AN INTRODUCTION TO THE OPERATION OF

CLOSURE AS DEFINED IN A GROUP

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TABLE OF CONTENTS

CHAPTER I

THE PROBLEM

It is the purpose of this study to demonstrate how the operation of closure may be logically introduced in group theory as an operation on the elements of a group.

Closure is usually defined in connection with partially ordered sets. As an example of a lattice, the student is invariably offered the structure of the subgroups of a group with the operations of intersection and closure equated to the meet and join operations of lattice theory. The student is familiar with the operation of intersection. However, the operation of closure may not be familiar, since it is not even named in such comprehensive studies as W. R. Scott's "Group Theory".

This study shows that the concept can properly be introduced quite early in a study of groups, and, by presenting theorems which relate to the basic properties of closure, demonstrates that closure can be logically used in the development of a presentation of group theory. Hence, this study should be meaningful to any student that has had an introduction to the theory of groups, and should be an aid to any study of closure, whether related specifically to groups or not.

<u>Definition of symbols used</u>. The following symbols will be used throughout the body of this thesis:

$\Rightarrow$: only if

$\Leftrightarrow$: if and only if

$\in$: is an element of

$\ni$: such that

$\exists$: there exists

$\subseteq$: is a subset of

$\leqslant$: is a subgroup of

$/$: used in connection with the above implies denial, e.g., $\notin$: is not an element of

$\{R\}$: the set whose members are R

N : will always be used to represent the set of natural numbers.

Other symbols will be introduced as they are needed in the body of the paper.

CHAPTER II

DEVELOPMENT OF BACKGROUND NECESSARY TO INTRODUCE THE CONCEPT OF CLOSURE

Definition of a group.

Definition 1: A group is a set G for which an operation called multiplication is defined such that:

1. If $x \in G$, $y \in G$, then $xy \in G$. (The set is closed.)
2. For every x, y, z contained in G, $(xy)z = x(yz)$.
3. G contains a unique element e, called the identity of G, $\ni$ for all $x \in G$, $ex = xe = x$.
4. For every $x \in G$, $\exists$ a unique element $x^{-1} \in G \ni xx^{-1} = x^{-1}x = e$.

Theorem 1. If x is an element of a group then $(x^{-1})^{-1} = x$.

Proof: Let x be an element of a group G. Then $x^{-1} \in G$ and $(x^{-1})^{-1} \in G$ by property 4 of the definition of a group. Then

$$x^{-1}(x^{-1})^{-1} = e = x^{-1}x \quad \text{by property 4}$$
$$x[x^{-1}(x^{-1})^{-1}] = x[x^{-1}x] \quad \text{by property 1}$$
$$(xx^{-1})(x^{-1})^{-1} = (xx^{-1})x \quad \text{by property 2}$$
$$e(x^{-1})^{-1} = ex \quad \text{by property 4}$$
$$\text{and } (x^{-1})^{-1} = x \quad \text{by property 3 of a group.}$$

Definition 2: A complex is a subset of a group. [7:2]

Basic properties of sets, their union and intersection. If S_i is a complex, the following properties are obtainable from set theory:

1. $S_i \subseteq S_i$
2. If $S_1 \subseteq S_2$ and $S_2 \subseteq S_3$, then $S_1 \subseteq S_3$
3. $S_1 \subseteq S_2$ and $S_2 \subseteq S_1 \Leftrightarrow S_1 = S_2$.

Definition 3: The totality of all elements that lie in either S_1 or S_2 ... or S_n is the union of the S_i, and is denoted by $S_1 \cup S_2 \cup \ldots \cup S_n = \bigcup_{i=1}^{n} S_i$.

Definition 4: The totality of all elements that lie simultaneously in n given complexes $S_1, S_2, \ldots S_n$ is the intersection of the S_i, denoted $S_1 \cap S_2 \cap \ldots \cap S_n = \bigcap_{i=1}^{n} S_i$.

Again from a consideration of set theory the following properties are obtainable:

4. $S_1 \cup S_1 = S_1$.
5. $S_1 \cup S_2 = S_2 \cup S_1$.
6. $(S_1 \cup S_2) \cup S_3 = S_1 \cup (S_2 \cup S_3)$.
7. $S_1 \subseteq S_2 \Leftrightarrow S_1 \cup S_2 = S_2$.
8. $S_1 \cap S_1 = S_1$.
9. $S_1 \cap S_2 = S_2 \cap S_1$.
10. $(S_1 \cap S_2) \cap S_3 = S_1 \cap (S_2 \cap S_3)$.
11. $S_1 \subseteq S_2 \Leftrightarrow S_1 \cap S_2 = S_1$.
12. If $S_1 \subseteq S_2$ and $S_3 \subseteq S_4$, then $S_1 \cup S_2 \subseteq S_3 \cup S_4$. [7:2]

Definition 5: In a group G, the inverse complex S^{-1} of a non-empty complex S of G is the complex consisting of all the inverses of the elements of S. [7:19]

Theorem 2: If S_i and S_j are non-empty complexes of a group, then $(S_i^{-1})^{-1} = S_i$ and if $S_i = S_j$ then $S_i^{-1} = S_j^{-1}$.

Proof: (a) Let $x \in (S_i^{-1})^{-1}$. Then x is the inverse of an element of (S_i^{-1}) by the preceding definition, i.e., $x^{-1} \in S_i^{-1}$. But by the same definition $x^{-1} \in S_i^{-1} \Leftrightarrow x \in S_i$. Hence $x \in (S_i^{-1})^{-1} \Rightarrow x \in S_i$, i.e., $(S_i^{-1})^{-1} \subseteq S_i$. Let $x \in S_i$. Then $x^{-1} \in S_i^{-1}$ and $(x^{-1})^{-1} = x \in (S_i^{-1})^{-1}$, i.e., $S_i \subseteq (S_i^{-1})^{-1}$. Hence $S_i = (S_i^{-1})^{-1}$ by property 3 of sets.

(b) Let $S_i = S_j$ and let x^{-1} be an element of S_i^{-1}. Then x S_i by part (a) and the definition of an inverse complex. But if $x \in S_i$, then $x \in S_j$ by property 3 of sets. Hence $x^{-1} \in S_j^{-1}$, i.e., $S_i^{-1} \subseteq S_j^{-1}$. Interchanging the i's and j's in the above demonstration yields $S_j^{-1} \subseteq S_i^{-1}$. Hence $S_i^{-1} = S_j^{-1}$ by property 3 of sets.

Theorem 3: If S_i is a non-empty complex of a group g, then $(\bigcup_{i=1}^{n} S_i)^{-1} = \bigcup_{i=1}^{n} (S_i^{-1})$, and $(S_1 \cap S_2)^{-1} = S_1^{-1} \cap S_2^{-1}$.

Proof: (a) Let $x^{-1} \in \bigcup_{i=1}^{n} (S_i^{-1})$. Then $x^{-1} \in$ some S_i^{-1}, say S_j^{-1}, by definition 3. But $x^{-1} \in S_j^{-1} \Rightarrow x \in S_j$, by definition 5. Now if $x \in S_j$, then $x \in \bigcup_{i=1}^{n} S_i$ by definition of union. But $x \in \bigcup_{i=1}^{n} S_i \Rightarrow x^{-1} \in (\bigcup_{i=1}^{n} S_i)^{-1}$. Hence $\bigcup_{i=1}^{n} (S_i^{-1})$ $(\bigcup_{i=1}^{n} S_i)^{-1}$. Now let $x^{-1} \in (\bigcup_{i=1}^{n} S_i)^{-1}$. Then $x \in \bigcup_{i=1}^{n} S_i$ and hence $x \in$ some S_i, say S_j. But $x \in S_j \Rightarrow x^{-1} \in S_j^{-1}$ by definition of an inverse complex. But $S_j^{-1} \in \bigcup_{i=1}^{n} (S_i^{-1})$, hence $x^{-1} \in \bigcup_{i=1}^{n} (S_i^{-1})$, i.e., $(\bigcup S_i)^{-1} \subseteq \bigcup_{i=1}^{n}(S_i^{-1})$. Therefore $\bigcup_{i=1}^{n}(S_i^{-1}) = (\bigcup_{i=1}^{n} S_i^{-1})$ by property 3 of sets.

(b) Let $x^{-1} \in (S_1 \cap S_2)^{-1}$. Then $x \in (S_1 \cap S_2)$, thus $x \in S_1$ and S_2, which implies $x^{-1} \in S_1^{-1}$ and S_2^{-1}, i.e., $(S_1 \cap S_2)^{-1} \subseteq S_1 \cap S_2$. Now $x^{-1} \in S_1 \cap S_2 \Rightarrow x^{-1} \in S_1^{-1}$ and S_2^{-1}, but this $\Rightarrow x \in S_1$ and S_2, i.e., $x \in S_1 \cap S_2$, thus $x^{-1} \in (S_1 \cap S_2)^{-1}$, i.e., $S_1^{-1} \cap S_2^{-1} \subseteq (S_1 \cap S_2)^{-1}$, hence $(S_1 \cap S_2)^{-1} = S_1^{-1} \cap S_2^{-1}$.

Theorem 4: If S_1 and S_2 are non-empty complexes such that $S_1 \subseteq S_2$, then $S_1^{-1} \subseteq S_2^{-1}$.

Proof: By property 11 of sets, $S_1 \subseteq S_2 \Rightarrow S_1 = S_1 \cap S_2$. Taking inverses in this equality yields $S_1^{-1} = (S_1 \cap S_2)^{-1}$, and by theorem 3, $(S_1 \cap S_2)^{-1} = S_1^{-1} \cap S_2^{-1}$. But $S_1^{-1} = (S_1^{-1} \cap S_2^{-1}) \Rightarrow S_1^{-1} \subseteq S_2^{-1}$, again by property 11.

Definition 6: If $A_1, A_2, \ldots A_n$ are non-empty complexes of a group G, then the product $A_1 A_2 \ldots A_n$ is the complex of G which contains all the elements of the form $a_1 a_2 \ldots a_n$, where $a_i \in A_i$. [6:15]

Definition 7: A subgroup of a group G is a complex H of G whose elements form a group under the multiplication defined on G, and is denoted $H \leqslant G$.

Theorem 5: Necessary and sufficient conditions that a non-empty complex S of a group G be a subgroup are:

1. $SS \subseteq S$.
2. $S^{-1} \subseteq S$. [7:21]

Proof: The necessity of the conditions follow immediately from the definition of a group, the first condition from the fact that a group must be closed, and the second from the requirement that every element have an inverse.

Sufficiency: Condition 1 insures that S is closed; the associativity follows from that of G; condition 2 guarantees that every element of S has an inverse in S; and this coupled with condition 1 guarantees that the identity is in S.

Theorem 6: A non-empty complex S of G is a subgroup of G if and only if $S^{-1}S \subseteq S$. [7:21]

Proof: $S^{-1}S \subseteq S \Rightarrow e \in S$. Therefore $S^{-1}e = S^{-1} \subseteq S$ and condition 2 of the previous theorem is satisfied. By theorem 4, $S^{-1} \subseteq S \Rightarrow S \subseteq S^{-1}$, and hence by property 3 of sets, $S^{-1} = S$ and condition 1 of theorem 5 is satisfied.

Theorem 7: If A and B are subgroups of G then the complex $C = AB$ is a subgroup if and only if A and B commute. [2:57]

Proof: (a) If $C = AB = BA$, then $C^2 = (AB)(AB) = A(BA)B = A(AB)B = A^2B^2$. But since A and B are groups and hence are closed, $A^2 = A$ and $B^2 = B$. Therefore $C^2 = AB = C$. Now if $ab \in AB$, then $(ab)^{-1} = b^{-1}a^{-1} \in BA = AB$, i.e., $(AB)^{-1} \subseteq AB$. Consequently C is a group by theorem 5.

(b) Let $C = AB$ be a group and $ab \in AB$. Then by definition 6 $a \in A$ and $b \in B$. but if $a \in A$, $b \in B$, then $a^{-1} \in A$ and $b^{-1} \in B$, hence $a^{-1}b^{-1} \in AB$. Therefore $(a^{-1}b^{-1})^{-1} = ba \in AB$, i.e., $BA \subseteq AB$. Similarly, $b^{-1}a^{-1} \in BA$ and $(b^{-1}a^{-1})^{-1} = ab \in AB$ and $AB \subseteq BA$. Hence $AB = BA$.

CHAPTER III

THEOREMS AND DEFINITIONS RELATING TO CLOSURE

In this section we will develop the theorems necessary to provide a familiarity with the basic properties of closure and demonstrate how the concept can easily be integrated into a study of group theory.

Definition of closure.

Definition 8: Let S be a non-empty complex of a group G and $S_1 = S \cup S^{-1}$. The closure of the complex S, denoted $\langle S \rangle$, is the complex $S_2 = S_1^1 \cup S_1^2 \cup S_1^3 \ldots = \bigcup_{i \in N} S^i$, where N is the set of all natural numbers. [7:22]

As an example of closure, consider the group J of integers, the operation addition and the complex $S = \{2\}$. Then $S^{-1} = \{-2\}$, $S_1 = S \cup S^{-1} = \{2, -2\}$, $S_1^2 = \{-4, 0, 4\}$, $S_1^3 = \{-6, -2, 2, 6\}$, $S_1^4 = \{-8, -4, 0, 4, 8\}$, and $\langle S \rangle = \{\ldots -6, -4, -2, 0, 2, 4, 6, \ldots\}$.

The following is a useful alternate definition:

Theorem 8: A complex S_2 is the closure of a non-empty complex S if and only if S_2 is the set of all finite products $x_1 x_2 \ldots x_n$, where $x_i \in S \cup S^{-1}$, $i \in N$.

Proof: Let S_2 be the set of all finite products $x_1 x_2 \ldots x_n$, $x_i \in S_1 = S \cup S^{-1}$, $i \in N$. If $x \in S$ then from

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Proof: Let S_2 be the set of all finite products $x_1 x_2 \ldots x_n$, $x_i \in S_1 = S \cup S^{-1}$, $i \in N$. If $x \in S$ then from the definition of S_1, $x^{-1} \in S^{-1}$ and $x^{-1}x = e \in S_2$, $xe = x \in S_2$, and hence $S_1^1 \subseteq S_2$. Suppose $S_1^1 \cup S_1^2 \cup \ldots \cup S_1^n \subseteq S_2$. The elements of S_1^{n+1} are of the form $x_j x_1 x_2 \ldots x_n$, where x_j and $x_i \in S_1$. Therefore S_1^{n+1} is a set of finite products of the elements of S_1 and hence $S_1^{n+1} \subseteq S_2$ and consequently $S_1^1 \cup S_1^2 \cup \ldots \cup S^{n+1} \subseteq S_2$, i.e., $\langle S \rangle \subseteq S_2$. Let $c = x_1 x_2 \ldots x_n \in S_2$. Then $c \in S_2^n$ and hence in $\langle S \rangle$, therefore $S_2 \subseteq \langle S \rangle$ and $S_2 = \langle S \rangle$ by property 3 of sets.

Theorem 9: If $A \subseteq B$, then $\langle A \rangle \subseteq \langle B \rangle$.

Proof: If $A \subseteq B$, then $A^{-1} \subseteq B^{-1}$ by theorem 4, hence $A \cup A^{-1} \subseteq B \cup B^{-1}$ by property 12 of sets, hence $(A \cup A^{-1})^1 \subseteq (B \cup B^{-1})^1$. Suppose $\bigcup_{i=1}^{n}(A \cup A^{-1})^i \subseteq \bigcup_{i=1}^{n}(B \cup B^{-1})^i$. Consider $\bigcup_{i=1}^{n+1}(A \cup A^{-1})^i$. $\bigcup_{i=1}^{n+1}(A \cup A^{-1})^i = (A \cup A^{-1}) \cup \{\bigcup_{i=1}^{n}(A \quad A^{-1})^i\}$. Since $(A \cup A^{-1}) \subseteq (B \cup B^{-1})$ then by property 12 of sets $(A \cup A^{-1}) \cup [\bigcup_{i=1}^{n}(A \cup A^{-1})^i] \subseteq [(B \cup B^{-1})] \cup [\bigcup_{i=1}^{n}(B \cup B^{-1})^i]$. Hence $\bigcup_{i=1}^{n+1}(A \cup A^{-1}) \subseteq \bigcup_{i=1}^{n+1}(B \cup B^{-1})$, i.e., $\bigcup_{i \in N}(A \cup A^{-1}) \subseteq \bigcup_{i \in N}(B \cup B^{-1})$, i.e., $\langle A \rangle \subseteq \langle B \rangle$.

<u>Basic properties of closure</u>. The following two theorems present the most fundamental properties of closure.

Theorem 10: The closure of a complex of a group G is a unique subgroup of G.

Proof: Let S be a non-empty complex of a group G. Then $\langle S \rangle$ is the set of all finite products of the elements of $S_1 = S \cup S^{-1}$ by theorem 8. Hence $\langle S\rangle$ is a closed set, since the product of any two finite products is a finite product, i.e.,

$(x_{a_1}x_{a_2}\ldots x_{a_n})(x_{b_1}x_{b_2}\ldots x_{b_m}) = x_{a_1}x_{a_2}\ldots x_{a_n}x_{b_{n+1}}\ldots x_{b_{n+m}}$ $\in \langle S\rangle$. The identity element $e \in S$, since if $a \in S$, then $a^{-1} \in S^{-1}$, and $aa^{-1} = e \in S_1^2 \subseteq \langle S\rangle$. That every element of S has an inverse follows from the definition of $\langle S\rangle$, for if $a = x_1x_2\ldots x_n \in S_1^n$, then $x_n^{-1}x_{n-1}^{-1}\ldots x_1^{-1} = a^{-1} \in S_1^n$. The associativity follows from associativity of the multiplication defined for G. Hence $\langle S\rangle$ is a subgroup, and is unique by virtue of theorem 8.

The following theorem has been used in some works to define closure:

Theorem 11: The closure of a non-empty complex S is the smallest subgroup which contains S.[6:17]

Proof: Suppose $\exists$ a subgroup $S_1 \ni S \subseteq S_1$, and $\langle S\rangle \not\subseteq S_1$. Then $\exists$ an $a \in \langle S\rangle \ni a \notin S_1$. But a is of the form $x_1x_2\ldots x_n$, where $x_i \in S$ or $x_i \in S^{-1}$ by theorem 8. Therefore $a \in S_1$, since S_1 must contain S^{-1} and must be a closed set since its a subgroup.

We can see now that our definition of closure as restricted to a group does satisfy the more generalized definition below that is found in lattice theory.

An endomorphism $\phi(x) = \bar{x}$ of the partially ordered set P which maps an element x of P onto the element $\bar{x}$ of P is a closure operation of P if:

1. $x \subseteq y \Longrightarrow \bar{x} \subseteq \bar{y}$.
2. $x \subseteq \bar{x}$.
3. $\bar{\bar{x}} = \bar{x}$. [5:97]

Theorem 12: If H and K are non-empty complexes of a group, then $\langle H \cup K \rangle = \langle\langle H \rangle \cup \langle K \rangle\rangle$. [7:22]

Proof: Let $Y = \langle\langle H \rangle \cup \langle K \rangle\rangle = \bigcup_{i \in N} [\langle H \rangle \cup \langle K \rangle) \cup (\langle H \rangle \cup \langle K \rangle)^{-1}]^i$. Note that $(\langle H \rangle \cup \langle K \rangle)^{-1} = (\langle H \rangle^{-1} \cup \langle K \rangle^{-1}) = \langle H \rangle \cup \langle K \rangle$. Therefore $Y = \bigcup_{i \in N} (\langle H \rangle \cup \langle K \rangle)^i$. Suppose $\exists\ x \ni x \in Y$ and $x \notin \langle H \cup K \rangle$. Now x has the form $y_1 y_2 \ldots y_n$, where $y_i \in \langle H \rangle$ or $y_i \in \langle K \rangle$, i.e., $y_i \in H$ or H^{-1} or K or K^{-1}. But then each $y_i \in \langle H \cup K \rangle$, consequently $x \in \langle H \cup K \rangle$ and hence $Y \subseteq \langle H \cup K \rangle$. Also, since $H \subseteq \langle H \rangle$, $K \subseteq \langle K \rangle$, then $H \cup K \subseteq \langle H \rangle \cup \langle K \rangle$ by property 12 of sets and $\langle H \cup K \rangle \subseteq \langle\langle H \rangle \cup \langle K \rangle\rangle$ by theorem 9. Therefore $\langle H \cup K \rangle = \langle\langle H \rangle \cup \langle K \rangle\rangle$ by property 3 of sets.

It is worth noticing that the "dual" of this theorem, i.e., $\langle H \cap K \rangle = \langle\langle H \rangle \cap \langle K \rangle\rangle$ is not necessarily true, even if $H \cap K$ is non-empty. If $H = \{1,2,4\}$ and $K = \{1,3\}$, then from the lattice in the appendix we see that $\langle H \cap K \rangle = \{1\}$ and $\langle\langle H \rangle \cap \langle K \rangle\rangle = \{1,2,3\}$. If H and K are subgroups then the statement would be valid but trivial, since the intersection of two subgroups is a subgroup.

Definition 6: A non-empty complex S of a group G is a normal complex of G if and only if $x^{-1} S x \subseteq S$ for all

$x \in G$. If S is a subgroup of G, S is called a normal subgroup of G, denoted by $S \triangleleft G$. [2:22]

Normality Theorems. In this section it will be shown that normality is a property of complexes that is preserved by closure.

Theorem 13: The inverse of a normal complex is a normal complex, and the closure of a normal complex is a normal subgroup.

Proof: Let A be a normal complex. Let $a_i^{-1} \in A^{-1}$. But $a_i^{-1} \in A^{-1}$ if and only if $a_i \in A$, by definition of an inverse complex. Now for every $x \in G$ and every $a_i \in A$ $\exists$ $a_j \in A \ni x^{-1}a_ix = a_j$, since A is a normal complex. If $x^{-1}a_jx = a_j$, then $(x^{-1}a_jx)^{-1} = a_j^{-1} = x^{-1}a_i^{-1}x$. But if $a_j \in A$, then $a_j^{-1} \in A^{-1}$, hence A^{-1} is a normal complex. Let $a \in \langle A \rangle$. Then a is of the form $y_1y_2 \ldots y_n$, $y_i \in A \cup A^{-1}$, $i \in N$. Consider then $x^{-1}ax = x^{-1}(y_1y_2 \ldots y_n)x =$ $x^{-1}[y_1(xx^{-1})y_2(xx^{-1}) \ldots y_{n-1}(xx^{-1})y_n]x =$ $(x^{-1}y_1x)(x^{-1}y_2x) \ldots (x^{-1}y_nx) = y_{b_1}y_{b_2} \ldots y_{b_n} \in A$. Hence A is a normal subgroup.

Theorem 14: If $H \leqslant G$ and $K \triangleleft G$, then $\langle H \cup K \rangle = HK$. [3:63]

Proof: (a) HK is closed, i.e., if $h_1k_1, h_2k_2 \in HK$, then $h_1k_1h_2k_2 \in HK$, for, by letting $h_3 = h_1h_2$ so that $h_2 = h_1^{-1}h_3$, we have $h_1k_1h_2k_2 = h_1k_1(h_1^{-1}h_3)k_2 =$ $(h_1k_1h_1^{-1})h_3k_2 = k_3h_3k_2$, since $K \triangleleft G$, and $k_3h_3k_2 =$ $(h_3h_3^{-1})k_3h_3k_2 = h_3(h_3^{-1}k_3h_3)k_2 = h_3k_4k_2$, again since $K \triangleleft G$; but $h_3k_4k_2 = h_3k_5 \in HK$.

(b) Since H and K are groups, $hk \in HK$ only if $(hk)^{-1} = k^{-1}h^{-1} \in KH$. But if $hk \in HK$, then $(ek^{-1})(h^{-1}e) = k^{-1}h^{-1} \in HK$ by part a, hence $HK = KH$, and HK is a subgroup by virtue of theorem 7, and $\langle H \cup K\rangle \subseteq HK$ as a result of theorem 11. But $HK \subseteq (H \cup K)^2 \subseteq \langle H \cup K\rangle$. Therefore $\langle H \cup K\rangle = HK$.

It was demonstrated in the proof of theorem 12 that when working with the closure of the union of two subgroups that the definition of closure as applied could be modified from $\langle H \cup K\rangle = \bigcup_{i \in N}[(H \cup K) \cup (H \cup K)^{-1}]^i$ to $\langle H \cup K\rangle = \bigcup_{i \in N}(H \cup K)^i$. It might be assumed that a still less cumbersome definition of the closure of $H \cup K$ should be readily available, such as $\langle H \cup K\rangle = HK \cup KH$ or $(HK)^2$ or $(HK)(KH)$. While not denying the assumption, at least the above hypotheses are all false, as can be seen by considering the subgroups $H = \{1, 4\}$ and $K = \{1, 5\}$ in the group given in the appendix. The above hypotheses yield, respectively, $\{1, 4, 14, 15, 18\}$, $\{1,2,4,5,14,15,17,18\}$, and $\{1,4,14,15,17,18\}$, while $\langle H \cup K\rangle = \{1,2,3,4,5,6,13,14,15,16,17,18\}$.

Theorem 16: If Q, M, and P are normal subgroups of G such that $M \subseteq Q$, then $Q \cap \langle P \cup M\rangle = \langle (Q \cap P) \cup M\rangle$.

Proof: Since Q, M, and P are normal subgroups, the condition to be proven can be rewritten thusly: $Q \cap PM = (Q \cap P)M$. Now $Q \cap (PM)$ is the set of all $pm \ni pm = q$, or $p = qm^{-1}$. Since $M \subseteq Q$, $m^{-1} \in Q$ and hence

$p \in Q$ and $p \in Q \cap P$ and $pm \in (Q \cap P)M$. Conversely, $(Q \cap P)M$ is the set of all $qm \ni q \in Q$, $q \in P$, and $m \in M \subseteq Q$. Then $qm \in Q$ and $qm \in PM$, therefore $qm \in Q \cap PM$ and $Q \cap (PM) = (Q \cap P)M$. [5:18]

The above theorem is the modular condition from lattice theory and proves that the lattice of normal subgroups is modular.

<u>A theorem on Abelian subgroups</u>. The closure of the union of Abelian subgroups is not necessarily an Abelian subgroup, for an element from one subgroup may not commute with an element from another. Stronger conditions, however, yield the following theorem.

Theorem 17: If H and K are normal Abelian subgroups of G and $H \cap K = e$, then $\langle H \cup K \rangle$ is Abelian.

Proof: Since H and K are Abelian, it only needs to be shown that the elements of H and K commute with each other, i.e., that $h_i k_j = k_j h_i$, where $h_i \in H$, $k_j \in K$. Consider $h_i^{-1} k_j^{-1} h_i k_j$. $h_i^{-1}(k_j^{-1} h_i k_j) = h_i h_f \in H$, and $(h_i^{-1} k_j^{-1} h_i) k_j = k_f k_j \in K$, since H and K are normal subgroups. Therefore $h_i^{-1} k_j^{-1} h_i k_j \in H \cap K$, i.e., $h_i^{-1} k_j^{-1} h_i k_j = e$, and by multiplying on the left first by h_i and then by k_j we get $h_i k_j = k_j h_i$, i.e., $\langle H \cup K \rangle$ is Abelian.

Definition 10: A group G is a cyclic group if and only if there exists an element $g \in G$ such that $\langle g \rangle = G$.

<u>Theorems on cyclic groups and generators</u>. It can be proven that all subgroups of a cyclic group are cyclic.[6:35] Hence, we have as an immediate corollary the following:

Theorem 18: If H and K are subgroups of a cyclic group G, then $\langle H \cup K \rangle$ is cyclic.

That theorem 18 is not generally true for all groups G can be demonstrated by considering the subgroups $H = \{1,4\}$ and $K = \{1,13\}$ of the group in the appendix. Since $\langle 4 \rangle = H$, $\langle 13 \rangle = K$, H and K are cyclic subgroups. Now $\langle H \cup K \rangle = \{1,4,13,16\}$. But $\langle 1 \rangle = 1 = e$, $\langle 4 \rangle = H$, $\langle 13 \rangle = J$, and $\langle 16 \rangle = \{1,16\}$. Hence $\langle H \cup K \rangle$ is not cyclic.

Definition 11: A complex C of a group is independent if no element of C is contained in the closure of the remaining elements. If C is an independent complex of G such that $\langle C \rangle = G$, then C is an independent generator of G.

If G is the cyclic group of order 6 $\ni \langle g \rangle = G$, then all the independent generators of G, called the system of generators of G, are: $\{g\}$, $\{g^5\}$, $\{g^2, g^3\}$, and $\{g^3, g^4\}$. Notice that $g^6 = e$ is not in any independent generator of G, since the closure of any element will generate e.

Theorem 19: Let G' be the set containing those elements of G which are in the system of independent generators of a non-trivial group G. Then the elements of G not contained in G' form a subgroup H of G. [7:51]

Proof: (a) Since $G \neq e$, $e \in H$.

(b) If $x \in H$, $y \in H$, then $xy \in H$, for, supposing $\exists$ a complex $C \ni \{xy, C\}$ is an independent generator of G, it follows that $\langle\{x,y,C\}\rangle = G$; but $x \in H$ implies $\langle\{y,C\}\rangle = G$, and, since $y \in H$, $\langle\{C\}\rangle = G$, hence $xy \in H$.

(c) If $x \in H$ then it follows from $\langle\{x^{-1},C\}\rangle = G$ that $\langle\{x,C\}\rangle = G$, since $\langle\{x,C\}\rangle$ generate x^{-1}. But if $\langle\{x,C\}\rangle = G$, then $\langle C\rangle = G$, since $x \in H$. Hence $x^{-1} \in H$. The associativity of H follows from that of G, hence H is a subgroup of G. [7:51]

It can be proven that the subgroup H, defined in theorem 19, is a characteristic subgroup, and hence is normal. [7:51]

Definition 12: A homomorphism of a group G into a group H is a function T of G into H such that if $x \in G$ and $y \in G$, then $(xy)T = (x)T(y)T$. That T is a homomorphism of G into H will be denoted $T \in \text{Homo}(G,H)$. [5:24]

As an example of a homomorphism, let $G = \{1,2,3,4,5,6\}$ and $H = \{1,13\}$ from the group in the appendix.

Then $T \ni 1T = 1$
$2T = 1$
$3T = 1$
$4T = 13$
$5T = 13$
$6T = 13$ is a homomorphism of G onto H.

Homomorphism Theorems.

Theorem 20: If $T \in \text{Homo}(G,H)$, then $e_G T = e_H$, and if $xT = y$, then $(x^{-1})T = y^{-1}$.

Proof: Let $e_G T = m$ and $aT = n$, where $a \in G$. Then $(e_G a)T = (e_G T)(aT) = mn$. But $(e_G a)T = (a)T = n$, hence $mn = n$. Multiplying both sides on the right by n^{-1} yields $m = e_H$.

(b) Let $xT = y$, $(x^{-1})T = z$. Then $(xx^{-1})T = (x)T(x^{-1})T = yz$. But $(xx^{-1})T = (e_G)T = e_H$, hence $yz = e_H$ and $z = y^{-1}$.

Let us state the following two corollaries, each due solely to the definition of homomorphism:

Corollary 1: $(x_1 x_2 \ldots x_n)T = (x_1 x_2 \ldots x_{n-1})T x_n T = \ldots = x_1 T x_2 T \ldots x_n T$.

Corollary 2: $\{x_1, x_2, \ldots x_n\}T = \{x_1 T, x_2 T, \ldots x_n T\}$.

Theorem 21: If $T \in \text{Homo}(G,H)$ and J and K are complexes of G, then $\langle J \cup K \rangle T = \langle JT \cup KT \rangle$.

Proof: Now $\langle JT \cup KT \rangle = \bigcup_{i \in N} (JT \cup KT \cup J^{-1}T \cup K^{-1}T)^i = \bigcup_{i \in N} [(J \cup K \cup J^{-1} \cup K^{-1})T]^i = \bigcup_{i \in N} [(J \cup K \cup J^{-1} \cup K^{-1})^i T] = [\bigcup_{i \in N} (J \cup K \cup J^{-1} \cup K^{-1})^i]T = \langle J \cup K \rangle T$, the second, third and fourth equalities due to corollaries 2, 1, and 2, respectively.

Theorem 22: If H and K are complexes of the groups G and J, respectively, and if there exists $T \in \text{Homo}(G, J)$ such that $HT \subseteq K$, then $\langle H \rangle T \subseteq \langle K \rangle$.

Proof: Let $HT = L$, $L \subseteq K$. Then $\langle HT \rangle = \langle L \rangle$. Hence by theorem 21, $\langle H \rangle T = \langle HT \rangle = \langle L \rangle \subseteq \langle K \rangle$, the last inclusion following from theorem 8. But by theorem 10, $\langle H \rangle$ is a subgroup, thus $\langle H \rangle T \leqslant \langle K \rangle$.

Theorem 23: Let A and C be subgroups of G and B and D be subgroups of J. If there exists a $T \in \text{Homo}(G,J)$ such that $AT = B$ and $CT = D$, then $\langle A \cup C \rangle T = \langle B \cup D \rangle$.

Proof: By corollary 2, $(A \cup C)T = AT \cup CT$. Therefore $(A \cup C)T = AT \cup CT = B \cup D$, hence $\langle B \cup D \rangle = \langle AT \cup CT \rangle = \langle (A \cup C)T \rangle = \langle A \cup C \rangle T$.

An "obvious" converse of the above theorem is the following:

Let $A \leqslant G$, $C \leqslant G$, $B \leqslant J$ and $D \leqslant J$. If $\exists\, T \in \text{Homo}(G,J) \ni \langle A \cup C \rangle T = \langle B \cup D \rangle$ and $AT = B$, then $CT = D$. However, this converse is false. Even if we strengthen the conditions to make T a 1-1 homomorphism and A and B are normal subgroup subgroups, so that:

$\langle A \cup C \rangle T = (AC)T = BD$ and $AT = B$, CT does not have to equal D! The following example illustrates the preceeding discussion.

Let $\{G\} = \{J\}$ and $A = \{1, 2\}$, $B = \{1, 4\}$, $C = B$, and $D = \{1, 3\}$. Let T be the homomorphic mapping of G onto $J \ni$:

$GT = J$ and let G have the table:

$1T = 1$

$2T = 4$

$3T = 3$

$4T = 2$

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Then the above conditions are satisfied but $CT \neq D$.

The condition that $T \in$ Homo (G,J) in theorem 23 is very important, for it is possible to have even a 1-1 mapping $T \ni AT = B$ and $CT = D$ and still $\langle A \cup C \rangle T \neq \langle B \cup D \rangle$. To illustrate this fact, let $G = J$ be the group given in the appendix; let $A = \{1, 4\}$, $B = \{1, 13\}$, $C = D = \{1, 2, 3\}$, and T be the 1-1 homomorphic mapping of A onto B and C onto $D \ni$:

$1T = 1$

$2T = 2$

$3T = 3$

$4T = 13$. Note that T may not be $\in$ Homo(G,J). Now $\langle A \cup C \rangle = \{1, 2, 3, 4, 5, 6\}$, $\langle B \cup D \rangle = \{1, 2, 3, 13, 14, 15\}$, and their multiplication tables are:

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	3	1	6	4	5
3	3	1	2	5	6	4
4	4	5	6	1	2	3
5	5	6	4	2	3	1
6	6	4	5	3	1	2

	1	2	3	13	14	15
1	1	2	3	13	14	15
2	2	3	1	14	15	13
3	3	1	2	15	13	14
13	13	14	15	1	2	3
14	14	15	13	2	3	1
15	15	13	14	3	1	2

It may seem that with a "proper" extension of T, e.g., $5T = 14$, $6T = 15$, or $5T = 15$, $6T = 14$, that we could get $\langle A \cup C \rangle T = \langle B \cup D \rangle$. However, $\langle B \cup D \rangle$ is an Abelian group, while $\langle A \cup C \rangle$ is not, hence the hoped for extension of T cannot exist for the given subgroups, i.e., $T \not\in \mathrm{Homo}(G,J)$.

CHAPTER VI

SUMMARY

It has been demonstrated that the consideration of closure in group theory can add meaningfully to the development of the theory as well as providing a familiar foundation for a future study of lattice theory. It yields several important theorems to a discussion of normal subgroups and allows a very natural definition of cyclic groups. It has been shown that closure can be included in many facets of a presentation of the theory of groups, including the important homomorphism theorems.

It was the discussion which follows theorem 23 on homomorphisms, in fact, which led to the investigation which resulted in this paper. An isomorphism is a 1-1 homomorphism, and it seemed an easy assumption to make that the closures of the union of isomorphic subgroups should be isomorphic, i.e., if $A \cong B$ and $C \cong D$, then $\langle A \cup C \rangle \cong \langle C \cup D \rangle$. Upon finding that this assumption was invalid, a search was made for the necessary and sufficient conditions that would make it a valid hypothesis. Theorem 21 is a sufficient condition, but a condition that is both necessary and sufficient is still being sought.

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APPENDIX

MULTIPLICATION TABLE OF A TYPE 4 GROUP [1:20]

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	3	1	6	4	5	8	9	7	12	10	11	14	15	13	18	16	17	20	21	19	24	22	23
3	1	2	5	6	4	9	7	8	11	12	10	15	13	14	17	18	16	21	19	20	23	24	22
4	5	6	1	2	3	10	11	12	7	8	9	16	17	18	13	14	15	22	23	24	19	20	21
5	6	4	3	1	2	11	12	10	9	7	8	17	18	16	15	13	14	23	24	22	21	19	20
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19
7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	1	2	3	4	5	6
8	9	7	12	10	11	14	15	13	18	16	17	20	21	19	24	22	23	2	3	1	6	4	5
9	7	8	11	12	10	15	13	14	17	18	16	21	19	20	23	24	22	3	1	2	5	6	4
10	11	12	7	8	9	16	17	18	13	14	15	22	23	24	19	20	21	4	5	6	1	2	3
11	12	10	9	7	8	17	18	16	15	13	14	23	24	22	21	19	20	5	6	4	3	1	2
12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19	6	4	5	2	3	1
13	14	15	16	17	18	19	20	21	22	23	24	1	2	3	4	5	6	7	8	9	10	11	12
14	15	13	18	16	17	20	21	19	24	22	23	2	3	1	6	4	5	8	9	7	12	10	11
15	13	14	17	18	16	21	19	20	23	24	22	3	1	2	5	6	4	9	7	8	11	12	10
16	17	18	13	14	15	22	23	24	19	20	21	4	5	6	1	2	3	10	11	12	7	8	9
17	18	16	15	13	14	23	24	22	21	19	20	5	6	4	3	1	2	11	12	10	9	7	8
18	16	17	14	15	13	24	22	23	20	21	19	6	4	5	2	3	1	12	10	11	8	9	7
19	20	21	22	23	24	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
20	21	19	24	22	23	2	3	1	6	4	5	8	9	7	12	10	11	14	15	13	18	16	17
21	19	20	23	24	22	3	1	2	5	6	4	9	7	8	11	12	10	15	13	14	17	18	16
22	23	24	19	20	21	4	5	6	1	2	3	10	11	12	7	8	9	16	17	18	13	14	15
23	24	22	21	19	20	5	6	4	3	1	2	11	12	10	9	7	8	17	18	16	15	13	14
24	22	23	20	21	19	6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13

LATTICE OF A TYPE 4 GROUP 1:20