identity element, call it e. By computation it is seen

$$h_1 = (11) \quad q_1 = (11) \quad q_1' = (11)$$

$$h_2 = (22) \quad q_2 = (21) \quad q_2' = (21)$$

By further computation it is seen

$$m_{11}(11) = \gamma[(11)\ (11)\ (11)] = \gamma(11) = e$$
$$m_{12}(11) = \gamma[(11)\ (11)\ (21)] = \gamma(22) = 0$$
$$m_{21}(11) = \gamma[(21)\ (11)\ (11)] = \gamma(11) = e$$
$$m_{22}(11) = \gamma[(21)\ (11)\ (21)] = \gamma(12) = 0$$

$$(11) \longrightarrow \begin{pmatrix} e & 0 \\ e & 0 \end{pmatrix}$$

$$m_{11}(12) = \gamma[(11)\ (12)\ (11)] = \gamma(11) = e$$
$$m_{12}(12) = \gamma[(11)\ (12)\ (21)] = \gamma(22) = 0$$
$$m_{21}(12) = \gamma[(21)\ (12)\ (11)] = \gamma(11) = 0$$
$$m_{22}(12) = \gamma[(21)\ (12)\ (21)] = \gamma(12) = e$$

$$(12) \longrightarrow \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}$$

$$m_{11}(21) = \gamma[(11)\ (21)\ (11)] = \gamma(11) = 0$$
$$m_{12}(21) = \gamma[(11)\ (21)\ (21)] = \gamma(11) = e$$
$$m_{21}(21) = \gamma[(21)\ (21)\ (11)] = \gamma(11) = e$$
$$m_{22}(21) = \gamma[(21)\ (21)\ (21)] = \gamma(21) = 0$$

$$(21) \longrightarrow \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}$$

$$m_{11}(22) = \gamma[(11)\ (22)\ (11)] = \gamma(11) = 0$$
$$m_{12}(22) = \gamma[(11)\ (22)\ (21)] = \gamma(11) = e$$
$$m_{21}(22) = \gamma[(21)\ (22)\ (11)] = \gamma(11) = 0$$
$$m_{22}(22) = \gamma[(21)\ (22)\ (21)] = \gamma(11) = e$$

$$(22) \longrightarrow \begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix}$$

Chapter V

Summary

The question considered in this study was whether an arbitrary semigroup could be represented by a semigroup of matrices. The Rees Theorem showed that a completely 0-simple semigroup was representable. Not only can a completely 0-simple semigroup be represented, it's representation is an isomorphism. Green's theorem 2.2 provided the group for the Rees matrix elements. In the case of any other semigroup, the representation may or may not be an isomorphism.

In the more general case, the Schutzenberger group of an $\mathcal{H}$-class provided the group. In the case of an arbitrary semigroup, however, the representation may or may not be isomorphic.

It might be well to discuss the chain of events that lead from the elements s of the semigroup S^0 to the final elements of the semigroup of matrices.

Recall that S^0 was first partitioned off into its $\mathcal{D}$-classes, $\mathcal{L}$-classes and $\mathcal{R}$-classes. The $\mathcal{L}$-classes and $\mathcal{R}$-classes were indexed by the sets I and Λ. This indexing was then used to provide an indexing of the $\mathcal{H}$-classes formed by the intersection of the $\mathcal{L}$-classes and $\mathcal{R}$-classes.

In this manner, one of the $\mathcal{H}$-classes was given the index H_{11}. This $\mathcal{H}$-class was arbitrarily selected to form the Schutzenberger group. The elements of this group would become the elements of the matrices.

It was necessary then to set up a mapping from the elements of S^o to the $\gamma(t)$ of $\Gamma(H_{11})$.

Green's theorem 2.2 was used to calculate the q_λ that provided a mapping from the $\mathcal{H}$-class H_{11} to $H_{1\lambda}$ and the q'_λ that provided the mapping from $H_{1\lambda}$ to H_{11}. It can be readily seen that given any $s \in S^o$ and $h \in H_{11}$ that the product $hq_\lambda sq'_\mu$ is again an element of H_{11}. Breaking the product down $hq_\lambda \in H_{1\lambda}$ by definition of q_λ. But the condition that $H_{1\lambda}s=H_{1\mu}$ forces $hq_\lambda s$ to be in $H_{1\mu}$. The means used to calculate q'_λ then guarantees that $hq_\lambda sq'_\mu \in H_{11}$ and so $q_\lambda sq'_\mu \in T(H)$ and $\gamma(q_\lambda sq'_\mu) \in \Gamma(H)$.

Therefore the element in the row λ and the column μ of the matrix has the value $\gamma(q_\lambda sq'_\mu)$ if $H_{1\lambda}s=H_{1\mu}$ otherwise it is zero.

After the matrices have been calculated to represent the elements of S^o the question, how the preservation of operations is the result, occurs. To put this more precisely given $r,s,t \in S^o$ and the matrix representations A, B, C of r,s and t respectively. Why is it that if $r*s=t$ then $A \circ B=C$?

First recall that the matrices are row(or column) monomial. This means that when rows of A are "multiplied" by each of the columns of B the resulting summation will be concerned with at most one non-zero element. Only the case where a non-zero entry in C is calculated will be considered here.

Suppose that $a_{\lambda 1}, a_{\lambda 2} \ldots, a_{\lambda n}$ is an arbitrary row of matrix A and $b_{1\mu}, b_{2\mu} \ldots, b_{n\mu}$ is an arbitrary column of matrix B. Also suppose that $a_{\lambda k} b_{k\mu} \neq 0$. Then $a_{\lambda k} = \gamma(q_\lambda r q_k')$ and $b_{k\mu} = \gamma(q_k s q_\mu')$. So $a_{\lambda k} b_{k\mu} = \gamma(q_\lambda r q_k')\gamma(q_k s q_\mu')$

$$= \gamma(q_\lambda r q_k' \, q_k s q_\mu')$$

The reason for this is given in the proof of theorem 4.4. In order for $\gamma(q_\lambda r q_k' \, q_k s q_\mu')$ to be the element in the λth row the μth column of the matrix C, $hq_\lambda r q_k' \, q_k s q_\mu'$ must equal $hq_\lambda t q_\mu'$ for all $h \in H_{11}$. But q_k' and q_k are inverse mappings by theorem 2.2 and so r is mapped onto the element of H_{1k} that is mapped onto r. Therefore $hq_\lambda r q_k' \, q_k s q_\mu'$ $= hq_\lambda r s q_\mu' = hq_\lambda t q_\mu'$. It can be concluded from this that the "product" $a_{\lambda k} \, b_{k\mu} = c_{\lambda\mu}$.

The preceeding is not intended as a complete answer to how the representation is established or why the mapping is at least a homomorphism. It is hoped that it does provide a feel for some of the more bothersome questions that may enter ones mind as he flounders in the theory.

In the special case of representing a group as a group of matrices it was found that following the definitions all matrices were 1x1 due to the fact that there was only one $\mathcal{H}$-class.

A few added theorems are of interest and can be found in the book Algebraic Theory of Semigroups, by A.H. Clifford and G.B. Preston.

1. Let S be a regular Rees matrix semigroup $\mathcal{M}^0(G;I,\Lambda;P)$. Then the Schutzenberger representations of S corresponding to the $\mathcal{D}$-class $D=S\setminus 0$ can be taken to be $M_D(s)=Ps$ where s denotes an arbitrary element of S.

2. Let H and H' be $\mathcal{H}$-classes of a semigroup S both contained in the same $\mathcal{D}$-class of S. Then $\Gamma(H)$ is isomorphic to $\Gamma(H')$.

3. Any of the theorems on completely 0-simple semigroups are also true about completely simple semigroups.

4. For any of the theorems on Schutzenberger representations, there exists dual theorems on anti-representations. Anti-representations are anti-isomorphisms or anti-homomorphisms defined by $(a*b)\phi=(b\phi)\circ(a\phi)$.

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