A COMPARISON OF PROPERTIES OF CIRCLE IN EUCLIDEAN AND HYPERBOLIC GEOMETRY

A Thesis

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by

Aristides Rodosthenous Constantinides

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John Gerrietts

Approved for the Major Department

George L. Downing

Committee Member

Jane S. Harter

Committee Member

Marion P. Emerson

Committee Chairman

Harold E. Durst

Approved for the Graduate Council

PREFACE

The purpose of this paper is to help College students with major in Mathematics to see the main differences relative to properties of circles in Euclidean and Hyperbolic Geometry.

The writer of this paper wishes to express his sincere appreciation to Professor George L. Downing of the Emporia Kansas State College for the suggestion of the topic, and Professor C.E. John Gerriets of Emporia Kansas State College for his helpful assistance without which this paper would not have been possible.

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CHAPTER I

INTRODUCTION

1.1. Introduction. For Ancient Greeks, who were the main inventors of Plane Geometry, the circle was the most perfect of all plane figures. Probably this is the reason why they gave so much attention to this figure. The main theorems of Plane Geometry are related to it. Thus, in studying the properties of the circle, actually we are studying most of Plane Geometry.

1.2. Statement of the problem. The purpose of this paper is: (1) to show the evolution of knowledge of the properties of the circle from ancient years to the present; and (2) to determine which Euclidean theorems are valid in Hyperbolic Geometry, and to show how other theorems can be valid with some changes.

1.3. Importance of the paper. It may well be necessary "to remark that much of the Euclidean theory must be abandoned or greatly modified in Hyperbolic Geometry". ([4], p.100). Most of the Non-Euclidean geometry books include much about the differences between the properties of angles, lines, triangles and polygons in Euclidean and Hyperbolic Geometry, but none of them gives much attention to which Euclidean theorems about circles are valid or invalid in Hyperbolic Geometry. The circle is a basic figure of Geometry, and the present paper will present some ideas relating to this neglected area.

1.4. Undefined terms and symbols. The following terms and symbols have been taken as undefined in this paper:

1) points, denoted by capital letters A, B, C, . . .

2) angles, denoted by Greek small letters α, β, γ, . . ., or numbers 1, 2, 3, . . ., and the symbol " $\angle$ " $\angle\alpha$, $\angle 2$, . . .,

3) arcs, denoted by capital letters and the symbol " $\frown$ " $\overset{\frown}{AB}$, $\overset{\frown}{CD}$, . . .,

4) segments, denoted by capital letters and the symbol " $-$ " $\overline{AB}$, $\overline{CD}$, . . .,

5) rays, denoted by capital letters and the symbol " $\rightarrow$ " $\overrightarrow{AB}$, $\overrightarrow{CD}$, . . .,

6) lines, denoted by capital letters and the symbol " $\leftrightarrow$ " $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$, . . .,

7) measure of angles or arcs denoted by the small letter "m" $m\angle ABC$, $m\angle\beta$, $m\overset{\frown}{AB}$, . . .,

8) triangles, denoted by capital letters and the symbol " Δ " ΔABC, ΔDEF, . . .,

9) radius of the circle, denoted by the letter r.

1.5. Definition of terms. As in Euclidean Geometry, a circle is defined to be the set of all points which are at a constant distance (called the radius) from a fixed point (called the centre of the circle). Likewise, all of the other familiar terms connected with circles (diameter, arc, chord, secant, tangent, central angle, inscribed angle, minor arc, etc.) retain their usual meaning.

1.6. Organization of the remainder of the paper. The remainder of the paper has been divided into four parts.

1) Chapter II gives a brief history of the circle.

2) Chapter III gives a brief history of the discovery of Hyperbolic Geometry and some important elements of Hyperbolic Geometry which will be used in Chapter IV as references.

3) Chapter IV presents the most important Euclidean theorems and determines which of them are valid and which are invalid in Hyperbolic Geometry. For some invalid theorems, revisions are made to give similar theorems which are valid.

4) A summary of the paper is presented in Chapter V.

CHAPTER II

A BRIEF HISTORY OF THE CIRCLE

2.1. Introduction. The discovery of the properties of the circle is not a work of one day from one people in a special place. It is a work of thousands of years from different people in different places. The purpose of this chapter is to present briefly the history of the circle.

2.2. Babylon. The earliest known documents relating to the circle come from Babylon and Egypt. Those from Babylon are written on small clay tablets, some of them about the size of the hand. A table in the British Museum shows that such geometrical forms as circular segments were used in astrology or on talismans. For the mensuration of the circle, the Babylonians used $\pi = 3$. ([2] ,p.6). They divided the circumference into 360 parts and also made use of various geometric designs in their mural decorations. ([2] , p.6).

2.3. Egypt. The first definite knowledge that we have of Egyptian Mathematics comes to us from a manuscript copied on papyrus, a kind of paper used in the Mediterranean area in early times. This copy was made by someone called Ahmes, who probably lived about 1700 B.C. Ahmes gave a rule for finding the area of a circle, substantially as follows: Multiply the square of the radius by $(16/9)^2$ which is equivalent to taking π to the value 3.1605. ([1], p.27).

2.4. Ancient Greece. From Egypt and possibly from Babylon, geometry passed to the shore of Asia Minor and Greece. The scientific study of the subject begins with Thales (640-546 B.C.). One of the most remarkable of his geometrical achievements was proving that any angle inscribed in a semicircle must be a right angle. ([3], p.207). Plato (429-348 B.C.) and his school, developed the first systematic attempt to create exact definitions, axioms and postulates for geometry. In addition, he tried to find a square equivalent to a given circle. ([1], p.31). About 420 B.C. Hippias of Elis invented a certain curve called the quadratix, by means of which he could square the circle and trisect any angle. ([1], p.31). Euclid (300 B.C.) published systematic, rigorous proofs of the leading propositions of the geometry known at his time. ([3], p.28). Archimedes (287 B.C.) gave three important propositions for circle: (1) The area of a circle is the same as that of a right-angled triangle with base equal to the circumference, and vertical height equal to the radius a, which meant that the : Area of circle$=\frac{1}{2} r (2\pi r)$. (2) The ratio of the area of a circle to the square on its diameter is approximately 11:14 which means that $a^2 : 4a^2 :: 11:14$. (3) π is less than 3 1/7 but greater than 3 10/71. ([7], p.53).

2.5. India. Aryabhatta (476 A.D.) arrived at an approximation for π expressed in modern rotation as 3.1466 ([1] , p.36). He also found the area of circle to be the product of half of circumference by half of diameter. ([5] , p.15).

2.6. After 5th century. Al Kashi (Arab 1430 A.D.) wrote down the value of π as 3.141592653898732 ([7] , p.153). Wallis (England 1616-1703). One of his interesting discoveries was the relationship that

$$4/\pi = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \ldots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \ldots}$$

This is one of the earliest values of π found by such a means known as infinite product. ([7] , p.93). William Jones is usually recognized as being the first man to use π definitely in its present relationship and when Leonhard Euler (Sweden) used it in 1737 it came into general use ([7] , p. 155). In 1796 Gauss showed that it was possible, by the use of the straightedge and compasses alone, to inscribe a polygon of 17 sides in a circle. He even extended the solution to inscribe polygons of 257 and 65,537 sides ([8] , p.301).

CHAPTER III

IMPORTANT ELEMENTS IN HYPERBOLIC GEOMETRY

3.1. Introduction. The purpose of this chapter is to present briefly the history of Hyperbolic Geometry and those important postulates, theorems, corollaries of Hyperbolic Geometry which will be used in the next chapter as references.

3.2. History of Discovery of Hyperbolic Geometry. In the fourth century B.C. Euclid organized the early study of Geometry into his thirteen volumes called the Elements. He took a total of five "self-evident truths" which were the basis of his geometry. Since they were the very foundation of the system there was no way to prove them.

The first five postulates of Euclidean Geometry are:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line.
3. To draw a circle with any center and any distance as radius.
4. That all right angles are equal to one another.
5. That if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight

lines if produced indefinitely meet on that side on which are the angles less than two right angles.

The fifth postulate is equivalent to Playfair's axiom: Through a point not on a line, one and only one line can be drawn parallel to a given line.

Many Mathematicians, including the Greeks and Arabians, tried to prove this last postulate and no one could do it adequately. Around 1700, a Jesuit priest by the name of Giovanni Saccheri became interested in proving Euclid's fifth postulate. He used the method of indirect proof and the quadrilateral in figure 1 in an attempt to find a contradiction which would prove the fifth postulate.

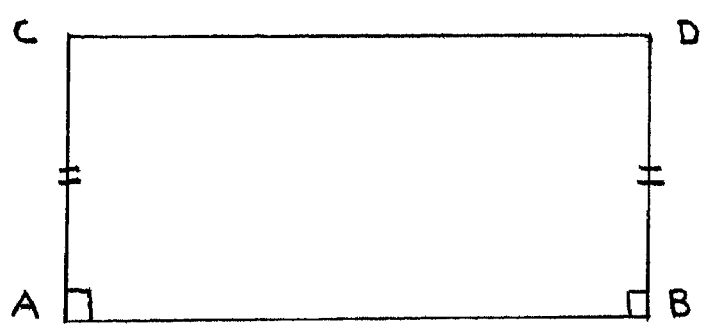


Figure 1

We know from elementary geometry that if two lines AC and BD of equal length are perpendicular to the same line AB, the line CD connecting the endpoints will form equal, in fact, right angles at C and D. Saccheri called the angles at C and D the summit angles and came up with three possibilities:

1. The summit angles are right angles.
2. The summit angles are obtuse angles.
3. The summit angles are acute angles.

By assuming each of these in turn, Saccheri came to the corresponding relation between the lines AB and CD. The first hypothesis, that the angles are right, resulted in equal lines which, of course, led to Euclid's parallel postulate. The second hypothesis, that if the obtuse angle when CD will be less than AB, turned out to be a contradiction, and was thereby eliminated. But when Saccheri came to the acute angle hypothesis, he could not find the desired contradiction. He had to trust the intuition in his last proof saying that the "hypothesis to the acute angle is absolutely false because repugnant to the nature of the straight line". We can say that Saccheri was probably not satisfied with this conclusion. But he was not able to produce a better contradiction.

About fifty years after Saccheri, three men working with the acute angle hypothesis finally developed a geometry different from Euclid's. The greatest of these men was Karl Friedrich Gauss. He adopted Saccheri's alternative of the acute angle, leaving the rest of Euclid's postulate alone. The result was a geometry just as rational and valid as Euclid's. Nicholas Lobachevsky, a Russian, and John Bolyai, a Hungarian, were not the prominent mathematicians that Gauss was, but at about the same time, they developed a geometry identical to that of Gauss.

Bolyai found that if we start with AB perpendicular to q (see figure 2) and allow AB to rotate about A in either direction, it will continue to cut q at first and then cease to cut it. He was thus led to postulate the existence of two lines through A which

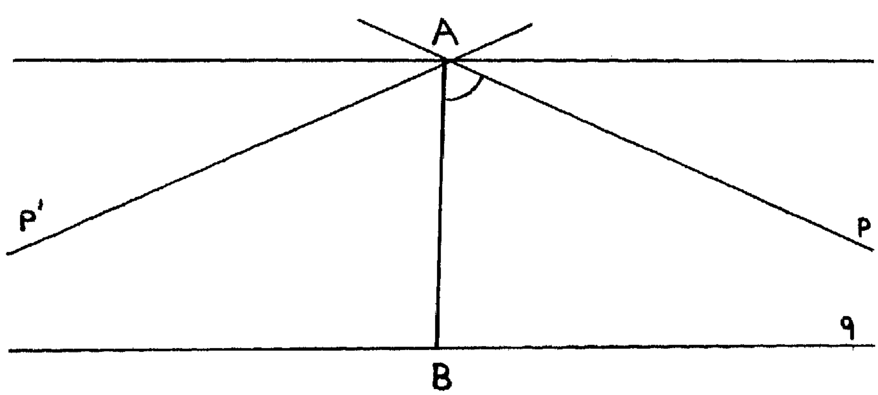


Figure 2

separate the lines which cut q from those which do not. Since for rotation of AB in either direction there is no last-intersecting line, these postulated lines must be the first on the non-intersecting lines. He defined also the angle of parallelism $\pi(AB)$ as the acute angle between AB and either of the parallels p, p'.

<u>Postulate 3-3</u>. Through a given point, not on a

given line, more than one line can be drawn not intersecting the given line.

Note: The figure 3 which is related to the above postulate shows the different kind of lines in Hyperbolic Geometry. There are lines such as AI which intersect ℓ. There are two lines AS, AP which are the first non-intersecting lines. There are other non-intersecting lines as FG, CD. Because some Non-Euclidean editions use different names for the same lines, the present paper will use these lines with the following names:

1. $\overrightarrow{AS}$, $\overrightarrow{AP}$ parallel lines (first non-intersecting lines).
2. FG, CD non-intersecting lines.
3. AH, AI intersecting lines.

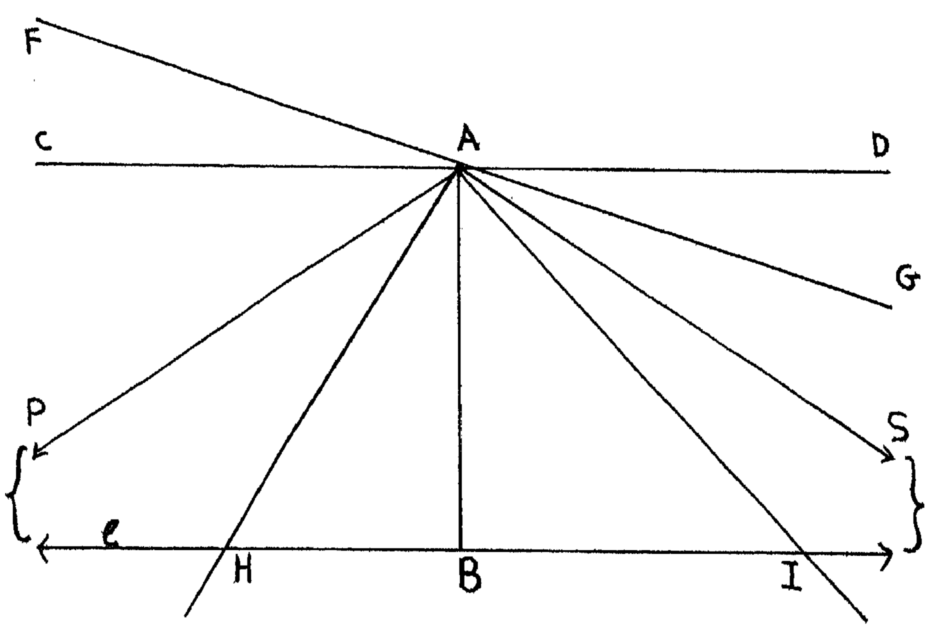


Figure 3

Theorem 3-4. Two non-intersecting lines have one and only one common perpendicular. ([4] , p.84)

Theorem 3-5. The line joining the midpoints of the base and summit of a Saccheri Quadrilateral is perpendicular to the both of them; the summit angles are equal and acute.

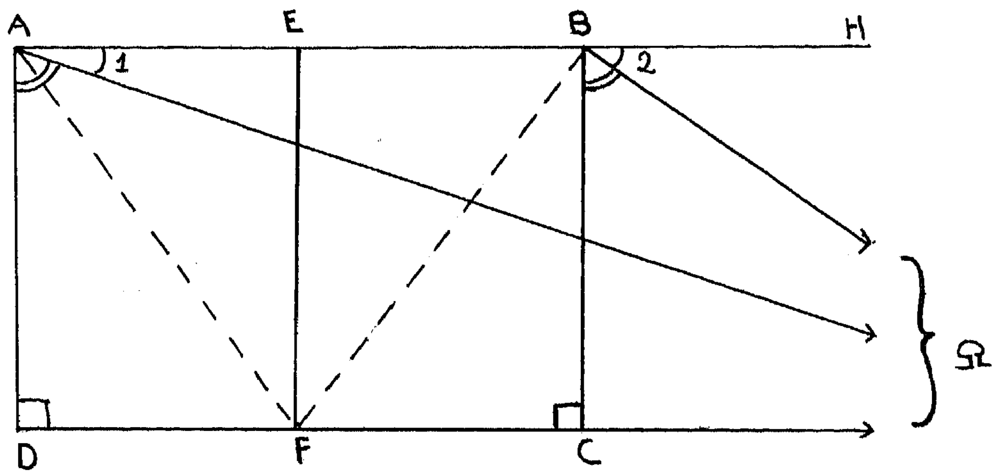


Figure 4

Proof: Let ABCD be a Saccheri Quadrilateral with right angles at D and C and $\overline{AD} \cong \overline{BC}$. Let E, F the midpoints of $\overline{AB}$ and $\overline{BC}$ respectively. Introduce $\overline{EF}$, $\overline{FA}$, $\overline{FB}$. Produce $\overline{EB}$ to H. Draw through A and B parallels to $\overline{DC}$ in direction $\overrightarrow{DC}$. From the congruent triangles ADF, BCF ($\overline{AD} \cong \overline{BC}$, $\overline{DF} \cong \overline{FC}$, $\angle ADF \cong \angle BCF$) it follows $\overline{AF} \cong \overline{BF}$, $\angle DAF \cong \angle FBC$, and $\angle AFD \cong \angle BFC$. From the triangles AEF, BEF which are also congruent ($\overline{AF} \cong \overline{BF}$, $\overline{AE} \cong \overline{EB}$, $\overline{EF} \cong \overline{EF}$) it follows

$\angle$ EFA $\cong \angle$ EFB, $\angle$ EBF $\cong \angle$ EAF, $\angle$ AEF $\cong \angle$ BEF. Because $\angle$ AEF $\cong \angle$ BEF and $\overline{AB}$ is a straight line it follows m$\angle$AEF = m$\angle$BEF = a right angle. Similarly because $\angle$AFD + $\angle$ EFA $= \angle$BFC + $\angle$ EFB and $\overline{DC}$ is a straight line it follows m$\angle$DFE = m$\angle$CFE = a right angle. Therefore the line joining the midpoints of the base and summit of a Saccheri Quadrilateral is perpendicular to both of them.

Since m$\angle$DAF + m$\angle$EAF = m$\angle$CBF + m$\angle$EBF it follows m$\angle$DAE = m$\angle$CBE. Because m$\angle$2 $>$ m$\angle$1 (exterior angle of the triangle ABΩ) and $\angle$DA$\Omega \cong \angle$CBΩ (angles of parallelism corresponding to equal distances) it follows m$\angle$2 + m$\angle$CBΩ $>$ m$\angle$1 + m$\angle$DAΩ and m$\angle$CBH $>$ m$\angle$DAE and consequently m$\angle$CBH $>$ m$\angle$CBE. From the inequality m$\angle$CBH $>$ m$\angle$CBE and because $\overline{AH}$ is a straight line it follows $\angle$EBC is acute. But $\angle$EBC $\cong \angle$EAD so $\angle$EAD is also acute. Therefore the summit angles are equal and acute.

<u>Theorem 3-6</u>. In a trirectangular quadrilateral the fourth angle is acute.

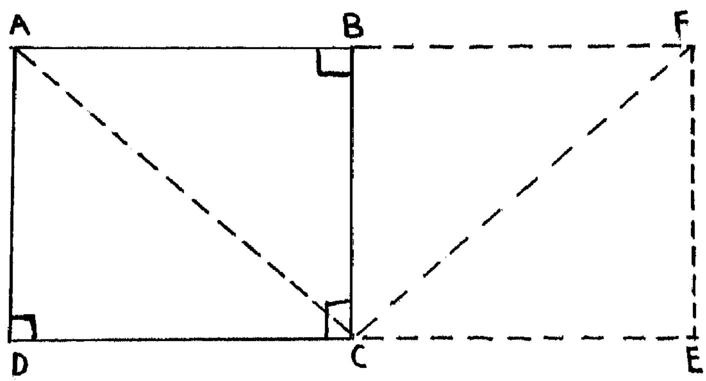


Figure 5

Proof: Let ABCD be a Lambert Quadrilateral with right angles at D, C, B. (Figure 5). We have to show that $m\angle BAD$ is acute. Produce $\overline{DC}$ through C to E so that $\overline{CE} \cong \overline{DC}$. At E draw $\overline{EF}$ perpendicular to $\overline{EC}$ and congruent to $\overline{AD}$. Introduce $\overline{FC}$, $\overline{FB}$, $\overline{AC}$. From the congruent triangles DAC, EFC ($\overline{AD} \cong \overline{FE}$, $\overline{DC} = \overline{CE}$, $\angle ADC \cong \angle FEC$) it follows $\overline{AC} \cong \overline{CF}$, $\angle DAC \cong \angle EFC$, $\angle ACD \cong \angle FCE$. Because $\angle ACB$ and $\angle FCB$ are complementary to equal angles $\angle ACD$ and $\angle FCE$ it follows $\angle ACB \cong \angle FCB$. The triangles ACB and FCB are congruent ($\overline{BC}$ common, $\angle ACB \cong \angle FCB$, $\overline{AC} \cong \overline{FC}$) so $\angle ABC \cong \angle FBC$ = a right angle therefore the points A, B, F are collinear and ADEF is a Saccheri Quadrilateral (Theorem 3-5) with acute angles at A and F. Therefore in a trirectangular quadrilateral the fourth angle is acute.

Theorem 3-7. The sum of the angles of every right triangle is less than two right angles.

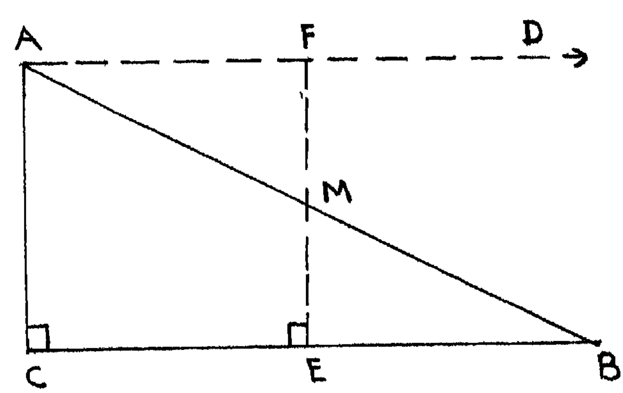


Figure 6

Proof: Let ACB be any triangle with right angle at C (Figure 6). At A construct $\angle BAD$ congruent to $\angle ABC$. Take the midpoint, M, of $\overline{AB}$ and draw $\overline{ME}$ perpendicular to $\overline{CB}$. On $\overrightarrow{AD}$ take $\overline{AF}$ equal to $\overline{EB}$ and draw MF. From the congruent triangles MEB, MFA ($\overline{AM} \cong \overline{MB}$, $\overline{AF} \cong \overline{EB}$, $\angle FAM \cong \angle MBE$) it follows $\angle AFM \cong \angle MEB$ = a right angle and $\angle FMA \cong \angle EMB$. Therefore points F, M, E are collinear and AFEC is a Lambert quadrilateral (Theorem 3-6) with acute angle at A. Since $m\angle DAC = m\angle DAB + m\angle BAC$ $<$ a right angle and $m\angle ABC = \angle DAB$ it follows that $m\angle ABC + m\angle BAC <$ a right angle. Therefore the sum of the angles of every right triangle is less than two right angles.

Corollary 3-7.1. The sum of the angles of every triangle is less than two right angles.

Corollary 3-7.2. The sum of the angles of every quadrilateral is less than four right angles.

Corollary 3-7.3. The exterior angle of every triangle is greater than the sum of the two opposite angles.

3-8. The Pythagorean Theorem. The Pythagorean theorem is invalid in Hyperbolic Geometry. ([4], p.144). In Hyperbolic Geometry we use a similar theorem which is also based on the sides of a right triangle. It is given by the formula $\cosh c = \cosh a \cosh b$ where c is the hypotenuse and a, b the other sides of the triangle.

3-9. Area and circumference of the circle. While it is true that there are 360° in a circle and that all central angles of 1° in a given circle subtend equal arcs, no longer is the circumference 2π times the radius. ([9], p. 76).

As in Euclidean Geometry, limits are required to obtain formulas for the circumference and the area of the circle. The circumference is the limit approached by the perimeter of an inscribed regular polygon of n sides when n becomes infinite, and the area is the limit approached by the area of the polygon.

It can be shown that the formulas relating the circumference and the area to the radius of the circle are

$C = 2\pi \sinh r \qquad A = 4\pi \sinh^2 r/2$ ([4], p.161-9)

when the standard unit of length and unit of area are used. ([9], p.27). It is beyond the scope of this paper to prove these formulas or to explain the meaning of standard unit of length and unit of area.

Theorem 3-10. If the three angles of one triangle are equal, respectively, to the three angles of a second, then the two triangles are congruent. ([4], p.83).

CHAPTER IV

IMPORTANT EUCLIDEAN THEOREMS RELATING TO CIRCLES IN HYPERBOLIC GEOMETRY

4.1. Introduction. The purpose of this chapter is to present the most important Euclidean theorems which pertain to circles and to classify them as to their validity in Hyperbolic Geometry. For those which are invalid in Hyperbolic Geometry an explanation why they are invalid is given. For some of them it is possible to give similar valid theorems in Hyperbolic Geometry.

The chapter is organized into three sections. In the first section there are those Euclidean theorems which are valid in Hyperbolic Geometry. The proofs are quite elementary and for the readers who are interested in these proofs the paper gives at the end of each theorem the book and the page in which there is a proof valid for both Geometries. The second section consists of those of the Euclidean theorems which are invalid in Hyperbolic Geometry and includes similar theorems in Hyperbolic Geometry. For each theorem there is first a proof in Euclidean Geometry and then an explanation why it is invalid in Hyperbolic Geometry. Then the similar theorem in Hyperbolic Geometry is given with its proof. The third section presents some other Euclidean theorems which are invalid in Hyperbolic Geometry but for which no similar theorems in Hyperbolic Geometry were proved. For each theorem the writer presents the proof in Euclidean Geometry and then follows it with an explanation of why the theorem is invalid in Hyperbolic Geometry.

4.2. Valid theorems in both Geometries.

Theorem 4-2.1EH. If a line and a circle are in the same plane, and if P is the foot of the perpendicular from the center of the circle to the line, then:

(1) Every point of the line is outside the circle, or

(2) P is on the circle and the line is tangent to the circle, or

(3) P is inside the circle, and the line intersects the circle in two points which are equidistant from P. ([7], p.302).

Theorem 4-2.2EH. Every line tangent to a circle is perpendicular to the radius drawn to the point of contact. ([6], p.304).

Theorem 4-2.3EH. Any line perpendicular to a radius at its outer end is tangent to the circle. ([6], p.304).

Theorem 4-2.4EH. A line through the center of a circle and perpendicular to a chord bisects the chord and its arc. ([3], p.227).

<u>Theorem 4-2.5EH.</u> The segment joining the center of the circle to the midpoint of a chord is perpendicular to the chord. ([6], p.304).

<u>Theorem 4-2.6EH.</u> The perpendicular bisector of a chord passes through the center of the circle. ([6], p.305).

<u>Theorem 4-2.7EH.</u> If a line has a point in the interior of a circle, then it intersects the circle in exactly two points. ([6], p.305).

<u>Theorem 4-2.8EH.</u> In the same circle, or in congruent circles, chords equidistant from the center are congruent. ([6], p. 305).

<u>Theorem 4-2.9EH.</u> In the same circle, or in congruent circles, any two congruent chords are equidistant from the center. ([6], p.305).

<u>Theorem 4-2.10EH.</u> In the same circle, or in congruent circles, if two chords are congruent, the corresponding minor arcs are congruent. ([6], p.312).

<u>Theorem 4-2.11EH.</u> In the same circle, or in congruent circles, if two arcs are congruent, then so are the corresponding chords. ([6], p.312).

<u>Theorem 4-2.12EH.</u> The two tangent segments to a circle from a given point are congruent, and form congruent angles with the line joining the point to the

center of the circle. ([6] , p.325).

Theorem 4-2.13EH. Two distinct circles can intersect in at most two points. ([6] , p.341).

Theorem 4-2.14EH. If two central angles of the same or congruent circles are congruent then their intercepted arcs are congruent. ([3] , p.214).

Theorem 4-2.15EH. If two arcs of a circle or congruent circles are congruent, then the central angles corresponding to these arcs are congruent. ([3] , p.215).

Theorem 4-2.16EH. In the same circle or in congruent circles, chords are congruent iff they have congruent central angles. ([3] , p.223).

Theorem 4-2.17EH. In a circle or in congruent circles, if two central angles have unequal measures, the greater central angle has the greater minor arc. ([3] , p.294).

Theorem 4-2.18EH. In a circle or in congruent circles, if two minor arcs are not congruent, the greater arc has the greater central angle. ([3] , p.295)

Theorem 4-2.19EH. In a circle or in congruent circles, the greater of the two noncongruent chords has the greater minor arc. ([3] , p.295).

Theorem 4-2.20EH. In a circle or in congruent circles, the greater of two noncongruent minor arcs has the greater chord. ([3], p.296).

Theorem 4-2.21EH. In a circle or in congruent circles, if two chords are not congruent, they are unequally distant from the center, the greater chord being nearer the center. ([3], p.296).

Theorem 4-2.22EH. In a circle or in congruent circles, if two chords are unequally distant from the center, they are not congruent, the chord nearer the center being the greater. ([3], p.296).

4.3. Invalid theorems in Hyperbolic Geometry - Similar Theorems

Theorem 4-3.1E. The measure of an inscribed angle is half the measure of its intercepted arc.

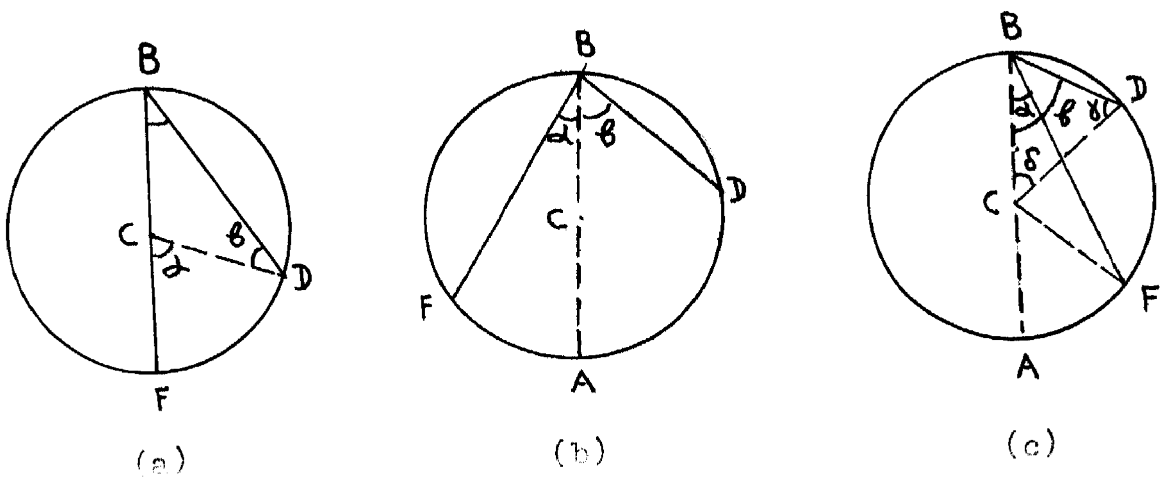


Figure 7

Proof: There are three possible cases, so a three part proof is needed.

Case I. Suppose one side of the angle is diameter (Figure 7a). Introduce $\overline{CD}$. $m\angle\alpha = m\angle B + m\angle\beta$ because $\angle\alpha$ is exterior angle of $\triangle CBD$. Since $m\angle\beta = m\angle B$ (angles at the base of isosceles triangle) $m\angle\alpha = 2m\angle B$. But $m\angle\alpha = m\,\widehat{FD}$ therefore $m\angle B = \frac{1}{2}m\widehat{FD}$.

Case II. Suppose the center of the circle lies in the interior of the angle (Figure 7b). Introduce $\overline{BA}$. $m\angle\alpha = \frac{1}{2}m\widehat{FA}$ and $m\angle\beta = \frac{1}{2}m\widehat{AD}$ (Case I). So $m\angle\alpha + m\angle\beta = \frac{1}{2}(m\,\widehat{FA} + m\,\widehat{AD})$ and since $m\angle\alpha + m\angle\beta = m\angle FBD$ and $m\,\widehat{FA} + m\,\widehat{AD} = m\,\widehat{FD}$, $m\angle FBD = \frac{1}{2}m\,\widehat{FD}$.

Case III. Suppose the center of the circle lies in the exterior of the angle (Figure 7c). Introduce $\overline{BA}$. $m\angle\alpha = \frac{1}{2}m\,\widehat{AF}$ and $m\angle\beta = \frac{1}{2}m\,\widehat{AD}$ (Case I). Since $m\angle B = m\angle\beta - m\angle\alpha$, $m\angle B = \frac{1}{2}(m\widehat{AD} - m\widehat{AF})$. But $m\widehat{AD} - m\widehat{AF} = m\widehat{FD}$ therefore $m\angle B = \frac{1}{2}m\,\widehat{FD}$.

Comments. In Hyperbolic Geometry this proof is invalid because we used in case I the formula $m\angle\alpha = m\angle B + m\angle\beta$ which is a contradiction with Corollary 3-7.3. The theorem is also invalid because in Hyperbolic Geometry we have the following similar theorem:

Theorem 4-3.1H. The measure of an inscribed angle is (a) less than the half of the measure of the intercepted arc when one side of the angle is diameter, or when the center of the circle lies in the interior of the angle and (b) less than, equal to, or greater than the half of the intercepted arc, when the centre of the circle is in the exterior of the angle, according as the defect of $\triangle$ BCD is greater than, equal to, or less than the defect of $\triangle$ BCF. (See figure 7c).

Proof: There are three possible cases, so a three part proof is needed.

Case I. Suppose one side of the angle is diameter (Figure 7a). Since $\angle\alpha$ is exterior angle of $\triangle$ CBD by Corollary 3-7.3 $m\angle\alpha > m\angle\beta + m\angle B$. Because $m\angle\beta = m\angle B$(angles at the base of isosceles triangle) $2m\angle B < m\angle\alpha$. But $m\angle\alpha = m\widehat{FD}$ therefore $m\angle B < \frac{1}{2}m\widehat{FD}$.

Case II. Suppose the centre of the circle lies in the interior of the angle (Figure 7b). From Case I $m\angle\alpha < \frac{1}{2}m\,\widehat{FA}$ and $m\angle\beta < \frac{1}{2}m\,\widehat{AD}$ so $m\angle\alpha + m\angle\beta < \frac{1}{2}\left(m\widehat{FA} + m\widehat{AD}\right)$. Since $m\angle\alpha + m\angle\beta = \angle$ FBD and $m\widehat{FA} + m\widehat{AD} = m\widehat{FD}$ it follows that $m\angle$ FBD $< \frac{1}{2}m\widehat{FD}$.

Case III. Suppose the centre of the circle lies in the exterior of the angle (Figure 7c). We show first the relation of the measure of the inscribed angle CBD with the associated $\triangle$ CBD and the measure of the intercepted arc $\widehat{AD}$. From the formula of the defect of the $\triangle$ CBD, $d = 180 - \delta - \beta - \gamma$ it follows $180 - \delta = d + (\beta + \gamma)$. Because $180 - \delta$ is equivalent to the measure of the arc $\widehat{AD}$ and $\angle\beta \equiv \angle\gamma$ (angles at the base of the isosceles $\triangle$ DBC) the last formula becomes $\beta = \frac{1}{2}m\widehat{AD} - d/2$. Suppose the point D is rotating around the circle from B to A. The area of the triangle CBD increases first, reaches at a maximum value and then decreases. Because in Hyperbolic Geometry the area of triangle is defined as the defect of the triangle, the defect of the triangle associated with the $\angle CBD$ increases first, reaches at a maximum point and then decreases.

Using the result of the last paragraph $\beta = \frac{1}{2}m\widehat{AD} - d/2$ we have $\alpha = \frac{1}{2}m\widehat{AF} - d_1/2$ where d_1 is the defect of the ΔCBF. Subtracting the two equations $\beta - \alpha = \frac{1}{2}(m\widehat{AD} - m\widehat{AF}) + (d_1/2 - d/2)$. Because $\beta - \alpha = m\angle FBD$ and $m\widehat{AD} - m\widehat{AF} = m\widehat{FD}$ it follows $m\angle FBD = \frac{1}{2}m\widehat{FD} + (d_1/2 - d/2)$. By examining this equation we find the following relations:

If $(d_1/2 - d/2) > 0$ it follows $m\angle FBD > \frac{1}{2}m\widehat{FD}$.

If $(d_1/2 - d/2) = 0$ it follows $m\angle FBD = \frac{1}{2}m\widehat{FD}$.

If $(d_1/2 - d/2) < 0$ it follows $m\angle FBD < \frac{1}{2}m\widehat{FD}$.

Thus an inscribed angle with the center of the circle in the exterior of the angle may be greater than, equal to, or less than half the intercepted arc.

Theorem 4-3.2E. The opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

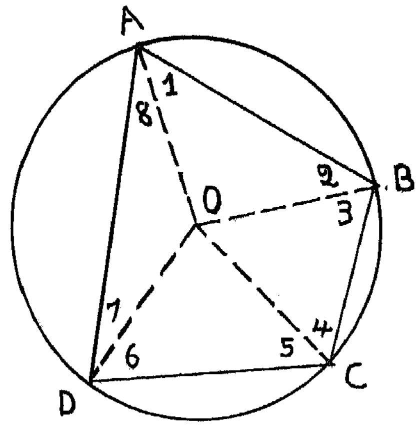


Figure 8

Proof: Let ABCD a quadrilateral inscribed in the circle O (Figure 8). Introduce radii $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, $\overline{OD}$. By theorem 4-3.1E $m\angle A = \frac{1}{2}m\widehat{DCB}$, $m\angle C = \frac{1}{2}m\widehat{DAB}$, $m\angle B = \frac{1}{2}m\widehat{ADC}$ and $m\angle D = \frac{1}{2}m\widehat{ABC}$. By addition property $m\angle A + m\angle C = \frac{1}{2}\left(m\ \widehat{DCB} + m\ \widehat{DAB}\right)$ and $m\angle B + m\angle D = \frac{1}{2}\left(m\ \widehat{ADC} + m\ \widehat{ABC}\right)$. Since $m\widehat{DCB} + m\widehat{DAB} = m\widehat{ADC} + m\widehat{ABC} = 4$ right angles it follows that $m\angle A + m\angle C = m\angle B + m\angle D = 2$ right angles.

Comments. In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle A = \frac{1}{2}m\ \widehat{DCB}$ which is a contradiction of theorem 4-3.1H. The theorem is also invalid because in Hyperbolic Geometry we have the following similar theorem:

Theorem 4-3.2H. In any quadrilateral inscribed in a circle the sum of one pair of opposite angles is equal to the sum of the other pair and both are less than 2 right angles.

Proof: From isosceles triangles OAB, OBC, OCD and ODA (Figure 8) we have $m\angle 1 = m\angle 2$, $m\angle 3 = m\angle 4$, $m\angle 5 = m\angle 6$ and $m\angle 7 = m\angle 8$. Using these equalities we find that $m\angle 1 + m\angle 8 + m\angle 4 + m\angle 5 = m\angle 2 + m\angle 3 + m\angle 6 + m\angle 7$. But $m\angle 1 + m\angle 8 + m\angle 4 + m\angle 5 = m\angle A + m\angle C$ and $m\angle 2 + m\angle 3 + m\angle 6 + m\angle 7 = m\angle B + m\angle D$ therefore $m\angle A + m\angle C = m\angle B + m\angle D$. Since the sum of the angles of the quadrilateral ABCD is less than 4 right angles (Corollary 3-7.2) it follows that $m\angle A + m\angle C = m\angle D + m\angle B$ $<$ 2 right angles.

Theorem 4-3.3E. Angles inscribed in the same arc are congruent.

Proof: Let A and E be inscribed angles in arc BAD of circle O (Figure 9). By theorem 4-3.1E $m\angle BAD = \frac{1}{2}m\widehat{BD}$ and $m\angle DEB = \frac{1}{2}m\widehat{BD}$ therefore $m\angle BAD = m\angle DEB$ and $\angle BAD \cong \angle DEB$.

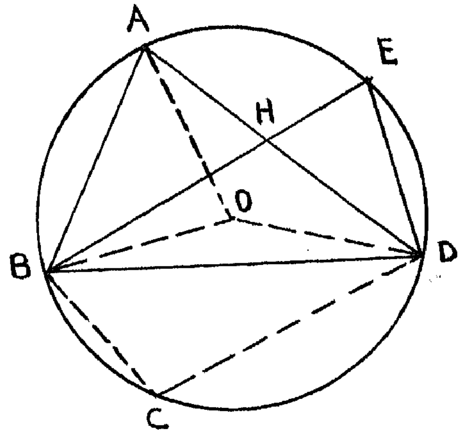


Figure 9

<u>Comments.</u> In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle BAD = \frac{1}{2} m\widehat{BD}$ which is a contradiction of theorem 4-3.1H. The theorem is also invalid because in Hyperbolic Geometry we have the similar theorem:

<u>Theorem 4-3.3H.</u> Angles inscribed in the same arc are congruent if and only if the non-intersecting sides of the angles are congruent.

<u>Proof:</u> (a) We first prove if $\angle BAD \cong \angle DEB$ then $\overline{AB} \cong \overline{ED}$ Take any point C in $\widehat{BD}$ (Figure 9). Introduce $\overline{BC}$, $\overline{CD}$. Assume that $\angle BAD \cong \angle DEB$ i.e. $m\angle BAD = m\angle DEB$. By theorem 4-3.2H $m\angle BAD + m\angle BCD = m\angle ABC + m\angle ADC$ and $m\angle BED + m\angle BCD = m\angle EBC + m\angle EDC$. Because $m\angle BAD = m\angle DEB$ (assumption), it follows that $m\angle ABC + m\angle ADC =$ $m\angle EBC + m\angle EDC$. But $m\angle ABC = m\angle ABE + m\angle EBC$ and $m\angle EDC = m\angle EDA + m\angle ADC$ so by substitution to the previous

equality we have

$m\angle ABE + m\angle EBC + m\angle ADC = m\angle EBC + m\angle EDA + m\angle ADC.$

By cancellation property $m\angle ABE = m\angle EDA$. $\triangle ABH \cong \triangle EHD$ ($\angle ABH \cong \angle EDH$, $\angle BAD \cong \angle HED$, $\angle AHB \cong \angle EHD$) theorem 3.10, therefore $\overline{AB} \cong \overline{ED}$.

(b) We now prove that if $\overline{AB} \cong \overline{ED}$ then $\angle BAD \cong \angle DEB$. Introduce $\overline{OA}$, $\overline{OB}$, $\overline{OD}$, $\overline{OE}$. Assume $\overline{AB} \cong ED$. From congruent triangles AOB, EOD we have $\angle ABO \cong \angle EDO$. Because $\angle OBD \cong \angle ODB$ (angles at the base of isosceles $\triangle ODB$) it follows that $\angle EDB \cong \angle ABD$. Consequently $\triangle EDB \cong \triangle ADB$ (two sides and included angle)and therefore $\angle BAD \cong \angle DEB$.

<u>Theorem 4-3.4E</u>. An angle inscribed in a semicircle is a right angle.

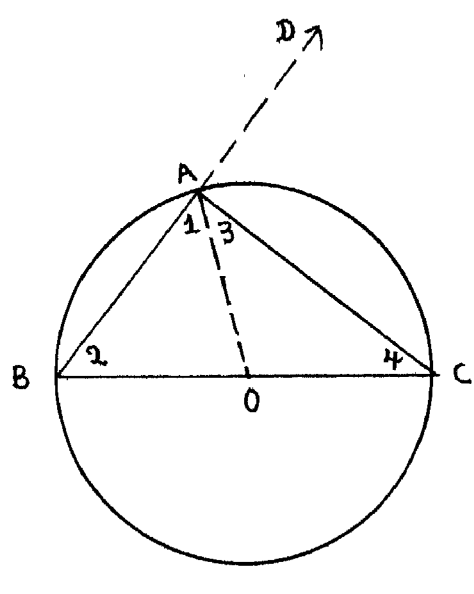


Figure 10

Proof: Let $\angle BAC$ be inscribed in semicircle $\widehat{BAC}$ (Figure 10). Introduce $\overline{AO}$ and produce $\overline{BA}$ to D. From the isosceles triangles OAB, OAC $m\angle 1 = m\angle 2$ and $m\angle 3 = m\angle 4$. Adding these equalities we have $m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$. Because $m\angle DAC = m\angle 2 + m\angle 4$ (exterior angle of $\triangle ABC$) it follows that $m\angle DAC = m\angle 1 + m\angle 3$ and consequently $m\angle DAC = m\angle BAC$ = a right angle.

Comments. In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle DAC = m\angle 2 + m\angle 4$ which is a contradiction of Corollary 3-7.3. The theorem is also invalid because in Hyperbolic Geometry we have the following similar theorem:

Theorem 4-3.4H. An angle inscribed in a semicircle is less than a right angle.

Proof: From the isosceles triangles OBA, OAC (Figure 10) it follows that $m\angle 1 = m\angle 2$, $m\angle 3 = m\angle 4$. By Corollary 3-7.1 $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 < 2$ right angles so $2(m\angle 1) + 2(m\angle 3) < 2$ right angles and $m\angle 1 + m\angle 3 = m\angle BAC <$ a right angle.

Theorem 4-3.5E. The measure of an angle formed by a secant ray and a tangent ray, with its vertex on the circle, is half the measure of the intercepted arc.

Proof: There are three possible cases, so a three part proof is needed.

<u>Case I</u>. When $\angle$BAC is acute (Figure 11a). Introduce diameter $\overline{AE}$ and $\overline{EC}$. By theorem 4-3.4E m $\angle 3$ = a right angle. Because $\angle 4$ is complementary to $\angle 1$ and $\angle 2$ it follows that m $\angle 1$ = m $\angle 2$. But m $\angle 2$ = $\frac{1}{2}$m $\overset{\frown}{AC}$ (Theorem 4-3.1E) consequently m $\angle 1$ = $\frac{1}{2}$m $\overset{\frown}{AC}$.

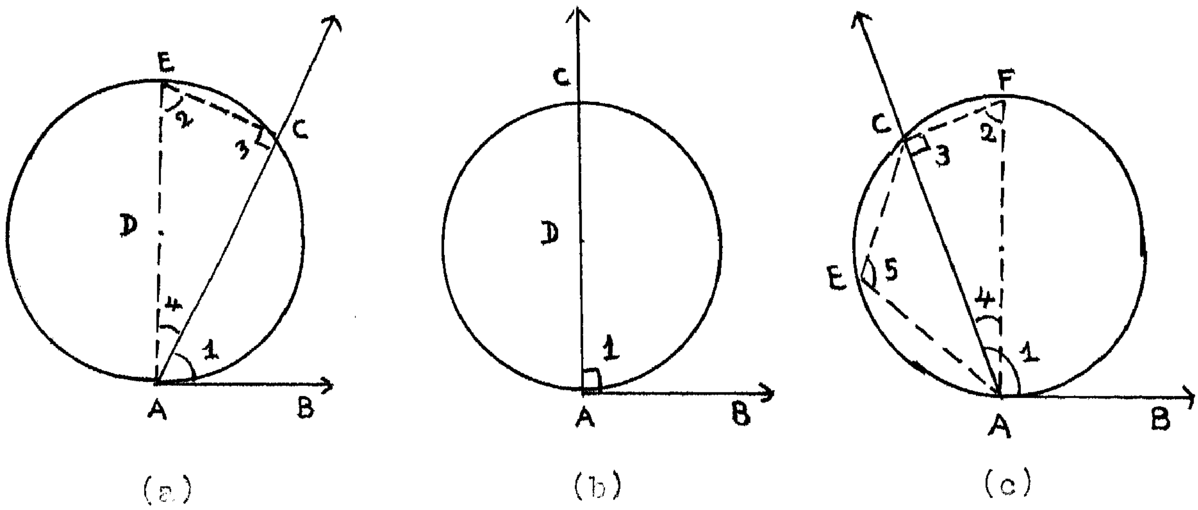


Figure 11

<u>Case II</u>. When $\angle$BAC is right angle (Figure 11b). Because m$\overset{\frown}{AC}$ is 2 right angles it follows that m $\angle 1$ = $\frac{1}{2}$m$\overset{\frown}{AC}$.

<u>Case III</u>. When $\angle$BAC is obtuse (Figure 11c). Choose any point E on minor arc AC. Introduce $\overleftrightarrow{CF}$, $\overline{CE}$, $\overline{EA}$. Because m $\angle 3$ = a right angle (Theorem 4-3.4E) m $\angle 2$ + m $\angle 4$ = a right angle. Consequently m $\angle 1$ + m $\angle 2$ = 2 right angles. By theorem 4-3.2E m $\angle 2$ + m $\angle 5$ = 2 right angles therefore m $\angle 1$ = m $\angle 5$. But m $\angle 5$ = $\frac{1}{2}$m $\overset{\frown}{AFC}$ (Theorem 4-3.1E) hence m $\angle 1$ = $\frac{1}{2}$m $\overset{\frown}{AFC}$.

Comments. In Hyperbolic Geometry the proofs for case I, III are invalid because we used the formula $m\angle 3$ = a right angle which is a contradiction with theorem 4-3.4H. The theorem is partly invalid because in Hyperbolic Geometry we have the following theorem:

Theorem 4-3.5H. The measure of an angle formed by a secant ray and a tangent ray, with its vertex on the circle is greater than, equal to, or less than one half of the intercepted arc, according as the angle is acute, right or obtuse.

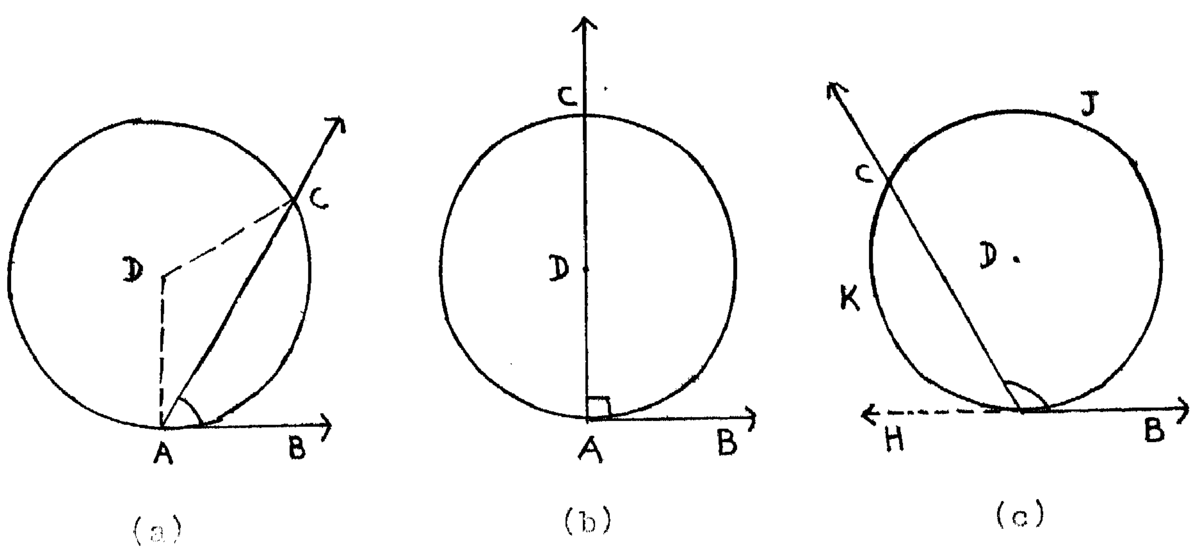


Figure 12

Proof:

Case I. When $\angle BAC <$ a right angle (Figure 12a). Introduce radii $\overline{DA}$ and $\overline{DC}$. $\angle BAC + \angle DAC$ = a right angle and $m\angle DAC = m\angle DCA$ (angles at the base of the isosceles $\triangle ADC$). Using addition property $2(\angle BAC) + \angle DCA + \angle DAC$ = 2 right angles. By Corollary 3-7.1 $m\angle ADC + m\angle DCA + m\angle DAC <$ 2 right angles. Comparing the last equation and last inequality we find that $m\angle BAC > \frac{1}{2}m\angle ADC$ and consequently $m\angle BAC \quad \frac{1}{2}m\widehat{AC}$.

<u>Case II</u>. When $\angle BAC$ is a right angle (Figure 12b). Because $m\widehat{AC}$ is 2 right angles it follows that $m\angle BAC = \frac{1}{2}m\widehat{AC}$.

<u>Case III</u>. When $\angle BAC$ is obtuse (Figure 12c). Produce $\overrightarrow{AB}$ in the opposite direction. Take on the minor arc any point K and on major arc any point J. From case I, $m\angle HAC > \frac{1}{2}m\widehat{AKC}$ so $2(m\angle HAC) > m\widehat{AKC}$. $m\angle HAC + m\angle CAB$ = 2 right angles hence $2(m\angle HAC) + 2(m\angle CAB)$ = 4 right angles. $m\widehat{AKC} + m\widehat{AJC}$ = 4 right angles. By transitive property and the last two equations $2(m\angle HAC) + 2(m\angle CAB) = m\widehat{AKC} + m\widehat{AJC}$. Using the inequality $2(m\angle HAC) > m\widehat{AKC}$ we find that $m\angle CAB < \frac{1}{2}m\widehat{AJC}$.

<u>Theorem 4-3.6E</u>. Parallel lines cut off congruent arcs on a circle.

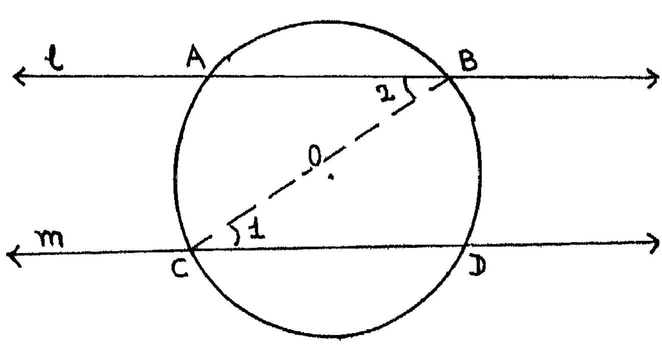
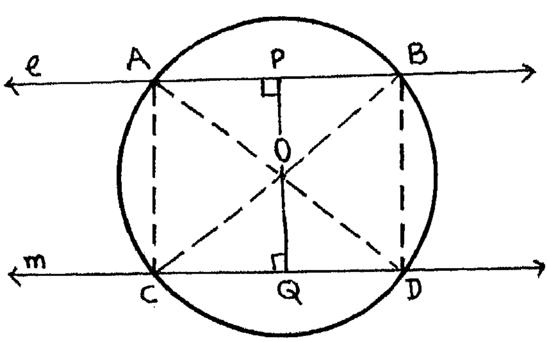


Figure 13

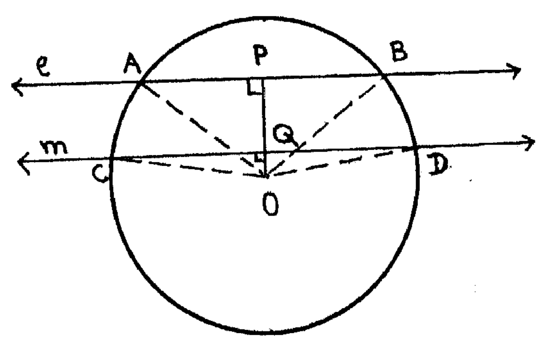
Proof: Let ℓ and m be parallel lines which cut the circle O at A, B and C, D respectively (Figure 13). Introduce $\overline{BC}$. By theorem 4-3.1E $m\angle 1 = \frac{1}{2}m\,\widehat{BD}$, $m\angle 2 = \frac{1}{2}m\,\widehat{AC}$. Since $m\angle 1 = m\angle 2$ (alternate interior angles) $m\widehat{BD} = m\widehat{AC}$ and $\widehat{BD} \cong \widehat{AC}$.

Comments. In Hyperbolic Geometry this proof is invalid because we used $m\angle 1 = \frac{1}{2}m\,\widehat{BD}$ which is a contradiction of theorem 4-3.1H. The theorem is also invalid because in Hyperbolic Geometry we have the following theorem:

Theorem 4-3.5H. (a) Non-intersecting lines cut off congruent arcs on a circle iff the common perpendicular passes through the centre of the circle and (b) parallel lines cut off non-congruent arcs.



(a)



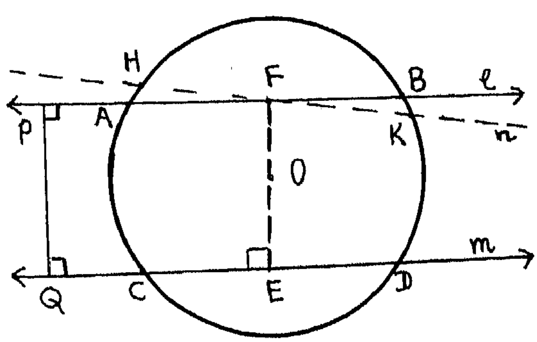
(b)

Figure 14

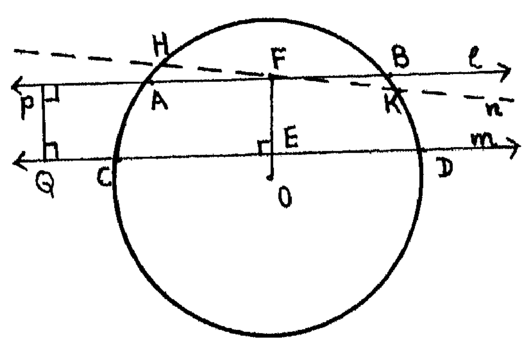
Proof:

Case I. When the common perpendicular passes through the centre of the circle (Figure 14a, b). Let ℓ, m be any two non-intersecting lines which cut the circle at points A, B and C, D respectively and their common perpendicular PQ passes through the center of the circle O. The theorem is valid whether the center is between the parallel lines or not and the following proof applies to both cases. Introduce $\overline{OB}$, $\overline{BD}$, $\overline{OD}$, $\overline{OA}$, $\overline{AC}$ and $\overline{OC}$. $\angle POB \cong \angle POA$ because $\Delta POB \cong \Delta POA$ ($\overline{PO}$ common, $\overline{OB} \cong \overline{OA}$ radii and $\angle APO$, $\angle BPO$ right angles). Similarly $\angle QOC \cong \angle QOD$. From these two equalities and because $\overline{PQ}$ is a straight line $\angle AOC \cong \angle BOD$. Therefore $\widehat{BD} \cong \widehat{AC}$.

Case II. When the common perpendicular does not pass through the centre of the circle (Figure 15a, b). The theorem is valid whether the non-intersecting lines include or not the center of the circle and the following proof applies to both cases. Let ℓ, m be any two



(a)



(b)

Figure 15

non-intersecting lines which cut the circle at points A, B and C, D respectively and their common perpendicular PQ does not pass through the center of the circle O. From center O draw perpendicular to line m. Produce this line to cut line ℓ. Let E, F the two points of intersection. Examination of the quadrilateral PFEQ shows that it is a Lambert Quadrilateral (Theorem 3-6) with acute angle at F. At point F draw perpendicular line n to $\overline{FE}$. Since $\angle$PFE is acute this line will cut the circle at a point H on the arc $\widehat{AB}$. Similarly since the $\angle$BFE is obtuse the line n will cut the circle in a point K on the arc $\widehat{BD}$. Because FE is common perpendicular to the lines n and m it follows from case I that $\widehat{HC} \cong \widehat{KD}$. Since $m\widehat{AC} < m\widehat{HC}$ and $m\widehat{KD} < m\widehat{BD}$ it follows that $m\widehat{AC} < m\widehat{BD}$.

<u>Case III</u>. When the lines are parallels (Figure 16). Let ℓ, m be any two parallel lines which cut the circle O at points A, B and C, D respectively.

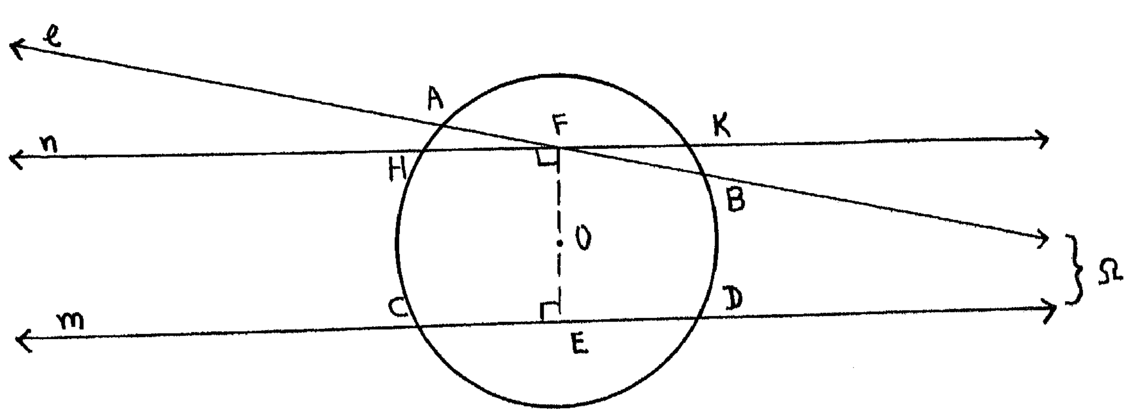


Figure 16

Let E the foot of the perpendicular from O to line m. Produce this perpendicular to cut ℓ. Call this point F. At the point F draw line n perpendicular to $\overline{FE}$. Let H, K the points where this line cut the circle. $\widehat{HC} \equiv \widehat{KD}$, $\widehat{AC} > \widehat{HC}$ and $\widehat{KD} > \widehat{BD}$ and consequently $\widehat{AC} > \widehat{BD}$.

<u>Theorem 4-3.7E</u>. There always exists a circle passing through three given non-collinear points.

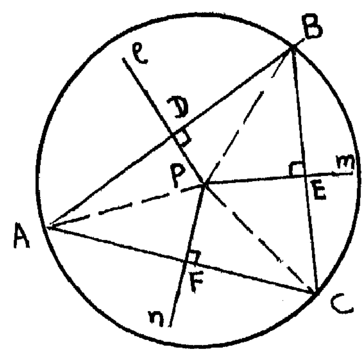


Figure 17

<u>Proof</u>: Let A, B, C be three non-collinear points. Introduce $\overline{AB}$, $\overline{BC}$, $\overline{CA}$. Let D, E be the midpoints of the $\overline{AB}$, $\overline{BC}$ respectively. Draw perpendiculars ℓ, m to these lines at D and E. These lines will meet in a point. If they do not then they will be parallel and consequently BA will be parallel to BC. But this is impossible and so lines ℓ, m will meet in a point. Let this point be P. Introduce $\overline{PA}$, $\overline{PB}$. From the point.

congruent triangles PDB, PAD we see that $\overline{PA} \cong \overline{PB}$. Introduce now $\overline{PC}$. From the congruent triangles PBE, PCE we see that $\overline{PB} \cong \overline{PC}$ and therefore $\overline{PA} \cong \overline{PB} \cong \overline{PC}$. Therefore P is the center of a circle with radius PA and which passes through A, B and C.

Comments. This proof is invalid because the lines ℓ, m may not meet in Hyperbolic Geometry. In Hyperbolic Geometry this theorem is invalid because we have the following similar theorem:

Theorem 4-3.7H. There does not always exist a circle passing through three given non-collinear points.

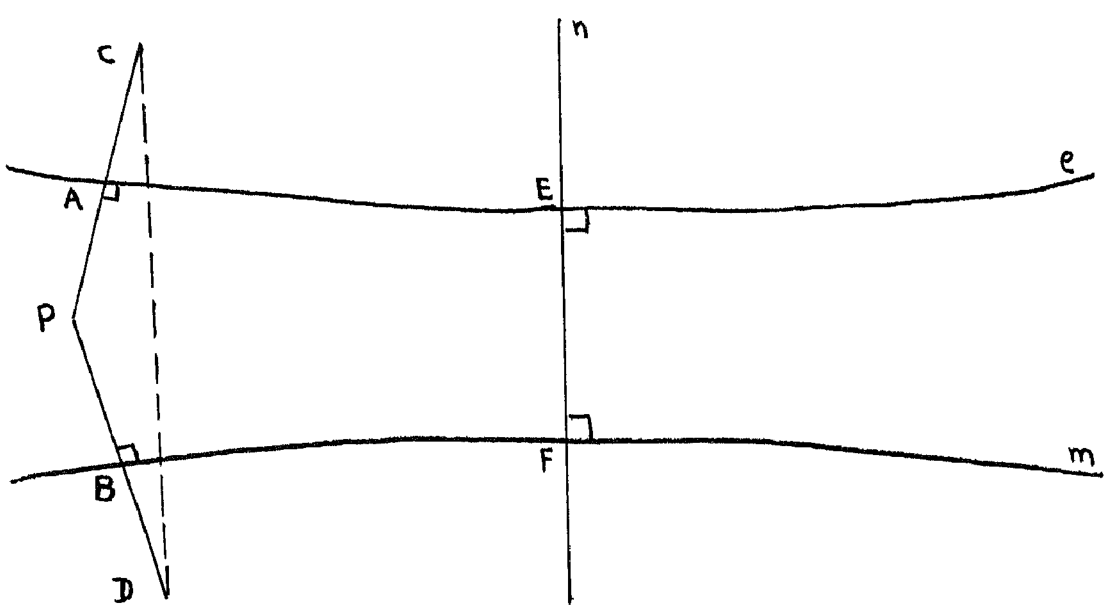


Figure 18

Proof: As in Euclidean Geometry if three points lie on a circle, then they are necessarily the vertexes of a triangl and consequently the perpendicular bisectors of the sides of the triangle pass through the center of the circle. To prove the above theorem we need three points which are the vertexes of a triangle two of whose sides are bisected at right angles by a pair of parallel lines for no circle can pass through three such points. Let ℓ, m be two non-intersecting lines with common perpendicular n. Take any point P not on ℓ, m or n. Draw from P perpendiculars to the lines ℓ, m. Let A, B be the points of contact. Extend $\overline{PA}$ to C so that $\overline{PA} \equiv \overline{AC}$ and $\overline{PB}$ to D so that $\overline{PB} \equiv \overline{BD}$. Introduce $\overline{CD}$, the points P, C, D do not lie on a line because in this case this line would be perpendicular to ℓ, m. This is impossible because n is the only common perpendicular to ℓ, m (Theorem 3-4). So P, C, D are non-collinear points are there is no circle on P, C and D because in this case, as we said in the beginning of the proof, the perpendicular bisectors ℓ, m must pass through the center of the circle.

<u>Theorem 4-3.8E.</u> The measure of an angle formed by two secants intersecting in the interior of a circle is half the sum of the measures of the arcs intercepted by the angle and its vertical angle.

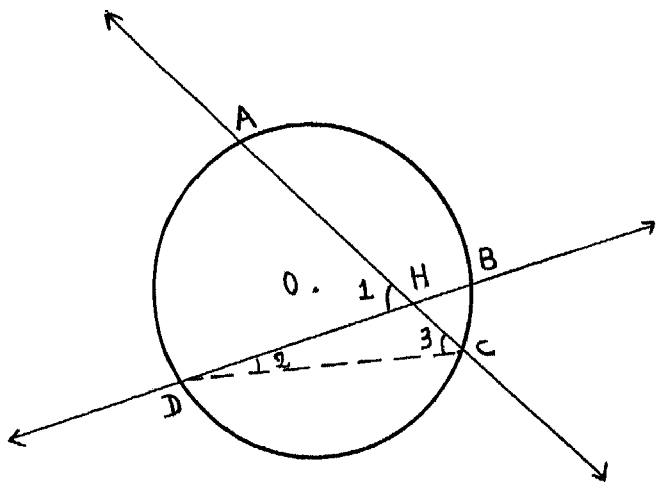


Figure 19

Proof: Let $\overleftrightarrow{AC}$, $\overleftrightarrow{DB}$ intersecting in the interior of the circle O (Figure 19). Introduce $\overline{DC}$. $m\angle 1 = m\angle 3 + m\angle 2$ ($m\angle 1$ exterior angle of the Δ HCD). By theorem 4-3.1E $m\angle 3 = \frac{1}{2}m\widehat{AD}$ and $m\angle 2 = \frac{1}{2}m\widehat{BC}$. Consequently $m\angle 1 = \frac{1}{2}(m\widehat{AD}+m\widehat{BC})$.

Comments. In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle 1 = m\angle 3 + m\angle 2$ which is a contradiction of Corollary 3-7.3. The theorem is also invalid because in Hyperbolic Geometry we have the following theorem:

Theorem 4-3.8H. The measure of an angle formed by two secants intersecting in the interior of a circle is related to the sum of the measures of the arcs intercepted by the angle and its vertical angle as follows:

(a) When the vertex of the angle is the center of the circle or when one side is diameter and the other intersects it at right angle, the angle is half the sum of the measures of the intercepted arc.

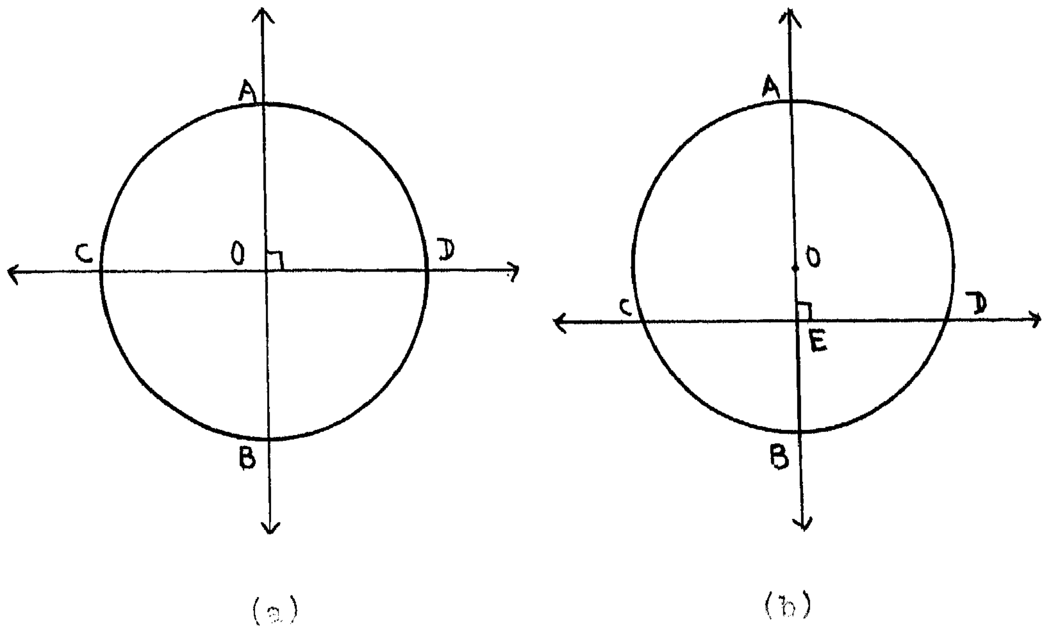
(b) When one side is diameter, the angle, is less than or greater than the half of the sum of the intercepted arcs according as it is acute or obtuse.

(c) When neither side is diameter and the angle contains the center and is divided into two angles which are not obtuse by the line joining the vertex to the center then the angle is less than the half of the intercepted arc.

(d) When neither side is diameter and the angle contains the center and is divided into two angles one of which is obtuse, by the line joining the vertex to the center, then the angle may be equal to, less than or greater than the half of the intercepted arc.

<u>Proof</u>:

(a) When the vertex of the angle lies on the center of the circle or when one line is diameter and the other intersects it at right angle, the angle is half the sum of the measures of the intercepted arc.



(a)

(b)

Figure 20

Case I. When the vertex lies at the center of the circle (Figure 20a). Because $\angle AOC$, $\angle BOD$ are central angles $m\angle AOC = m\widehat{AC}$, $m\angle BOD = m\widehat{BD}$. From this and because $m\angle AOC = m\angle BOD$ (vertical angles) it follows that $m\angle AOC = \frac{1}{2}(m\widehat{AC}+m\widehat{BD})$.

Case II. When the vertex of the angle is not at the center of the circle (Figure 20b). Because $\overline{AB}$ is perpendicular to the chord $\overline{CD}$ it follows that $m\widehat{BD} = m\widehat{BC}$ and $m\widehat{AC} = m\widehat{AD}$ and consequently $m\widehat{AC} + m\widehat{BD} = m\widehat{AD} + m\widehat{BC} = 2$ right angles. Since $\angle AEC$ is a right angle it follows that $m\angle AEC = \frac{1}{2}(m\widehat{AC}+m\widehat{BD})$.

(b) When one side of the angle is diameter(Figure 21), the angle, is less than or greater than the half of the sum of the intercepted arcs according as it is acute or obtuse.

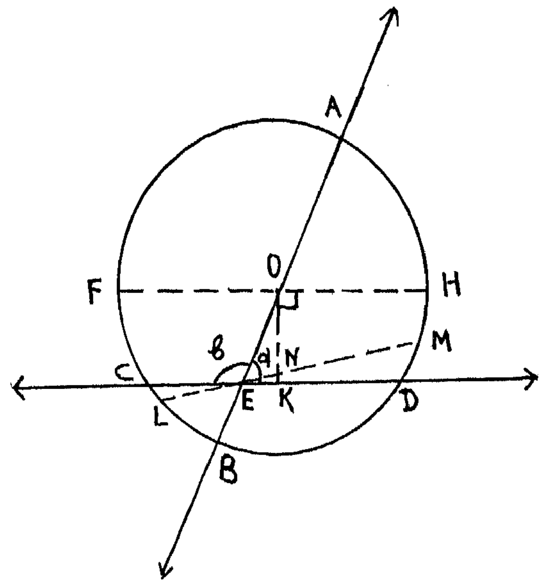


Figure 21

Let α be any acute angle with one side a diameter and β any obtuse angle with one side a diameter. Then $\angle AED$ or $\angle BEC$ can represent α and $\angle AEC$ or $\angle BED$ can represent β. Since $\angle AED = \angle BEC$ and $\angle BED = \angle AEC$ assume $\alpha = \angle AED$ and $\beta = \angle AEC$. Construct at the center of the circle a diameter $\overline{FH}$ such that $\angle FOE \equiv \angle AED$. Draw perpendicular to the diameter $\overline{FH}$ at the center of the circle. Let K the point that this perpendicular meets $\overline{CD}$. Since m $\angle FOK$ = a right angle it follows that m $\angle OKE <$ a right angle and consequently $\angle OKD$ is obtuse. At the point K draw chord $\overline{LM}$ perpendicular to $\overline{OK}$. Let M, L be the points in which this line cuts the circle and N the point of intersection with $\overline{OK}$. Comparing the figures ONMH, ONLF we see that $\overline{HM} \equiv \overline{FL}$ and $\widehat{HM} \equiv \widehat{FL}$. Since $\widehat{DH} > \widehat{HM}$ and $\widehat{FC} < \widehat{FL}$ it follows that $\widehat{HD} > \widehat{FC}$. From part (a) case I we know that $m\angle FOB = \frac{1}{2}(m\widehat{FB} + m\widehat{AH})$. Because $m\widehat{AH} + m\widehat{FB} < m\widehat{AD} + m\widehat{BC}$ it follows that $m\angle AED < \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$. Since $\angle AEC$ is supplementary of the $\angle AED$ it follows that $m\angle AEC > \frac{1}{2}(m\widehat{AFC} + m\widehat{BD})$.

(c) When the two sides are not diameters and the angles formed by the line joining the center of the circle and the vertex of the angle are not obtuse, by the line joining the vertex to the center then the angle is less than the half of the intercepted arc.

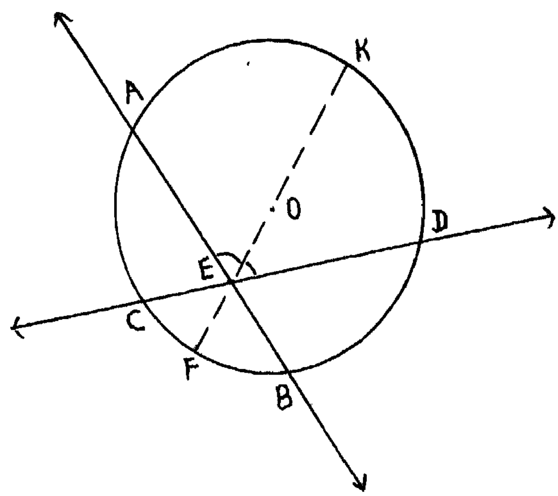


Figure 22

Let $\angle AED$ in Figure 22 be an angle such that $\angle AEO$ and $\angle OED$ are acute or right. Join points O, E. Produce this line. Let F, K be the points in which it cuts the circle. From parts a and b $m\angle KEA \leqslant \frac{1}{2}(m\widehat{AK} + m\widehat{FB})$ and $m\angle KED \leqslant \frac{1}{2}(m\widehat{KD} + m\widehat{CF})$ (but not both equal since both angles cannot be right) consequently $m\angle KEA + m\angle KED = m\angle AED < \frac{1}{2}(m\widehat{AKD} + m\widehat{CFB})$ Because $\angle AEC$ is supplementary angle of $\angle AED$ it follows that $m\angle AEC > \frac{1}{2}(m\widehat{AC} + m\widehat{BD})$.

(d) When the two sides are not diameter and the angle contains the center and is divided into two angles one of which is obtuse, by the line joining the vertex to the center, then the angle's relation to the arc is undetermined.

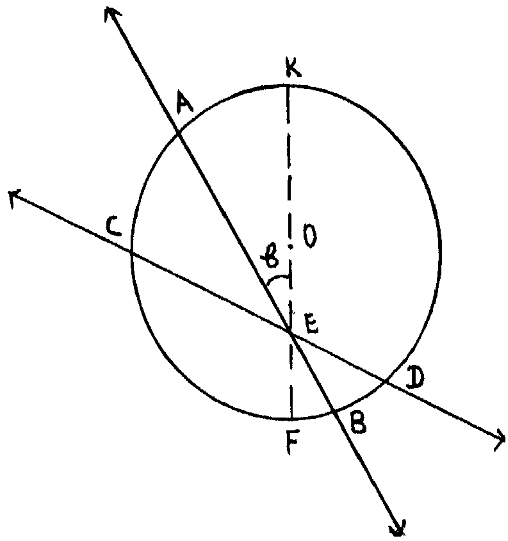


Figure 23

Let $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ be the two lines and $\angle KED > $ a right angle (Figure 23). From part b $m\angle KED > \frac{1}{2}(m\widehat{KD} + m\widehat{CF})$ which means that $m\angle KED = \frac{1}{2}(m\widehat{KD} + m\widehat{CF}) + \varepsilon$ where $\varepsilon > 0$. And $m\angle b < \frac{1}{2}(m\widehat{AK} + m\widehat{FB})$ which may be written as $\frac{1}{2}(m\widehat{AK} + m\widehat{FB}) - m\angle b > 0$. Suppose that the points A is moving towards K. The angle b becomes smaller and smaller and $\frac{1}{2}(m\widehat{AK} + m\widehat{FB}) - m\angle b \rightarrow 0$. But there exists a b^* such that $\frac{1}{2}(m\widehat{AK} + m\widehat{FB}) - m\angle b^* = \varepsilon$ and $m\angle b^* = \frac{1}{2}(m\widehat{AK} + m\widehat{FB}) - \varepsilon$. Adding this angle to $\angle KED$ we have $m\angle KED + m\angle b^* = \frac{1}{2}(m\widehat{KD} + m\widehat{CF}) + \varepsilon + \frac{1}{2}(m\widehat{AK} + m\widehat{FB}) - \varepsilon = \frac{1}{2}(m\widehat{AKD} + m\widehat{CFB})$. Since $m\angle KED + m\angle b = m\angle AED$ it follows $m\angle AED = \frac{1}{2}(m\widehat{AKD} + m\widehat{CFB})$. Obviously if angle $b > b^*$ then $\angle AED > \frac{1}{2}(m\widehat{AKD} + m\widehat{CFB})$ and if angle $b < b^*$ then $\angle AED < \frac{1}{2}(m\widehat{AKD} + m\widehat{CFB})$.

<u>Theorem 4-3.9E.</u> The measure of the angle formed by two tangents drawn from an external point to a circle is half the difference of the measures of the intercepted arcs.

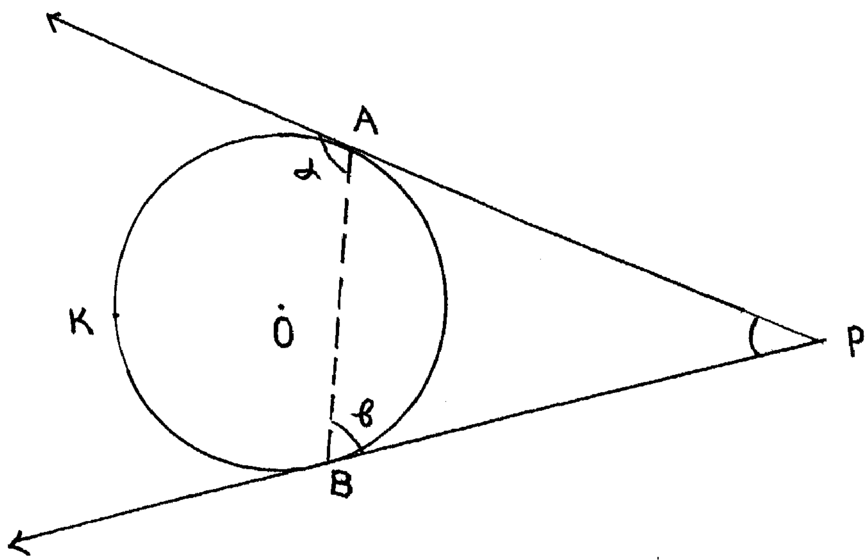


Figure 24

<u>Proof:</u> Let $\overrightarrow{PA}$, $\overrightarrow{PB}$ tangents to the circle O from a point P outside the circle (Figure 24). Introduce $\overline{AB}$. Take any point K in the major arc. $m\angle\alpha = m\angle P + m\angle\beta$. By subtraction property this becomes $m\angle P = m\angle\alpha - m\angle\beta$. By theorem 4-3.1E $m\angle\alpha = \frac{1}{2}m\widehat{AKB}$ and $m\angle\beta = \frac{1}{2}m\widehat{AB}$. Hence $m\angle P = \frac{1}{2}(m\widehat{AKB} - m\widehat{AB})$.

<u>Comments.</u> In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle\alpha = m\angle P + m\angle\beta$ which is a contradiction of Corollary 3-7.3. The theorem is also invalid because in Hyperbolic Geometry we have the following similar theorem:

<u>Theorem 4-3.9H</u>. The measure of the angle formed by two tangents drawn from an external point to a circle is less than the half of the difference of the measures of the intercepted arcs.

<u>Proof</u>: We need to prove that $m\angle P < \frac{1}{2}(m\widehat{AKB} - m\widehat{AB})$ (Figure 24). By Corollary 3-7.3 $m\angle\alpha > m\angle P + m\angle\beta$. It follows that $m\angle P < m\angle\alpha - m\angle\beta$. Since for any position of P, $\alpha > 90^0$ and $\beta < 90^0$ by theorem 4-3.5H $m\angle\alpha < \frac{1}{2}mAKB$ and $m\angle\beta > \frac{1}{2}mAB$. Substituting the last two inequalities to $m\angle P < m\angle\alpha - m\angle\beta$ we find $m\angle P < \frac{1}{2}(m\widehat{AKB} - m\widehat{AB})$.

<u>Theorem 4-3.10E</u>. The measure of the angle formed by a secant and a tangent intersecting outside a circle is half the difference of the measures of the intercepted arcs.

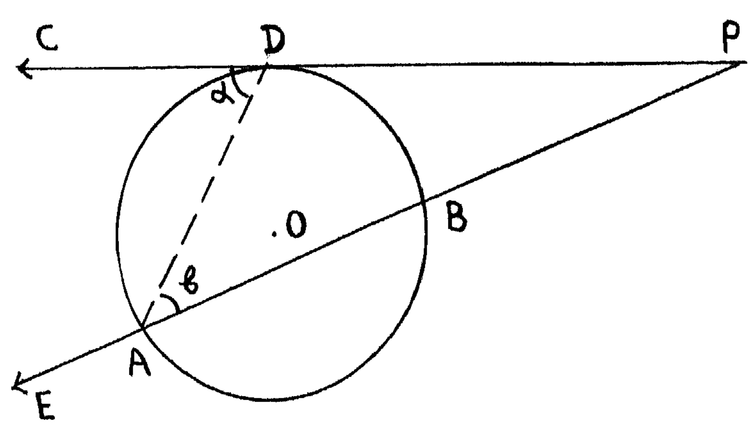


Figure 25

Proof: Let $\overrightarrow{DC}$ be tangent to the circle O at D and $\overrightarrow{PE}$ cut the circle at B, A (Figure 25). Introduce $\overline{DA}$. $m\angle\alpha = m\angle P + m\angle\beta$ (exterior angle of Δ DPA). By subtraction property this becomes $m\angle P = m\angle\alpha - m\angle\beta$. By theorem 4-3.5E $m\angle\alpha = \frac{1}{2}m\widehat{DA}$ and by theorem 4-3.1E $m\angle\beta = \frac{1}{2}m\widehat{DB}$. Substituting the last two equalities to $m\angle P = m\angle\alpha - m\angle\beta$ we find $m\angle P = \frac{1}{2}(m\widehat{DA} - m\widehat{DB})$.

Comments. In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle\alpha = m\angle P + m\angle\beta$ which is a contradiction of Corollary 3-7.3. The theorem is also invalid because in Hyperbolic Geometry we have the following similar theorem:

Theorem 4-3.10H. The measure of the angle formed by a secant and a tangent intersecting outside a circle is less than the half of the difference of the measures of the intercepted arcs when the center of the circle lies in the interior of the angle or on the secant.

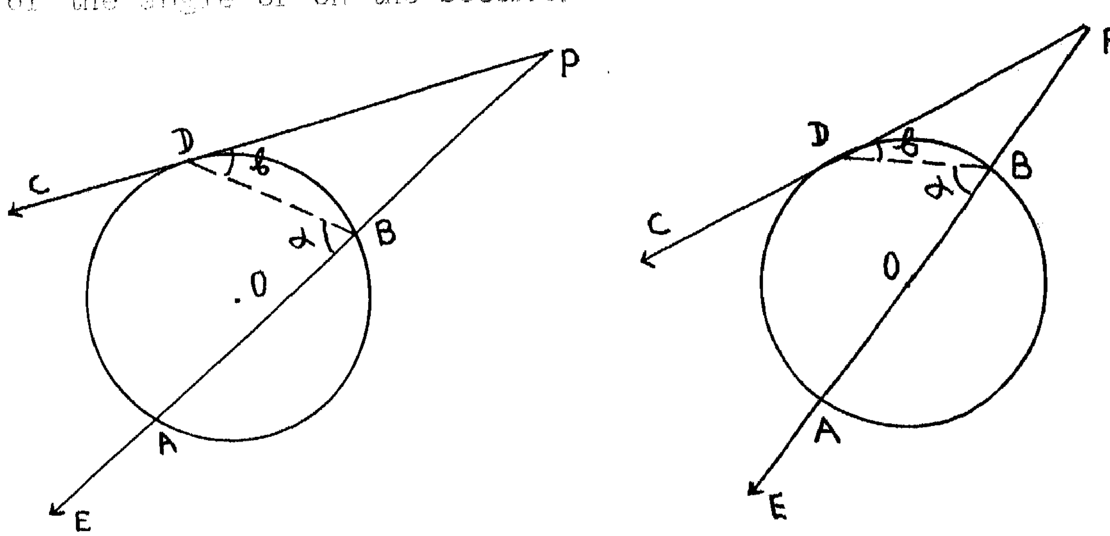


Figure 26

Proof: Let $\overrightarrow{PC}$ be tangent to the circle O at D and $\overrightarrow{PE}$ cut the circle at B, A (Figure 26). Introduce $\overline{DB}$. By Corollary 3-7.3 $m\angle\alpha > m\angle P + m\angle\beta$. Using subtraction property this becomes $m\angle P < m\angle\alpha - m\angle\beta$. By theorem 4-3.1H $m\angle\alpha < \frac{1}{2}m\widehat{AD}$ and because β is always acute, by theorem 4-3.5H $m\angle\beta > \frac{1}{2}m\widehat{DB}$. Substituting the last two inequalities to the first one we have $m\angle P < \frac{1}{2}(m\widehat{AD} - m\widehat{DB})$.

Comments. When the center of the circle lies in the exterior of the angle the above proof is invalid because the measure of angle α, as secant moves towards PC changes from less than to equal and then to greater than the half of the intercepted arc (see Theorem 4-3.1H).

Theorem 4-3.11H. When the standard unit of length is used (see 3-9) the circumference of a circle of given radius is always greater than its circumference as determined in Euclidean Geometry.

Proof: We consider the case in which the standard unit of length is used. Let O a circle with radius r and C_E, C_H the circumferences in Euclidean and Hyperbolic Geometry respectively. The circumferences are given by the formulas $C_E = 2\pi r$ and $C_H = 2\pi \sinh r$ (section 3-9). Substitute the following series to the formula $C_H = 2\pi \sinh r$

$$\sinh r = r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots$$

$$C_H = 2\pi\left(r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots\right)$$

$$C_H = 2\pi r + 2\pi\left(\frac{r^3}{3!} + \frac{r^5}{5!} + \ldots\right)$$
$$= C_E + 2\pi\left(\frac{r^3}{3!} + \frac{r^5}{5!} + \ldots\right)$$

Since the expression in the parenthesis is always positive, it follows that the circumference in Hyperbolic Geometry of given circle is greater than the circumference in Euclidean Geometry.

<u>Theorem 4-3.12H.</u> The ratio of the circumference of a circle to the diameter (a) is not constant, (b) exceeds π (c) approaches π when the diameter approaches 0.

<u>Proof:</u> We consider the case in which the standard unit of length is used. Let O be a circle with radius r and circumference C_H and $p = \frac{C_H}{2r}$ the ratio of the circumference to the diameter. From section 3-9 we have $C_H = 2\pi \sinh r$. So $p = \frac{2\pi \sinh r}{2r}$. Substitute the following series to our ratio

$$\sinh r = r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots$$

$$p = \frac{2\pi}{2r}\left(r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots\right) \qquad (1)$$

$$= \frac{\pi}{r}\left(r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots\right) \qquad (2)$$

Let us examine now the behavior of p with different values of r. Let r = 1, 2, 3. Using tables for the sinh we find $p_1 = \frac{2\pi \sinh (1)}{2\cdot 1} = 1.1752\,\pi$ $\quad p_2 = \frac{2\pi \sinh (2)}{2\cdot 2} = 3.6265\,\pi$

$p_3 = \frac{2\pi \sinh (3)}{2\cdot 3} = 10.018\,\pi$

From these results one can conclude that the ratio is not constant.

From step 2 we find $p = \pi + \pi \left(\frac{r^2}{3!} + \frac{r^4}{5!} + \ldots\right)$.

Because the expression in the parenthesis is always positive it follows that the ratio exceeds π, $p > \pi$.

Using the expression of the ratio of the last paragraph we have

$$\lim_{r \to 0} p = \lim_{r \to 0} \left\{ \pi + \pi \left[\frac{r^2}{3!} + \frac{r^4}{5!} + \ldots \right] \right\}$$

$$= \lim_{r \to 0} \pi + \lim_{r \to 0} \pi \left[\frac{r^2}{3!} + \frac{r^4}{5!} + \ldots \right]$$

$$= \pi$$

Therefore the ratio p approaches π when the diameter approaches 0.

Theorem 4-3.13H. When the standard unit of length is used (see section 3-9) the area of a circle of given radius is always greater than its area as determined in Euclidean Geometry.

Proof: We consider the case in which the standard unit of length is used. Let O a circle with radius r and A_E, A_H the areas in Euclidean and Hyperbolic Geometry respectively. The areas are given by the formulas

$$A_E = \pi r^2 \qquad \text{and } A_H = 4\pi \sinh^2 \frac{r}{2} \qquad \text{(section 3-9)}$$

Substitute the following series in the formula for A_H

$$\sinh r = r + \frac{r^3}{3!} + \frac{r^5}{5!} + \ldots$$

we find

$$A_H = 4\pi\left(\frac{r}{2}\right)^2 \frac{\sinh^2 r/2}{(r/2)^2} = \pi r^2 \left(\frac{\sinh r/2}{r/2}\right)^2$$

$$= \pi r^2 \left(\frac{\frac{r}{2} + \frac{(r/2)^3}{3!} + \frac{(r/2)^5}{5!} + \ldots}{r/2} \right)^2$$

$$= \pi r^2 \left(1 + \frac{(r/2)^2}{3!} + \frac{(r/2)^4}{5!} + \ldots \right)^2$$

$$= A_E \left(1 + \frac{(r/2)^2}{3!} + \frac{(r/2)^4}{5!} + \ldots \right)^2$$

Because the expression in the parenthesis is always greater than 1 it follows that $A_H > A_E$ which means that the area of a circle of given radius is greater than that found in Euclidean Geometry.

.4. Invalid theorems in Hyperbolic Geometry-No similar theorems.

Theorem 4-4.1E. If $\overline{PT}$ is a tangent segment to a circle, nd $\overline{PB}$ segment intersecting the circle at A and B then $\overline{PA} \cdot \overline{PB} = (\overline{PT})^2$.

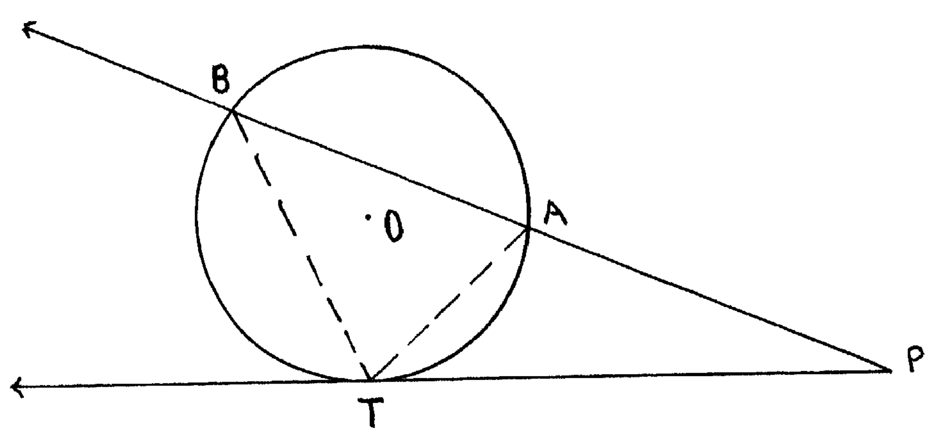


Figure 27

Proof: Introduce the chords $\overline{AT}$, $\overline{TB}$. From theorem 4-3.1E $m\angle ABT = \frac{1}{2}m\widehat{AT}$. From theorem 4-3.5E $m\angle ATP = \frac{1}{2}m\widehat{AT}$. It follows $\angle ABT \cong \angle ATP$. The triangles BTP, ATP are similar ($\angle P \cong \angle P$, $\angle ATP \cong \angle ABT$, $\angle BTP \cong \angle PAT$) therefore $\frac{PB}{PT} = \frac{PT}{PA}$ and $\overline{PA} \cdot \overline{PB} = (\overline{PT})^2$.

Comments. In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle ABT = \frac{1}{2}m\widehat{AT}$ which is a contradiction of theorem 4-3.1H.

The theorem is also invalid because it is based on the Pythagorean theorem which is invalid in Hyperbolic Geometry (see section 3-8). Let us see an example. Consider the case for which the segment $\overline{PB}$ passes through the center of the circle (Figure 28).

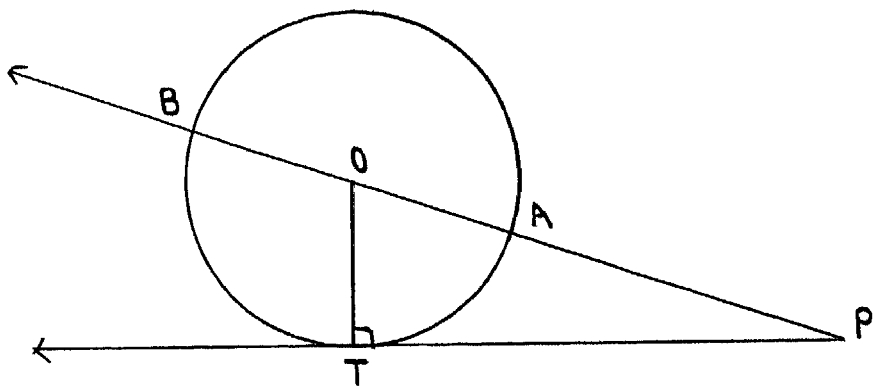


Figure 28

Using the formula of the above theorem we find:

$$
\begin{aligned}
PT^2 &= PA \cdot PB \\
&= PA\ (PA + AB) \\
&= PA\ (PA + 2AO) \\
&= PA^2 + 2\ PA \cdot AO \\
&= PA^2 + 2\ PA \cdot AO - AO^2 + AO^2
\end{aligned}
$$

$$= (PA + AO)^2 - AO^2$$
$$= PO^2 - OT^2$$
$$\text{and } PO^2 = PT^2 + OT^2$$

The last step is actually the conclusion of the Pythagorean theorem in the right triangle PTO. Therefore the above theorem is invalid in Hyperbolic Geometry.

<u>Theorem 4-4.2E.</u> If two chords intersect within a circle, the product of the segments of one chord equals the product of the segments of the other.

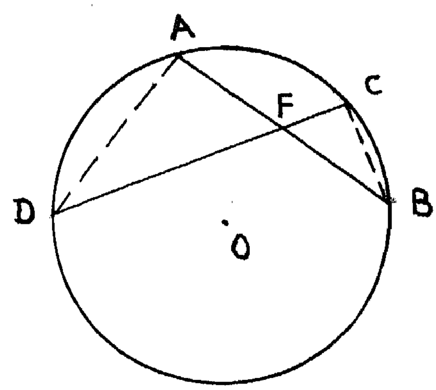


Figure 29

<u>Proof:</u> Let $\overline{AB}$, $\overline{CD}$ chords intersecting in the interior of the circle O (Figure 29). We need to prove that $\overline{AF} \cdot \overline{FB} = \overline{CF} \cdot \overline{FD}$. Introduce $\overline{AD}$, $\overline{BC}$. From $m\angle D = \frac{1}{2}m\widehat{AC}$, $m\angle B = \frac{1}{2}m\widehat{AC}$ (Theorem 4-3.1E) it follows $\angle D \cong \angle B$. The triangles FAD, FCB are similar ($\angle AFD \cong \angle CFB$, $\angle D \cong \angle B$) therefore

$\frac{AF}{CF} = \frac{FD}{FB}$ and $\overline{AF} \cdot \overline{FB} = \overline{CF} \cdot \overline{FD}$.

Comments. In Hyperbolic Geometry this proof is invalid because we used the formula $m\angle D = \frac{1}{2}m\widehat{AC}$ which is a contradiction of theorem 4-3.1H.

The theorem is also invalid because it is based on the Pythagorean theorem which is invalid in Hyperbolic Geometry (see section 3-8). Let us see an example. Consider the case for which one chord passes through the center of the circle and the other intersects it at right angle in the half distance of the radius (Figure 30).

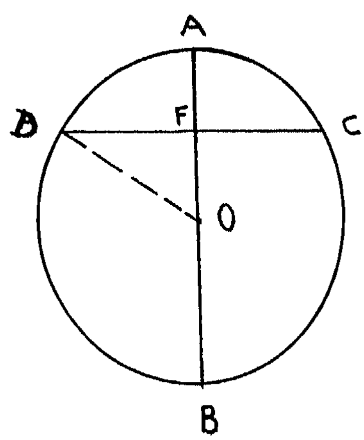


Figure 30

Using the formula of the above theorem we find:

$$CF \cdot FD = AF \cdot FB$$

$$FD^2 = AF(FO + OB)$$

$$= r/2(r/2 + r)$$

$$= 3/4\ r^2$$

$$= 4/4\ r^2 - r^2/4$$
$$= DO^2 - FO^2$$
and $DO^2 = FD^2 + FO^2$

The last step is actually the conclusion of the Pythagorean theorem in the right triangle DFO. Therefore the above theorem is invalid in Hyperbolic Geometry.

Theorem 4-4.3E. If from the vertex angle of a triangle a straight line be drawn perpendicular to the base, the product of the sides of the vertex angle is equal to the product of the perpendicular and the diameter of the circle described about the triangle.

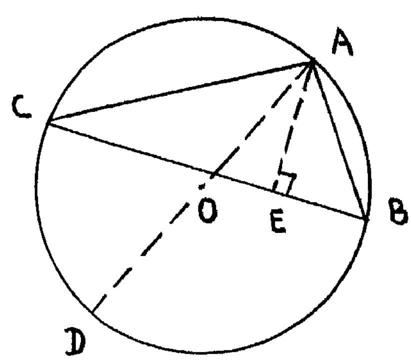


Figure 31

Proof: Let ΔABC inscribed in circle O (Figure 31). Introduce diameter $\overline{AD}$ and $\overline{AE}$ perpendicular to $\overline{BC}$. Join $\overline{BD}$.

We need to prove that $\overline{AC} \cdot \overline{AB} = \overline{AE} \cdot \overline{AD}$. Because ABD is a right angle (Theorem 4-3.4E) it follows $\angle ABD \cong \angle AEC$. From theorem 4-3.3E we know that $\angle ADB \cong \angle ACB$. Therefore the triangles ACE, ADB are similar ($\angle ABD \cong \angle AEC$, $\angle ADB \cong \angle ACB$) and $\frac{AC}{AD} = \frac{AE}{AB}$ or $\overline{AC} \cdot \overline{AB} = \overline{AE} \cdot \overline{AD}$.

Comments. In Hyperbolic Geometry this proof is invalid because we considered ABF as a right angle which is a contradiction with theorem 4-3.4H.

The theorem is also invalid because it is based on the Pythagorean theorem which is invalid in Hyperbolic Geometry. Let us see an example. Consider the case for which the triangle is isoceles and one side passes through the center of the circle (Figure 32).

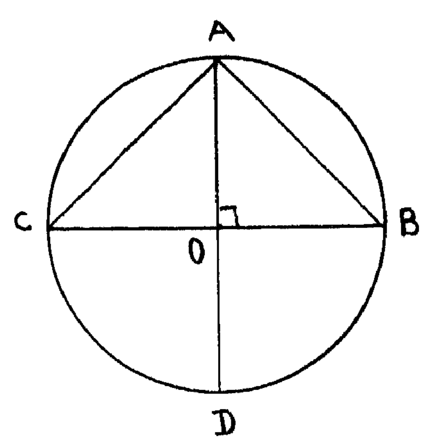


Figure 32

Using the formula of the above theorem we find:

$$AC \cdot AB = AO \cdot AD$$

$$AB^2 = AO \cdot 2(AO)$$

$$= 2(AO)^2$$

$$\text{and } AB^2 = AO^2 + OB^2$$

The last step is actually the conclusion of the Pythagorean theorem in the right triangle AOB. Therefore the above theorem is invalid in Hyperbolic Geometry.

CHAPTER V

CONCLUSION

5.1. Summary. The purpose of this paper was to present the evolution of knowledge of the properties of the circle and to determine which Euclidean theorems for circles are valid in Hyperbolic Geometry and to show how some theorems can be valid with some changes.

In Chapter I, defined and undefined terms which were to be used in this paper, were stated.

Chapter II gave a brief history of the circle from ancient years to the present.

Chapter III presented a brief history of the discovery of Hyperbolic Geometry, concetrating on the events taking place after Saccheri developed the three hypotheses for the Saccheri quadrilateral. Also presented in this chapter were some important elements of the circle in Hyperbolic Geometry that were used in Chapter IV as references.

Chapter IV was a development of the following: (a) the theorems about circles which are valid in both Geometries (b) the Euclidean theorems which are invalid in Hyperbolic Geometry but for which it was possible to give similar theorems in Hyperbolic Geometry and (c) the Euclidean theorems which are invalid in Hyperbolic Geometry but for which similar theorems for Hyperbolic Geometry were not discovered by the writer.

5.2. Suggested Research. There are two main areas for further research suggested by the writer.

A. (I) In Chapter IV, it was shown that the following theorems are invalid in Hyperbolic Geometry. For the student interested in pursuing these ideas in more detail, it may prove fruitful to try to discover theorems similar to the following:

a) If PT is a tangent segment to a circle and PB segment intersecting the circle at A and B then $PA \cdot PB = (PT)^2$.

b) If two chords intersect within a circle, the product of the segments of one chord equals the product of the segment of the other.

c) If from the vertex angle of a triangle a straight line can be drawn perpendicular to the base, the product of the sides of the vertex angle is equal to the product of the perpendicular and the diameter of the circle described about the triangle.

(II) Another question which might be considered is: Is the following theorem valid in Hyperbolic Geometry? If not, then perhaps a similar theorem in Hyperbolic Geometry can be discovered and proved.

The measure of the angle formed by the two secants intersecting outside a circle is half the difference of the measures of the intercepted arcs.

B. In Chapter IV, the fact that the Pythagorean theorem was invalid in Hyperbolic Geometry was used to prove that some Euclidean theorems were invalid in Hyperbolic Geometry. There are many other Euclidean theorems which are based on the Pythagorean theorem. A student interested in this area may find similar theorems in Hyperbolic Geometry for these theorems.

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