ON BOUNDS FOR EIGEN-VALUES OF NON-NEGATIVE MATRICES

511

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Inderjit Singh

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Approved for the Major Department

J. Stanley Laughlin

Approved for the Graduate Council

 Truman Hayes

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CHAPTER I

THE PROBLEM AND DEFINITIONS

1.1 STATEMENT OF THE PROBLEM

The purpose of this thesis was to study bounds of eigen-values of non-negative matrices. This study extended non-negative matrices to a special class of non-negative matrices and certain theorems of this class were introduced.

1.2 DEFINITION OF TERMS

For any nxn matrix, A, $\phi(\lambda)= \det(A-\lambda I_n)$, is called characteristic equation of A. The values of λ satisfying the equation $\phi(\lambda)= \det(A-\lambda I_n)=0$ are called eigen-values of A. Eigen-values of a matrix are bounded if there exist real constants s_1 and s_2 such that $s_2 \leq |\lambda| \leq s_1$, for all values of λ. ($|\lambda|$ = modulus of λ if λ is complex = absolute of λ if λ is real = $\sqrt{a^2 + b^2}$ where $\lambda = a + bi$). s_1 and s_2 are called bounds for the eigen-values of the matrix A. A matrix is non-negative if all its elements are non-negative.

1.3 ORGANIZATION OF STUDY

The next section of this chapter deals with basic definitions and terms to be used in succeeding chapters. Chapter II contains basic information about eigen-values of general matrices. Effects of certain operations on eigen-values of these matrices are also shown. Many results and theorems on bounds for eigen-values of general matrices are stated with an extensive study of eigen-values of non-negative matrices in

Chapter III. All results are proved and examples are given. Bounds for eigen-values of a special class of non-negative matrices are discussed in Chapter IV. Chapter V contains the concluding remarks of this study.

1.4 INTRODUCTION

Most of the definitions and terms will be defined in the chapter in which they are used. Before proceeding to discuss eigen-values of matrices, certain elementary ideas about matrices are needed. A matrix is a rectangular array of numbers or functions. A matrix A, also denoted by (a_{ij}), is a square matrix if it has the same number of columns as rows. Unless otherwise stated matrices discussed in this paper are square matrices. A matrix A is over a field F if all of its elements are members of the field F. Unless otherwise stated, matrices will be assumed to be over the complex field.

Let A be a matrix of order n

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Consider the equation $Ax = \lambda x$, where λ is a scalar, A is a matrix and x is a non-zero vector. $Ax - \lambda x = 0$; $(A - \lambda I_n)$ $x = 0$; where I_n is identity matrix of order n. $(A - \lambda I_n)\ x = 0$

is a homogeneous system of equations (The system of equations in n unknowns $\sum_{j=1}^{n} a_{ij} x_j = 0$, ($i = 1, 2, \ldots, n$,) or in matrix notation, $Ax = 0$ is called a system of homogeneous linear equations). This homogeneous system of linear equations has non-trivial solutions if and only if $\det (A - \lambda I_n) = 0$.

Written in matrix form:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & a_{3n} \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$\phi(\lambda) = \det (A - \lambda I_n) = 0$, is called the characteristic equation of A. All values of λ satisfying the equation $\det (A - \lambda I_n) = 0$ are called eigen-values of A. The vectors x satisfying $Ax = \lambda x$, for these values of λ are called eigen-vectors of A.

The vector x (which is a column matrix) satisfying the equation $Ax = \lambda_1 x$, is called the corresponding eigen-vector of λ_1. If $Ax = \lambda x$, gives $x = 0$ for all values of λ, then A is said to have a trivial solution. If $Ax = \lambda x$ gives non-zero vectors for at least one value of λ then the solution is non-trivial. There are three elementary operations used to simplify a matrix:

(i) Interchange of any two parallel lines.

(ii) Multiplying any line with a non-zero constant.

(iii) Addition of a scalar multiple of one line to a parallel line.

It must be noted that the third type of operation does not change the value of the determinant of the matrix. A matrix A is said to be non-singular if its determinant is not equal to zero. Otherwise it is said to be singular. A matrix is said to be of order n if it has n rows and n columns. A matrix is said to be of rank r if and only if it has at least one non-singular submatrix of order r. The three elementary transformations applied to a matrix A result in a matrix of the same order and of the same rank.

CHAPTER II

GENERAL MATRICES AND BOUNDS FOR THEIR EIGEN-VALUES

General matrices and basic properties of general matrices concerning eigen-values are discussed in this chapter. Information about eigen-values of special matrices are given. Many theorems about bounds for eigen-values of general matrices are stated and all localization theorems are discussed in detail as another approach to bounds for eigen-values of general matrices.

2.1.1 Let A be any nxn matrix whose expanded form is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

and let $\phi(\lambda)$ be defined as an expanded form of a characteristic equation as:

$$\phi(\lambda) = \det(A - \lambda I_n) = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

It easily follows from properties of determinants that a matrix A and transpose of A have the same eigen-values. It has been further stated that multiplication of the elements of a column (row) of A by a non-zero constant and division of elements of the corresponding row (column) by the same constant, leaves the eigen-values of the matrix A unchanged.

2.1.2 RANK AND EIGEN-VALUES

The rank of a matrix has definite effect on its eigen-values. If a matrix A is non-singular then obviously A has only non-zero eigen-values and A has only trivial solution. $\lambda = 0$ is an eigen-value of A if and only if A is singular.

2.1.3 THEOREM

Let A be of rank r and of order n, then A has at least $n - r$ zero eigen-values.

2.1.4 THEOREM

The equation $Ax = \lambda x$ has non-trivial solution x if and only if λ is an eigen-value of A. There exists at least one value of λ and corresponding non-zero x such that this equation is satisfied.

Definitions: A matrix is real if all its elements are real numbers. A matrix A is symmetric if the transpose of A is A itself (i.e. $A^T = A$). Matrix A is a Hermitian matrix if A is transposed and conjugated is still A itself (i.e. $A^* = A$).

2.1.5 SPECIAL MATRICES

There is quite a lot of information about eigen-values of special matrices. Diagonal and triangular matrices exhibit their eigen-values on the main diagonal. All eigen-values of a nilpotent matrix (i. e. $A^b = 0$ for some integer b) are equal to zero. While all those of idempotent matrices (i. e. $A^2 = A$) are equal to 0 or 1. All eigen-values of a unitary matrix (i. e. $A^* = A^{-1}$) lie on the unit circle in the complex plane. Eigen-values of Hermitian matrices lie on real axis while those of skew Hermitian lie on imaginary axis. Eigen-values of a real symmetric matrix are real. A^{-1} is called the multiplicative inverse of A (i. e. $AA^{-1} = 1$).

2.1.6 THEOREM

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct or not distinct eigen-values of A then the eigen-values of A^{-1} (if A^{-1} exists) are $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$.

2.1.7 THEOREM

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen-values of A then eigen-values of Ak are $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ where k is a constant.

2.2.1 BOUNDS FOR MAXIMAL EIGEN-VALUES OF GENERAL MATRICES

As far as eigen-values of a general matrix are concerned nothing specific can be said about their bounds and location in the complex plane. They can obviously lie anywhere in the complex plane. However eigen-values of a matrix can be thought of in terms of simple function of its elements. Therefore bounds on eigen-values of a matrix depend on the elements of the matrix. Bounds, of course, are real constants. Most of the results on bounds of general matrices, in this section are stated and not proved since their importance is historical.

2.2.2 HIRSCH'S THEOREM

If $A = (a_{ij})$ is an nxn matrix and $r = \max. (|a_{ij}|)$ then $|\lambda| \leqq nr$ for all values of λ (where λ is an eigen-value of A).

2.2.3 SCHUR'S INEQUALITY

If $A = (a_{ij})$ is any nxn matrix with eigen-values λ_p, ($p = 1, 2, \ldots, n$), then:

$$\sum_{p=1}^{n} |\lambda_p|^2 \leq \sum_{i=1}^{n} |a_{ii}|^2$$

Let $R_i = \sum_{j=1}^{n} |a_{ij}| \qquad (i = 1, 2, \ldots, n)$

$C_j = \sum_{i=1}^{n} |a_{ij}| \qquad (j = 1, 2, \ldots, n)$

$R = \max. \quad (R_i)$

$C = \max. \quad (C_j)$

2.2.4 FROBENIUS THEOREM

Frobenius proved that if λ is any eigen-value of A and R and C as defined above, then:

$$\lambda \leqq \min. (R,C) \qquad \left[\text{e.g. } \min. \ (3,5) = 3 \right]$$

2.2.5 PERRON'S THEOREM

If $C_1, C_2, C_3, \ldots, C_n$ are any positive, real numbers and R is the greatest of n numbers,

$$R_r = \frac{C_1 \left|a_{r1}\right| + C_2 \left|a_{r2}\right| + \ldots + C_n \left|a_{rn}\right|}{C_r}$$

$$(r = 1, 2, \ldots, n)$$

then, $\left|\lambda\right| \leqq R$.

2.3.1 LOCALIZATION OF EIGEN-VALUES OF GENERAL MATRICES

Localization of the eigen-values of a matrix in the complex plane is another approach to the problem of finding bounds for the eigen-values of the matrix. The thing of interest here is a curve bounding the region that contains all eigen-values of the matrix. Here is a theorem which is called the Hadamard Theorem, Minkowski Theorem and Levy-Desplanque Theorem.

Define: $P_i = \sum_{\substack{j=1 \\ i \neq j}}^{n} \left|a_{ij}\right| \quad (i, j = 1, 2, \ldots, n).$

2.3.2 LEVY-DESPLANQUE THEOREM

If $A = (a_{ij})$ is a complex n-square matrix and $\left|a_{ii}\right| > P_i$ for $i = 1, 2, \ldots, n$, then $\det (A) \neq 0$.

Proof: Suppose $\det (A) = 0$. Then $Ax = 0$ has non-trivial solutions (A Theorem). Let $x = (x_1, x_2, \ldots, x_n)$ be a solution. There exists r for which $\left|x_r\right| \geqq \left|x_i\right|$ for all i. Since $Ax = 0$, $\sum_{j=1}^{n} a_{rj}\, x_j = 0$. $\sum_{j=1}^{n} a_{rj}\, x_j = 0$ or $\sum_{\substack{j=1 \\ j\neq r}}^{n} a_{rj}\, x_j + a_{rr}\, x_r = 0$.

$$a_{rr}\, x_r = -\sum_{\substack{j=1 \\ j\neq r}}^{n} a_{rj}\, x_j$$

$$\left|a_{rr}\right| \left|x_r\right| = \left| - \sum_{j\neq r}^{n} a_{rj}\, x_j \right| = \left| \sum_{j\neq r}^{n} a_{rj}\, x_j \right|$$

$$\leq \sum_{j\neq r}^{n} |a_{rj}| |x_j| \leq \sum_{j\neq r}^{n} |a_{rj}| |x_r| = |x_r| |P_r|$$

$|a_{rr}| \leq P_r$ which is a contradiction, (det (A) $\neq$ 0).

Q. E. D.

2.3.3 GERSGORIN'S THEOREM

The eigen-values of an n square complex matrix A, lie in the closed region of the z plane consisting of all circular discs $|z - a_{ii}| \leq P_i$ $(i = 1, 2, \ldots, n)$ ... (i).

Proof: Let λ_p be an eigen-value of A. det ($A-\lambda_p I_n$) = 0. Following the proof of Levy-Desplanque Theorem:

$|a_{ii} - \lambda_p| \leq P_i$ for at least one i

or $|\lambda_p - a_{ii}| \leq P_i$ for at least one i

Since λ_p is an arbitrary eigen-value of A, this in-equality is true for all eigen-values of A.

All eigen-values are contained in the union of n circular discs: $|z - a_{ii}| \leq P_i$ (i = 1, 2, 3, ... , n).

Definition: A matrix is irreducible if it can not be brought to the form $\left[\begin{array}{c|c} K & O \\ \hline L & M \end{array}\right]$ by simultaneous row and column permutations. Otherwise it is called reducible. If a matrix A, is irreducible, then all eigen-values of A lie inside the union of n circular discs of $|z - a_{ii}| \leq P_i$ (i = 1, 2, 3, ... , n).

2.3.4 OVALS OF CASSINI

Let A = (a_{ij}) be a matrix of order n with real or complex elements and P_k as defined before. Each eigen-value w of A lies in the interior or on the boundary of at least one of the $\frac{n(n-1)}{2}$ ovals of Cassini:

$$\left|z - a_{kk}\right| \left|z - a_{\lambda\lambda}\right| \leqq P_k P_\lambda \quad (\lambda, k = 1, 2, \ldots, n; k \neq \lambda)$$

... (i).

Proof: Since w is an eigen-value of A, Ax = wx or $\sum a_{k\lambda} x_\lambda = wx_k$ (k = 1, 2, ..., n) ... (ii) has non-trivial solution (Theorem) ($x_1, x_2, x_3, \ldots, x_n$).

Suppose $\left|x_r\right| \geqq \left|x_s\right| \geqq \max. (\left|x_v\right|)$ for (v = 1, 2, ..., n; $v \neq r$, $v \neq s$) consider the r-th equation in (ij)

$$\sum_{v=1}^{n} a_{rv} x_v = w x_r$$

$$\text{or } \sum_{v=1}^{n} a_{rv} x_v - a_{rr} x_r = wx_r - a_{rr} w_r$$

$$\text{or } \sum_{\substack{v=1 \\ v \neq r}}^{n} a_{rv} x_v = (w - a_{rr}) \quad x_r \ \ldots \text{(iv)}$$

Similarly s-th equation can be written as

$$\sum_{\substack{v=1 \\ v \neq s}}^{n} a_{sv} x_v = (w - a_{ss}) \quad x_s \ \ldots \text{(v)}$$

If $x_s = 0$, then also $x_v = 0$ ($x_v = 0$ for all v) for every $v \neq r$. It follows from (iv) that $w = a_{rr}$. Since $x_r \neq 0$, therefore w lies in the oval $\left|z - a_{vv}\right| \left|z - a_{ss}\right| \leqq P_r P_s$ ... (vi). This proves the theorem if $x_s = 0$.

Now suppose $x_s \neq 0$. By multiplying (iv) and (v) :

$$(w - a_{vv})(w - a_{ss}) x_r x_s = \Big(\sum_{\substack{v=1 \\ v \neq r}}^{n} a_{rv} x_v\Big) \Big(\sum_{\substack{v=1 \\ v \neq s}}^{n} a_{sv} x_v\Big).$$

Hence $\left|w - a_{rv}\right| \left|w - a_{ss}\right| x_r x_s \leqq \left|x_r\right| \sum_{\substack{v=1 \\ v \neq r}}^{n} a_{rv}$.

$$\left(\sum_{\substack{v=1\\ v\neq r}}^{n} a_{sv}\right) = |x_r|\,|x_s|\,|P_r|\,|P_s| \cdot |w - a_{rr}|\,|w - a_{ss}| \leq |P_r|\,|P_s| =$$

$P_r\ P_s$. Therefore w lies in the oval $|z - a_{rr}|\,|z - a_{ss}| \leq P_r\ P_s$

Q. E. D.

Every point of the oval (vi) lies in at least one of the circular discs, $|z - a_{rr}| \leq P_r$ and $|z - a_{ss}| \leq P_s$.

In other words the union circular discs contains the union of ovals and therefore this theorem of ovals of Cassini is an improvement over Gersgorin's Theorem.

Example:

$$\text{Let } A = \begin{bmatrix} 7 + 3i & -4 - 6i & -4 \\ -1 - 6i & 7 & -2 - 6i \\ 2 & 4 - 6i & 13 - 3i \end{bmatrix}, \text{ be the matrix.}$$

$$\phi(\lambda) = \begin{bmatrix} 7 + 3i - \lambda & -4 - 6i & -4 \\ -1 - 6i & 7 - \lambda & -2 - 6i \\ 2 & 4 - 6i & 13 - 3i - \lambda \end{bmatrix} = 0$$

which gives eigen-values as 9, 9 + 9i and 9 - 9i. These Modulii are 9 and 12.73.

The bounds given by:

Hirsch's Theorem: $|\lambda| \leq 40.03$

Schur's Inequality: $\sum_{i=1}^{3} |\lambda_i|^2 = 405 \leq \sum |a_{ij}|^2 = 486$

Frobenius Theorem: 23.10

Perron's Theorem: 22.55

Gersgorin's Discs: $|z - 7 - 3i| \leqq 11.21$, $|z - 13 + 3i| \leqq 9.21$

contain two eigen-values each.

2.4 SUMMARY

General matrices and their eigen-values were discussed in this chapter. Several theorems stated on bounds give a feel for the problems going to be discussed in the next chapters. Localization theorems are discussed in detail as another approach to the problem of finding bounds for eigen-values of matrices.

CHAPTER III

BOUNDS FOR MAXIMAL EIGEN-VALUE OF NON-NEGATIVE MATRICES

This chapter is exclusively concerned with non-negative matrices and bounds for its eigen-values. Many results have been proven about bounds mainly by Frobenius, Ledermann, Ostrowski and Alfred Braur. This chapter gives an extensive study of their results and proofs. Each result is exemplified taking different matrices. Unless otherwise stated all matrices considered in this chapter are non-negative.

Definition: An eigen-value, r of a matrix A, is called a maximal eigen-value of A if $r \geqq |\lambda|$, for all eigen-values, λ of A.

Definition: $S = \max. (S_i)$, $s = \min. (S_i)$

S_i defined before as the sum of absolutes of elements in ith row of A.

Alfred Braur has proved that every positive matrix has at least one positive eigen-value. And in another theorem he has proven that the co-ordinates of an eigen-vector belonging to the maximal eigen-value of the positive matrix can be chosen as positive numbers.

3.1 FROBENIUS THEOREM:

$s \leqq r \leqq S$ where r is maximal eigen-value of a matrix, A.

Proof: (i) Let $x = (x_1, x_2, \ldots, x_3)$ be the positive eigen-vector corresponding to eigen-value r and A be positive matrix by standard continuity argument.

Suppose x_p = max. (x_i) and x_* = min. (x_i). $Ax = rx$ or $\sum_{j=1}^{n} a_{ij}\ x_j = rx_j$ (i = 1, 2, ..., n). Considering p-th row:

$$rx_p = \sum_{j=1}^{n} a_{pj}\ x_j \leq \sum_{j=1}^{n} a_{pj}\ x_p = x_p\ S_p$$

$$rx_p \leqq x_p\ S_p \qquad \therefore\ r \leqq S_p$$

since $S_p \leqq S$ by definition

$r \leqq S$

(ii) Again $\sum_{j=1}^{n} a_{ij}\ x_j = rx_j$ (i = 1, 2, ..., n)

Consider $*$ th row

$$rx_* = \sum_{j=1}^{n} a_{*j}\ x_j \geqq \sum_{j=1}^{n} a_{*j}\ x_* = x_*\ S_*$$

$$rx_* \geqq x_*\ S_* \qquad r \geqq S_*$$

$r \geqq s$

From (i) and (ii) $s \leqq r \leqq S$

Q. E. D.

Example: Let A be the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 5 & 7 \\ 7 & 0 & 3 & 4 & 3 \\ 2 & 3 & 6 & 7 & 1 \\ 2 & 4 & 5 & 1 & 1 \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$

Row sums: $\{15, 17, 19, 13, 5\}$ $s = 5$, $S = 19$.

Frobenius Theorem: $5 \leqq r \leqq 19$

Column Sums: $\{11, 9, 18, 18, 13\}$

By Frobenius Theorem: $9 \leqq r \leqq 18$ using columns and using both rows and columns: $9 \leqq r \leqq 18$.

Definition: A non-negative matrix is row Stochastic if all row-sums of A equal 1.

For non-negative matrices, the result of the Frobenius Theorem seems to be the best possible result. It is noticed that if A is row Stochastic matrix then $s = S = r = 1$.

3.2 LEDERMANN'S THEOREM

If $m = \min. (a_{ij})$ and $\rho = \max\limits_{S_i < S_j} . (\frac{S_i}{S_j})$ and not all S_i are equal then:

$$s + m \left(\frac{1}{\sqrt{\rho}} - 1\right) \leqq r \leqq S - m\left(1 - \sqrt{\rho}\right).$$

Proof: Let x be corresponding positive eigen-vector to an eigen-value r. $x = (x_1, x_2, \ldots, x_n)$. Obviously not all x_i are equal.

Let $x_p = \max. (x_i)$ and $x_* = \min. (x_i)$.

$$Ax = rx$$

$$\sum_{j=1}^{n} a_{ij} x_j = rx_i \quad (i = 1, 2, \ldots, n)$$

Considering p-th row

$$rx_p = \sum_{j=1}^{n} a_{pj} x_j \leqq \sum_{j=1}^{n} a_{pj} x_p = x_p S_p$$

$$r \leqq S_p$$

As proved in the Frobenius Theorem $r \geqq S_*$

$$S_* \leqq r \leqq S_p$$

or $$\frac{S_*}{S_p} < 1$$

$$rx_* = \sum_{j=1}^{n} a_{*j}\, x_j < x_p\, S_* \qquad \text{(i)}$$

$$rx_p = \sum_{j=1}^{n} a_{pj}\, x_j > x_*\, S_p \qquad \text{(ii)}$$

From (i) and (ii)

$$\frac{x_*}{x_p} < \frac{x_p\, S_*}{x_*\, S_p}$$

or $$\left(\frac{x_*}{x_p}\right)^2 < \frac{S_*}{S_p}$$

or $$\frac{x_*}{x_p} < \sqrt{\frac{S_*}{S_p}} \leqq \sqrt{\rho} \qquad \text{(iii)}$$

$$r = \sum_{j=1}^{n} \frac{a_{pj}\, x_j}{x_p} \qquad \left[\text{using (ii)}\right]$$

or $$r = \left(\frac{a_{p1}\, x_1 + a_{p2}\, x_2 + \dots + a_{pn}\, x_n}{x_p}\right) = \left(\frac{a_{p1}\, x_1}{x_p} + a_{p2}\, \frac{x_2}{x_p} + \right.$$

$$\left. \dots + \frac{a_{p*}\, x_*}{x_p} + \frac{a_{pp}\, x_p}{x_p} + \dots + \frac{a_{pn}}{x_p}\right)$$

Looking at $\frac{x_i}{x_p}$, it is found that $\frac{x_i}{x_p} \leqq 1$ for all n ($i = 1, 2, \dots, n$).

$$r \leqq a_{p1}\frac{x_1}{x_p} + a_{p2}\frac{x_2}{x_p} + \ldots + a_{p*}\sqrt{\rho} + \ldots + a_{pn}\frac{x_n}{x_p}$$

$$r \leqq (a_{p1} + a_{p2} + \ldots + a_{pn}) + a_{p*}\sqrt{\rho} - a_{p*}$$

$$r \leqq S_p - a_{p*} (1 - \sqrt{\rho})$$

$$r \leqq S - m (1 - \sqrt{\rho})$$

Similarly

$$r = \sum_{j=1}^{n} \frac{a_{*j}\, x_j}{x_*}$$

$$r \geqq (a_{*1} + a_{*2} + \ldots + a_{*p}\frac{x_p}{x_*} + \ldots + a_{*n})$$

$$r \geqq (a_{*1} + a_{*2} + \ldots + a_{*n}) + a_{*p}\frac{1}{\sqrt{\rho}} - a_{*p}$$

$$r \geqq S_* + a_{*p}\left(\frac{1}{\sqrt{\rho}} - 1\right) \geqq S_* + m\left(\frac{1}{\sqrt{\rho}} - 1\right).$$

$$r \geqq s + m\left(\frac{1}{\sqrt{\rho}} - 1\right)$$

$$s + \left(m\frac{1}{\sqrt{\rho}} - 1\right) \leqq r \leqq S - m (1 - \sqrt{\rho}) \qquad \text{Q. E. D.}$$

Example: Let A be the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

$s = 4$, $S = 9$ (rows)

$s = 5$, $S = 8$ (columns)

$m = 1$

$S_i = \{6, 4, 9\}$, $S_j = \{5, 6, 8\}$

$\sqrt{\rho} = \sqrt{\frac{4}{5}} = .89$, $m = 1$

Ledermann's Theorem: $4.13 \leqq r \leqq 8.89$ (rows)

$5.13 \leqq r \leqq 7.89$ (columns)

Except in the case of Stochastic or generalized Stochastic matrices, Ledermann's result gives better bounds for r since $(1 - \sqrt{\rho})$ is always positive. If the matrix is non-negative then Ledermann's result is reduced to that of Frobenius. Ostrowski further improved the results of Ledermann for positive matrices.[1]

3.3 OSTROWSKI'S THEOREM

If $A > 0$ and $s < S$ then,

$$s + m \left(\frac{1}{\delta} - 1 \right) \leq r \leq S - m (1 - \delta)$$

$$\text{where } \delta = \sqrt{\frac{s-m}{S-m}}$$

and s, S, m and other notations as defined before.

Proof: Suppose r is maximal eigen-values of A and $(x_1, x_2, \ldots, x_n)$, is the corresponding eigen-vector. For simplicity assume that $1 = x_1 \geqq x_2 \geqq \ldots \geqq x_n$. This can be achieved by pre-multiplication of A by a permutation matrix and post-multiplication by its inverse. Let k and t be any subscripts.

$$rx_k = Ax_k = \sum_{j=1}^{n} a_{kj}\, x_j = a_{k1}\, x_1 + a_{k2}\, x_2 + \ldots + a_{kn}\, x_n$$

$$rx_k = a_{k1} + a_{k2}\, x_n + a_{k3}\, x_n + \ldots + a_{kn}\, x_n = a_{k1} + x_n \sum_{j=2}^{n} a_{kj}$$

$$rx_k = a_{k1} + (S_k - a_{k1})\, x_n = x_n\, S_k + (1 - x_n)\, a_{k1}$$

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[1]Walter Ledermann, "Bounds for the Greatest Latent Root of a Positive Matrix," Journal of London Mathematical Society, XXV (1950), pp. 265-268.

$$r \geqq \frac{x_n S_k + (1 - x_n) m}{x_k} \qquad (I)$$

$$rx_t = \sum_{j=1}^{n} a_{tj} x_j = a_{t1} x_1 + a_{t2} x_2 + \ldots + a_{tn} x_n$$

$$\leqq (a_{t1} + a_{t2} + \ldots + a_{tn-1}) + a_{tn} x_n = \sum_{j=1}^{n} a_{tj} + a_{tn} -$$

$$a_{tn} + x_n a_{tn}$$

$$\leqq S_t - (1 - x_n) a_{tn}$$

therefore $r \leqq \dfrac{S_t - (1 - x_n) m}{x_t}$ (II)

In particular is $S_k = S$ and $S_t = s$, then results (I) and (II) will yield

$$r \geqq \frac{x_n S + (1 - x_n) m}{x_k} \geqq x_n S + (1 - x_n) m$$

and $r \leqq \dfrac{s - (1 - x_n) m}{x_t} \leqq \dfrac{s - (1 - x_n) m}{x_n}$

or $r \leqq \dfrac{(s-m)}{x_n} + m$

Therefore $x_n (S - m) + m \leqq r \leqq \dfrac{s-m}{x_n} + m$

or $x_n (S - m) \leqq r - m \leqq \dfrac{s-m}{x_n}$

or $x_n^2 \leqq \dfrac{s-m}{S-m}$ and therefore $x_n \leqq \sqrt{\dfrac{s-m}{S-m}} = \delta$

Putting $k = n$ and $t = 1$ in (I) and (II):

$$S_n + (\frac{1}{x_n} - 1) m \leqq r \leqq S_1 - (1 - x_n) m$$

and $s + (\frac{1}{\delta} - 1) m \leqq r = S - (1 - \delta) n$

Ostrowski's result is sharper than that of Ledermann since:

$$\frac{s-m}{S-m} \leqq \frac{s}{S} \leqq \max. \frac{S_i}{S_j}\,, \quad \text{where } S_i < S_j$$

therefore $\frac{s-m}{S-m} \leqq \max. \frac{S_i}{S_j}$

$$\delta^2 \leqq \rho \qquad \text{since } \delta \leqq 1,\ \rho \leqq 1$$

$$\frac{1}{\delta} = \frac{1}{\sqrt{\rho}}$$

therefore $s + m\left(\frac{1}{\delta} - 1\right) \geqq s + m\left(\frac{1}{\sqrt{\rho}} - 1\right)$ which is a better bound. Similarly

$S - (1 - \delta)\, m \leqq S - (1 - \sqrt{\rho})$ which also is a better bound. Therefore Ostrowski's result is sharper than that of Ledermann.

Example: Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$s = 3,\ S = 6,\ m = 1$ (rows)

$s = 4,\ S = 6,\ m = 1$ (columns)

$$\delta = \frac{s-n}{S-n}$$

$3.59 \leqq r \leqq 5.63$ (rows)

$4.48 \leqq r \leqq 5.77$ (columns)

Alfred Braur improved Ostrowski's results over bounds for maximal root of a positive matrix.[2]

3.4 BRAUR'S THEOREM

If $A = (a_{ij}) > 0$ and r, s, S, and m as defined earlier then:

$$s + m\,(h - 1) \leqq r \leqq S - m\,(1 - \frac{1}{g})$$

where $$g = \frac{S - 2m + \sqrt{S^2 - 4m\,(S - s)}}{2\,(s - m)},$$

$$h = \frac{-\,s + 2m + \sqrt{s^2 + 4m\,(S - s)}}{2m}$$

Proof: Assume without loss of generality that $S_1 = S$ and $S_n = s$. Let B be the matrix obtained from A by multiplying the elements in the last row of A by g and those in the last column of A by $\frac{1}{g}$ so that the last row sum of B is the smallest row sum. Then obviously A and B are similar and have the same eigen-values. Then i th row of B, (i = 1, 2, 3, ..., n-1), is

$$\sum_{j=1}^{n} a_{ij} - a_{in} + \frac{a_{in}}{g} = S_i - a_{in}\left(1 - \frac{1}{g} \right) \leqq$$

$$\leqq S - m\left(1 - \frac{1}{g} \right) = K_1 \quad \text{(say)}$$

The n th row sum of B is equal to

$$\sum_{j=1}^{n} a_{nj}g - a_{nn}g + a_{nn} = g\,s - a_{nn}\,(g - 1) \leqq g\,s - m\,(g - 1) =$$

K_2 (say)

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[2] A. Ostrowski, "Bounds for the Greatest Latent Root of a Positive Matrix," Journal of London Mathematical Society, XXVII (1952), pp. 253-256.

g as defined in the statement of this theorem.

It will now be proved that $K_1 = K_2$

$$g = \frac{S - 2m + \sqrt{(S - 2m)^2 + 4m\ (s - m)}}{2\ (s - m)}$$

which gives $g^2\ (s-m) - (S-2m)\ g - m = 0$ $\left[\because\ ag^2 + bg + c = 0 \quad g = \frac{-b - \ b^2 - 4ac}{2a} \right]$

$$Sg - mg + m = g^2 s + mg - mg^2$$

$$S - m\ (1 - \frac{1}{g}) = gs - m\ (1 - g)$$

$$K_1 = K_2$$

Therefore for this value of g all the row sums of B are bounded by $S - m\ (1 - \frac{1}{g})$ and using Frobenius's Theorem:

$$r \leqq S - m\ (1 - \frac{1}{g})$$

In order to obtain the lower bound one might construct a matrix C by dividing the elements in the first row of A by h and multiplying those in the first column by h so that the first row sum is the greatest row sum. The first row sum of C is $(a_{11} + \frac{a_{12}}{h} + \frac{a_{13}}{h} + \ldots + \frac{a_{1n}}{h}) = \sum_{j=1}^{n} \frac{a_{1j}}{h} - \frac{a_{11}}{h} + a_{11} =$

$\frac{S}{h} + a_{11}\ (1 - \frac{1}{h}) \geqq \frac{S}{h} + m\ (1 - \frac{1}{h}) = K_3$ (say)

The i-th row sum of C, ($i = 2, 3, \ldots, n$), is

$\sum_{j=1}^{n} a_{ij} - a_{i1} + a_{i1}h = S_i + a_{i1}\ (h-1) \geqq s + m\ (h-1) = K_4$ (say)

When h has the value given in the statement of this theorem

$$h = \frac{-s + 2m + \sqrt{s^2 + 4m\,(S-s)}}{2m}$$

it easily follows that $K_3 = K_4$

$$\left[\because \quad h^2 m + (s-2m)\,h + m - s = 0 \quad \text{which gives} \quad \frac{s}{h} + m\left(1 - \frac{1}{h}\right) = s + m\,(h-1)\right]$$

Thus all of the row sums of C are bounded below by $s + m\,(h-1)$ and again by the Frobenius Theorem:

$$r \geqq s + m\,(h-1)$$

Therefore $s + m\,(h-1) \leqq r \leqq S - m\left(1 - \frac{1}{g}\right)$

Q. E. D.

Example: Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$s = 3,\ S = 6,\ m = 1$ (rows)

$s = 4,\ S = 6,\ m = 1$ (columns)

$h = 1.78,\ g = 2.23$ (rows)

$h = 1.45,\ g = 1.55$ (columns)

Therefore by the Braur's Theorem:

$3.78 \leqq r \leqq 5.31$ (rows)

$4.45 \leqq r \leqq 5.64$ (columns)

Let P and Q for which it is assumed that the numbers $S > s \geqq nm > 0$ are prescribed and whose maximal eigen-values attain the upper and lower bounds in the inequality.

$$s + m\,(h-1) \leqq r \leqq S - m\left(1 - \frac{1}{g}\right).$$

Let $$P = \left[\begin{array}{c|c} P_1 & P_2 \\ \hline P_3 & m \end{array}\right]$$

where every element is not less than m, the row sums of the n-1 square matrix P_1 are equal to S-m, all the elements of P_2 are equal to m while those of P_3 add up to s-m. Then the matrix

$$\left[\begin{array}{c|c} P_1 & \frac{1}{g}P_2 \\ \hline gP_3 & m \end{array}\right]$$

has the same eigen-values. Each of its n-1 row sums is equal to $S - m + \frac{m}{g} = K_1$ and its last row sum is $gs - gn + m = K_2 = K_1$

Hence by the Forbenius Theorem the maximal root of P is K_1

Similarly,

$$Q = \left[\begin{array}{c|c} m & Q_2 \\ \hline Q_3 & Q_4 \end{array}\right]$$

where every element is not less than m, the row sums of the (n-1) square submatrix Q_4 are all equal to s-m, all the elements of Q_3 are equal to m, while those of Q_2 add up to S-m. Now if the first row is divided by h and the first column is multiplied by h the resulting matrix is similar to Q and has the same eigen-values and is a generalized row Stochastic matrix with the row sum $K_3 = K_4$ and therefore maximal eigen-value of Q is K_3.

Thus Alfred Braur's bounds for the positive matrices are the best possible using S, s, and m.[3]

3.5 SUMMARY

The main problem discussed in this chapter was bounds for maximal root for non-negative and positive matrices. If bounds for a non-negative matrix is S, then all eigen-values of the matrix lie in the interval $-S \leqq \lambda \leqq S$, which easily follows from Gersgorin's Theorem. It has been realized that bounds depend upon the elements in a matrix. If a matrix is non-negative, then the result of Frobenius is the best. If a matrix is positive, then the result of Alfred Braur is the sharpest using S, s, and m. If a non-negative matrix belongs to a class of matrices, then better bounds for eigen-values can be obtained.

Example: Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

The bounds given by:

Frobenius: $4 \leqq r \leqq 10$ (rows)

$5 \leqq r \leqq 10$ (columns)

[3]Alfred Braur, "The Theorems of Ledermann and Ostrowski on Positive Matrices," Duke Mathematical Journal, XXIV (1957), pp. 265-274.

Ledermann:	$4.225 \leqq r \leqq 9.816$	(rows)
	$4.414 \leqq r \leqq 9.707$	(columns)
Ostrowski:	$4.732 \leqq r \leqq 9.577$	(rows)
	$4.500 \leqq r \leqq 9.667$	(columns)
Braur:	$5.167 \leqq r \leqq 9.360$	(rows)
	$5.86 \leqq r \leqq 9.527$	(columns)

The actual value of r to 4 significant figures is 7.531.

CHAPTER IV

BOUNDS FOR MAXIMAL EIGEN-VALUES OF TWO SPECIAL CLASSES OF NON-NEGATIVE MATRICES

Two special classes of non-negative matrices, namely power-positive matrices and matrices satisfying the inequality $0<A^2 \leqq A$ are discussed in this chapter. Some theorems on both classes have been proven and statements about bounds of their eigen-values were made.

4.1.1 POWER-POSITIVE MATRICES

A matrix with real elements of which a positive-power (i.e. natural number) is a positive matrix is called a power-positive matrix. If only even powers of such a matrix are positive then the matrix is called power-positive of even exponent, otherwise power-positive of odd exponent. Every power-positive matrix has a greatest eigen-value r which is the maximal eigen-value. If A is power-positive of an odd exponent, then r is positive. If A is power-positive of an even exponent then it may be positive or negative. If r is negative, then the matrix -A has the greatest positive eigen-value -r. Hence it is sufficient to consider such power-positive matrices which have positive maximal eigen-value. While the maximal eigen-value of a positive matrix is greater than the greatest main diagonal element, this is not always true for maximal eigen-value of a power-positive matrix. There exist matrices with positive maximal eigen-value r for which r is greater than the greatest row sum.

The aim of this thesis was to see that the power-positive matrices have the most important properties of positive matrices.

Definition: An eigen-value λ_1 of a matrix A is simple if λ_1 is distinct from all other eigen-values of A.

Definition: If the maximal eigen-value λ_1 of a matrix is simple then all eigen-vectors are of the form (cx_1, cx_2, ..., cx_n), where (x_1, x_2, ..., x_n) is an eigen-vector belonging to λ_1.

4.1.2 BRAUR'S THEOREM

Every power-positive matrix has a real eigen-value λ_1 which is simple. Its absolute value is greater than the absolute values of all the other eigen-values. The coordinates of an eigen-vector belonging to λ_1 ($r = |\lambda_1|$) can be chosen as positive numbers.

Proof: Let $A = (a_{ij})$ be a power-positive matrix of the order n with eigen-values $\lambda_1, \lambda_2, \ldots, \lambda_n$. A^K is positive for the positive integer K. Then maximal eigen-value of A^K is positive. Since the roots of A^K are

$$\lambda_1^K > \max. \left(\lambda_2^K , \lambda_3^K , \ldots, \lambda_n^K\right) ,$$

hence $$|\lambda_1| > \max.\left(|\lambda_2| , |\lambda_3| , \ldots, |\lambda_n|\right),$$

and λ_1 is the maximal eigen-value of A. Obviously λ_1 is real.

Since neither λ_1 nor its complex conjugate can be one of the numbers $\lambda_2, \lambda_3, \ldots, \lambda_n$, it follows that $\lambda_1 = \bar{\lambda}_1$ and therefore λ_1 is real and the simple eigen-value of A.[1]

Let ($x_1, x_2, \ldots, x_n$) be an eigen-vector belonging to λ_1 of A. Since this eigen-value λ_1 is simple, all eigen-vectors are of the form $(cx_1, cx_2, \ldots, cx_n)$.

It is well-known that an eigen-vector belonging to the eigen-value λ_1 of A is also an eigen-vector belonging to eigen-value λ_1^K of A^K. On the other hand, since λ_1^K is a simple eigen-value, the set $(cx_1, cx_2, \ldots, cx_n)$ is the set of all eigen-vectors belonging to λ_1^K. Since λ_1^K is maximal eigen-value of A^K (a positive matrix) all coordinates of the given vector ($cx_1, cx_2, \ldots, cx_n$), have the same sign. Therefore coordinates of an eigen-vector belonging to the eigen-value λ_1 can be chosen as positive numbers.[2]

Q. E. D.

4.1.3 THEOREM

A power-positive matrix of an odd exponent has the positive maximal eigen-value λ_1.

Proof: A^K is positive where K is odd. By Theorem

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[1]Alfred Braur, "On the Characteristic Roots of Power-Positive Matrices," Duke Mathematical Journal, XXVIII (1961), pp. 291-196.

[2]Alfred Braur, "Limits for Characteristic Roots of a Matrix," Duke Mathematical Journal, XV (1948), pp. 871-877.

4.1.2, the maximal eigen-value of A is real and λ_1^K is positive.

Q. E. D.

4.1.4 THEOREM

The maximal eigen-value λ_1 of a power-positive matrix lies between the greatest and the smallest row sums of A.

Proof: Let $S_1, S_2, \ldots, S_n$ be the row sums of A. Assume $S_1 \geqq S_2 \geqq \ldots \geqq S_n$. Consider the system of linear equations belonging to λ_1 with regard to columns.

$$\sum_{i=1}^{n} a_{ij} Y_i = \lambda_1 Y_j \qquad (j = 1, 2, \ldots, n)$$

Adding these equations we have,

$$S_1 Y_1 + S_2 Y_2 + \ldots + S_n Y_n = \lambda_1 (Y_1 + Y_2 + \ldots + Y_n).$$

It may be assumed that $Y_1, Y_2, \ldots, Y_n$ are positive,

$$S_1 (Y_1 + Y_2 + \ldots + Y_n) \geqq \lambda_1 (Y_1 + Y_2 + \ldots + Y_n)$$
$$\geqq S_n (Y_1 + Y_2 + \ldots + Y_n),$$

Therefore $S_1 \geqq \lambda_1 \geqq S_n$

4.1.5 THEOREM

If all the elements of a row in the power-positive matrix A are all non-positive (non-negative), then the maximal eigen-value is negative (positive).

Proof: Suppose that the first row of A has only non-positive elements.

Let $(x_1, x_2, \ldots, x_n)$ be an eigen-vector with negative coordinates belonging to the maximal eigen-value γ.

$$\sum_{j=1}^{n} a_{ij}\, x_j = \lambda\, x_i \qquad \text{For } i = 1,$$

$$a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n = \lambda\, x_1$$

The left hand side of the above equation is positive. $\lambda\, x_1$ is positive and λ is negative.

A similar proof holds if all the elements of a row are non-negative.

4.1.6 THEOREM

If all the elements of a column of the power-positive matrix A are non-positive (non-negative), then the maximal eigen-value is negative (positive).

Proof: The Proof is similar to the previous theorem and it would only be repetitious to present it here.

4.1.7 STATEMENT

The proceeding theorems show that many properties of eigen-values of positive matrices hold for power matrices of odd exponents. But the maximal eigen-value of such a matrix is not always greater than the main diagonal element.

4.1.8 MATRICES SATISFYING $0 < A^2 \leqq A$

Now consider matrices satisfying the inequality $0 < A^2 \leqq A$. The inequality $A \leqq B$ means that every element of A is less than or equal to the corresponding element of B. Of course, both matrices are of the same order.

Examine A^2 in the expanded form:

$$B = A^2 = \sum_{\substack{i=1 \\ j=1}}^{n} a_{ij}\, a_{ji}$$

Consider b_{11}

$$b_{11} = (a_{11}^{2} + a_{12}\, a_{21} + a_{13}\, a_{31} + \ldots + a_{1n}\, a_{n1})$$

By the above inequality

$$b_{11} = a_{11}^{2} + a_{12}\, a_{21} + a_{13}\, a_{31} + \ldots + a_{1n}\, a_{n1} \leqq a_{11}$$

$$a_{11} < 1 \quad \text{and}$$

$$a_{11} + \frac{a_{12}\, a_{21}}{a_{11}} + \frac{a_{13}\, a_{31}}{a_{11}} + \ldots + \frac{a_{1n}\, a_{n1}}{a_{11}} \leqq 1.$$

which gives the result that

$$a_{ij} \leqq \frac{1}{n} \qquad \text{if } i \neq j$$

Therefore

$$0 < A^2 \leqq A \Longrightarrow \begin{cases} a_{ij} < 1 & \text{if } i = j \\ a_{ij} \leqq \frac{1}{n} & \text{if } i \neq j \end{cases}$$

where n is the order of A.

Now certain proofs of some theorems on bounds for maximal eigen-values of this kind of matrices will be presented.

It has been examined that if $n \geqq 3$ then $a_{ii} \leqq \frac{2}{n}$.

4.1.9 THEOREM

If $0 < A^2 \leqq A$ then the maximal eigen-value r of A lies between the least row sum and $\frac{n+1}{n}$, where n is the order of A and $n \geqq 3$.

Proof: Since A is positive matrix, therefore the maximal eigen-value r of A lies between the least row sum and the greatest row sum. (Frobenius Theorem)

Therefore $r \geqq s$ when s is the least row sum of A. Suppose the first row sum of A is the greatest row sum, if

$n \geqq 3$ and $A^2 \leqq A$ then:

$$a_{ij} \leqq \frac{2}{n} \quad \text{if } i = j$$

$$a_{ij} \leqq \frac{1}{n} \quad \text{if } i \neq j$$

$$r \leqq S_1 = a_{11} + a_{12} + \ldots + a_{1n}$$

$$r \leqq \frac{2}{n} + \left(\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n} \right)$$

$$r \leqq \frac{2}{n} + \frac{n-1}{n} = \frac{n+1}{n}$$

$$s \leqq r \leqq \frac{n+1}{n}$$

Q. E. D.

This usually increases efficiency in finding bounds since looking at the order of the matrix is enough to find the bounds for maximal eigen-values.

Example: Let A be the matrix such that $0 < A^2 \leqq A$

$$A = \begin{bmatrix} 1/3 & 1/4 & 1/5 \\ 1/5 & 1/3 & 1/6 \\ 1/4 & 1/5 & 1/7 \end{bmatrix}$$

By the above theorem:

$$.593 \leqq r \leqq 1.333$$

4.2.0 THEOREM

If $0 < a\,A^2 \leqq A$ then the maximal eigen-value of A lies between a_{jj}, the greatest main diagonal element, and $\frac{n+1}{na}$ (where $a > 1$).

Proof: Since $a\,A^2 \leqq A$, therefore the obvious inequalities are:

$$a_{ij} \leqq \frac{2}{na} \qquad \text{if } i = j$$

$$a_{ij} \leqq \frac{1}{na} \qquad \text{if } i \neq j$$

Following the pattern of the previous theorem, it can be easily shown that $a_{jj} \leqq r$, where a_{jj} is the greatest main diagonal element of A and

$$r \geqq \frac{2}{na} + \frac{n-1}{na} = \frac{n+1}{na}$$

$$a_{jj} \leqq r \leqq \frac{n+1}{na}$$

Q. E. D.

Example: Let A be the matrix such that $0 < a\,A^2 \leqq A$

$$A = \begin{bmatrix} .100 & .150 & .100 \\ .150 & .150 & .100 \\ .100 & .100 & .150 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} .042 & .045 & .040 \\ .047 & .054 & .045 \\ .040 & .052 & .035 \end{bmatrix}$$

$$a\,A^2 \leqq A \text{ where } a = 1.9$$

By theorem 4.2.0 $\quad .150 \leqq r \leqq \frac{3+1}{3 \times 1.9}$

$$.150 \leqq r \leqq .667$$

This theorem is sharper than the previous theorem since according to the previous theorem the bounds for the maximal root are $.150 \leqq r \leqq 1.333$.

One does not have to obtain A^2 to find the bounds if $a_{ij} \leqq \frac{1}{an}$ for all i and j then the bounds can be obtained by the above theorem. i. e. $a_{jj} \leqq r \leqq \frac{n+1}{na}$

Summary: The bounds for the maximal eigen-values of power-positive matrices and matrices satisfying $0 < A^2 \leqq A$ were discussed in this chapter. Similarity of the properties of power-positive matrices to those of positive matrices were shown. Two new theorems on bounds of matrices satisfying $0 < A^2 \leqq A$ were proved.

CHAPTER V

CONCLUSION

5.1 SUMMARY

The primary purpose of this thesis has been accomplished by detailed discussion of the bounds for eigen-values of non-negative matrices. The study was carried deep into two special classes of matrices, power-positive matrices and matrices satisfying $0 < A^2 \leqq A$. Properties of power-positive matrices were discussed in the form of theorems in order to show that power-positive matrices behave like positive matrices in many ways (not in all ways). Two new theorems stating:

1. If $0 < A^2 \leqq A$ and $n \geqq 3$ then the maximal eigen-value of A lies between the least row sum of A and $\frac{n+1}{n}$ (where n is the order of A).
2. If $0 < a\,A^2 \leqq A$ for the positive real number a and $n \geqq 3$, then the eigen-values of A lie between the greatest main diagonal element a_{jj} and $\frac{n+1}{na}$

have been proven. Examples were given for each.

The results obtained give easy and quick solutions for the bounds of maximal eigen-value. The bounds can be obtained without considering the row sums or the characteristic equation by the second theorem which is sharper than the first as it gives better bounds.

5.2 SUGGESTIONS FOR FURTHER STUDY

R. E. Demarr of the University of New Mexico presented a paper on the bounds for eigen-values of non-negative matrices before the American Mathematical Society meeting (September 1968). He proved a conjecture stating that if $0 < A^2 \leqq A$ then all eigen-values of A lie between 1 and $\frac{1-\sqrt{2}}{2}$. His proof is not published. An attempt to prove this conjecture can lead to an interesting research.

This problem could further be extended to a theorem saying if $0 < a A^2 \leqq A$, for some real positive number $a > 0$ then all eigen-values of A lie between $\frac{1}{a}$ and $\frac{1-\sqrt{2}}{2a}$. If this theorem is true, it would be an improvement over Demarr's conjecture.

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