ORTHOGONAL POLYNOMIALS

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CHAPTER 1

INTRODUCTION

The main purpose of this paper is to find a best approximating polynomial of a given degree for a continuous function defined on the interval $[a, b]$. In order to do this, one needs to understand metric spaces, normed linear spaces and inner product spaces.

Chapter 2 illustrates the construction of linear spaces and its subspaces. In Chapter 3, normed linear spaces are introduced. It is also shown that every metric space has a unique completion. Chapter 4 introduces Hilbert spaces and the Gram-Schmidt process. A normed linear space may be complete with respect to one norm but not complete with respect to another one. This concept is illustrated by an example in Chapter 5. Orthogonal sets are introduced in Chapter 6, and some special cases constructed in Chapter 7. In Chapter 8, it is shown that the best approximating polynomial of a given degree to a continuous function on a closed interval does exist and is unique. The application of min-max approximation is also given in Chapter 8.

CHAPTER 2

LINEAR SPACES

In this chapter, the concepts of a linear space and a basis for a linear space are discussed.

2.1 LINEAR SPACES

DEFINITION. A linear space V is a nonempty additive abelian group together with a function $F \times V \to V$, where F is a field, defined by $(\alpha, x) \to \alpha x$; satisfying the following properties, for all $\alpha, \beta \in F$, $x, y \in V$.

(i) $\alpha(x + y) = \alpha x + \alpha y$;

(ii) $(\alpha + \beta)x = \alpha x + \beta x$;

(iii) $(\alpha\beta)x = \alpha(\beta x)$;

(iv) $1x = x$.

EXAMPLE 1.1. The set of all real numbers, with addition and multiplication taken as the operations, is a linear space.

EXAMPLE 1.2. The set R^n of all n-tuples of real numbers is a linear space under the following operations.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, and define $x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$,

$$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n).$$

EXAMPLE 1.3. The set $C_2[a, b]$ of all bounded continuous real functions defined on $[a, b]$ is a linear space over R if the sum $h = f + g$ and the scalar product

$h' = \alpha \cdot f$ are the functions defined for each $x \in [a, b]$ by the equations

$$(f + g)(x) = f(x) + g(x),$$

$$(\alpha f)(x) = \alpha f(x).$$

This space will be discussed in some detail in Chapter 5.

EXAMPLE 1.4. Let P be the set of all polynomials, with real coefficients, defined on the interval $[-1, 1]$. P is a linear space over R, the real numbers, by the usual addition of two polynomials and the multiplication of a polynomial by a real number.

DEFINITION. Let V be a linear space over F. Let $v_1, v_2, \ldots, v_n$ belong to V and $c_1, c_2, \ldots, c_n$ be elements in F. Then the vector $c_1v_1 + c_2v_2 + \ldots + c_nv_n$ is said to be a linear combination of the vectors $v_1, \ldots, v_n$.

DEFINITION. Let S be a subset of V and $v_1, v_2, \ldots, v_n$ be vectors in V. The collection $v_1, v_2, \ldots, v_n$ is said to span S provided every vector in S can be written as a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

2.2 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

DEFINITION. A set of vectors $v_1, v_2, \ldots, v_n$ in V is said to be linearly dependent provided there exist scalars $c_1, c_2, \ldots, c_n$ in F, not all zero, such that $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$.

DEFINITION. A collection of vectors in a linear space V are linearly independent provided they are not linearly dependent.

EXAMPLE 2.1. The three vectors (1, 2, 3), (4, 1, 6), and (6, 5, 12) are linearly dependent in R^3.

EXAMPLE 2.2. (1, 0, 0), (0, 1, 0), and (0, 0, 1) are linearly independent in R^3.

EXAMPLE 2.3. In the linear space P of example 1.4, let $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$. Then these vectors are linearly independent.

DEFINITION. A collection of vectors $v_1, v_2, \ldots, v_n$ in V are said to form a basis for the linear space V provided they span V and are linearly independent.

EXAMPLE 2.4. (1, 0, 0), (0, 1, 0), and (0, 0, 1) form a basis for the linear space R^3.

EXAMPLE 2.5. (-1, 1) and (1, 1) form a basis for R^2.

THEOREM 2.1. The non-zero vectors $v_1, v_2, \ldots, v_n$ in a linear space V are linearly dependent if and only if at least one of the vectors v_k is a linear combination of the preceding ones.

PROOF. Suppose the vector v_k is a linear combination $v_k = a_1v_1 + a_2v_2 + \ldots + a_{k-1}v_{k-1}$ of the preceding ones. Thus

$$a_1v_1 + a_2v_2 + \ldots + a_{k-1}v_{k-1} + (-1)v_k = 0.$$

Hence vectors are linearly dependent.

Conversely, suppose that the vectors are linearly dependent. Then there exist scalars b_i, not all zero such that $b_1v_1 + b_2v_2 + \ldots + b_nv_n = 0$. Choose the last subscript k for which $b_k \neq 0$. One can solve for v_k as the linear combination

$$v_k = (-b_k^{-1}b_1)v_1 + (-b_k^{-1}b_2)v_2 + \ldots + (-b_k^{-1}b_{k-1})v_{k-1}.$$

Thus v_k is a linear combination of the preceding vectors, except in the case where $k = 1$. In this case $b_1v_1 = 0$, with $b_1 \neq 0$, so $v_1 = 0$, contrary to the hypothesis that none of the given vectors are zero.

THEOREM 2.2. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a linearly independent subset of a linear space V. If S is a basis for V, then every subset of V which properly contains S is linearly dependent.

PROOF. Let A be a subset of V which properly contains S. That is, A contains at least one vector v_{k+1} contained in V but not in S. Since S is a basis for V, then there exist scalars $a_1, \ldots, a_k$ such that v_{k+1} can be written as

$$v_{k+1} = a_1v_1 + a_2v_2 + \ldots + a_kv_k.$$

Therefore, A is linearly dependent by theorem 2.1.

THEOREM 2.3. Let n vectors span a linear space V containing m linearly independent vectors. Then $n \geq m$.

PROOF. Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of n vectors spanning V, and $X = \{x_1, x_2, \ldots, x_m\}$ be a subset of m linearly independent vectors in V. Since S spans V, x_1 is a linear combination of the v_i, so that the set $A_1 = \{x_1, v_1, v_2, \ldots, v_n\}$ still spans V and is linearly dependent. By theorem 2.1, some vector of A_1 must be a linear combination of its predecessors. This element cannot be x_1 since X is linearly independent. Hence some vector v_i is dependent on its predecessors $x_1, v_1, \ldots, v_{i-1}$. Deleting this vector, the set $S_1 = \{x_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$ still spans V.

Repeat this process. The set $A_2 = \{x_2, x_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\}$ spans V and is linearly dependent. Hence as before, some vector in A_2 is a linear combination of its predecessors. Because $x_1, \ldots, x_m$ are linearly independent, this vector cannot be x_1 or x_2, so it must be some v_j. Deleting this v_j, one has a new set.

$S_2 = \{x_2, x_1, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\}$ of n vectors spanning v. This process can be repeated m times, until the elements of X are exhausted. Each time an element of S is deleted. Hence S must have originally contained at least m elements.

THEOREM 2.4. Let V be a linear space for which $A = x_1, \ldots, x_n$ is a basis of V. Then every basis of V has exactly n elements.

PROOF. Let $A = \{x_1, \ldots, x_n\}$ and $B = \{y_1, \ldots, y_m\}$ be two bases for the linear space V. Since A spans V and B is linearly independent in V. Then $n \geq m$, by theorem 2.3. On the other hand, B spans V and A is linearly independent in V, so $m \geq n$. Hence $n = m$.

DEFINITION. The dimension of a linear space V is the number of vectors in a basis for V.

THEOREM 2.5. In a finite-dimensional linear space, every linearly independent set of vectors can be extended to a basis.

This theorem may be proved in a manner similar to the method in theorem 2.3.

COROLLARY 2.1. If a linear space V has dimension n, then (i) any $n + 1$ elements of V are linearly dependent, and (ii) no set of $n - 1$ elements spans V.

PROOF. The first part of this corollary is followed immediately by theorem 2.5 and theorem 2.2. The second part is the result of theorem 2.3.

2.3. SUBSPACES

DEFINITION. A nonempty subset S of a linear space V is a subspace if S is a linear space with respect to the operations defined in V.

DEFINITION. The coset of a subspace S of a linear space V is the set $x + S = \{x + s : s \in S$, x is fixed in $V\}$.

THEOREM 3.1. The set of all cosets of a subspace in V is a linear space under the operations defined by

$$(x + S) + (y + S) = (x + y) + S,$$

$$\alpha(x + S) = \alpha x + S,$$

for all $x, y \in V$ and $\alpha \in F$. This linear space is denoted by V/S and is called the quotient space of V with respect to S.

PROOF. For $x = y \in V$, $-y + x = 0 \in S$. It implies $(-y + x) + S = S$. Hence $x + S = y + S$. Therefore, the operations are well-defined.

Since V is an abelian group, S is a normal subgroup of V. Thus V/S form an abelian group. V/S is also satisfied the following conditions:

(i) $\alpha(x + S + y + S) = \alpha(x + y + S) = \alpha(x + y) + S$
$= \alpha x + \alpha y + S = \alpha x + S + \alpha y + S$
$= \alpha(x + S) + \alpha(y + S)$;

(ii) $(\alpha + \beta)(x + S) = (\alpha + \beta)x + S = \alpha x + \beta x + S$
$= \alpha x + S + \beta x + S$
$= \alpha(x + S) + \beta(x + S)$;

(iii) $(\alpha\beta)(x + S) = (\alpha\beta)x + S = \alpha(\beta x) + S$
$= \alpha(\beta x + S) = \alpha(\beta(x + S))$;

(iv) $1 \cdot (x + S) = 1 \cdot x + S = x + S$.

Therefore V/S is a linear space.

DEFINITION. Let S and T be two subspaces of a linear space V. Then the sum of S and T is the set $S + T$ which contains all the vector $s + t$ for each $s \in S$ and $t \in T$.

Figure 2.1 is a geometrical interpretation of a quotient space.

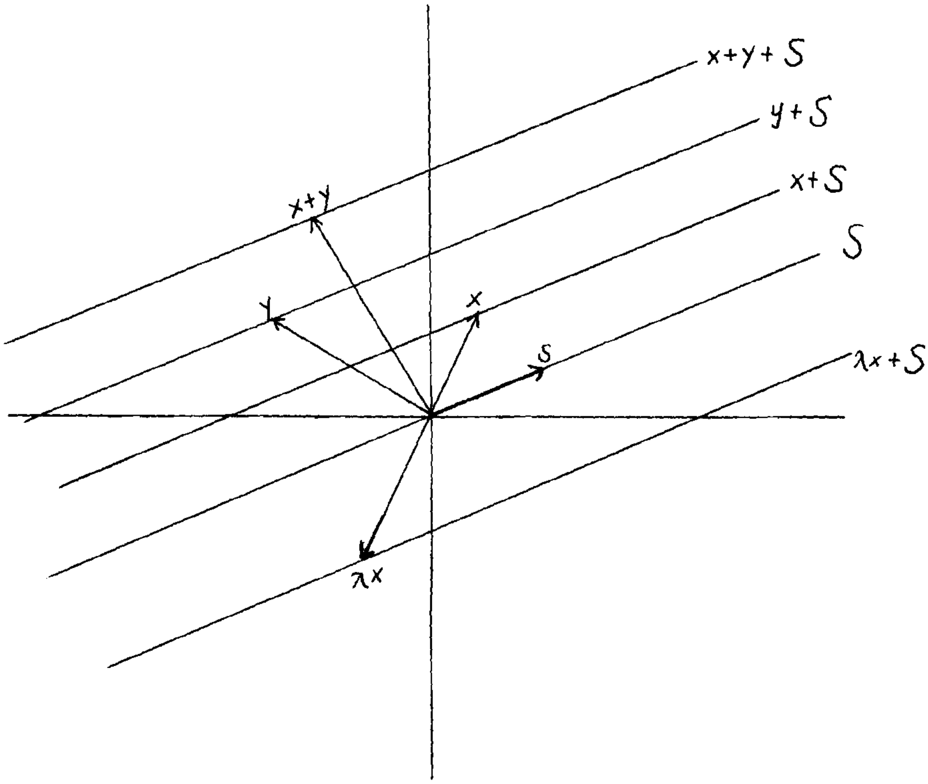

Fig. 2.1.

THEOREM 3.2. Let S and T be two subspaces of a linear space V. Then $S \cap T$ and $S + T$ are subspaces of V.

DEFINITION. Let S and T be two subspaces of a linear space V. V is a direct sum of S and T, denoted by $S \oplus T$, if for each vector v in V, there exists unique elements $s \in S$, $t \in T$ such that $v = s + t$.

THEOREM. 3.3. Let V be a linear space with subspace S and T, and $V = S + T$. Then $V = S \oplus T \Leftrightarrow S \cap T = \{0\}$.

PROOF. Assume $V = S \oplus T$. Then for each vector v in V $v = s + t$ is uniquely determined by $s \in S$ and

$t \in T$. Suppose there exists a non-zero vector $u \in S \cap T$. Then $u = 0 + u$ or $u = u + 0$. This contradicts the uniqueness. Therefore, $S \cap T = \{0\}$.

Next, assume $S \cap T = \{0\}$. Let $v = s + t$ for some $s \in S$ and $t \in T$. Suppose there exists another $s' \in S$ and $t' \in T$ such that $v = s' + t'$. Then $s + t = s' + t'$, and $-s' + s = t' - t = u$. Evidently, u is in both S and T. This contradicts $S \cap T = \{0\}$. Hence $V = S \oplus T$.

DEFINITION. Let V and V' be two linear spaces over the same field F. A mapping f of V into V' is called a linear transformation if

$$f(x + y) = f(x) + f(y)$$

$$f(\lambda x) = \lambda f(x),$$

for all $x, y \in V$ and $\lambda \in F$.

DEFINITION. Let V and V' be two linear spaces over the same field. An isomorphism of V onto V' is a one-to-one linear transformation of V onto V'. Two spaces V and V' are isomorphic, denoted by $V \cong V'$, if there is an isomorphism $f: V \rightarrow V'$.

EXAMPLE 3.1. Let $S = \{(x, 0): x \in R\}$ and $T = \{(0, x): x \in R\}$ be two subspaces of R^2. Define f by $f: (x, 0) \rightarrow (0, x)$. Then f is an isomorphism of S onto T.

DEFINITION. Let $f: V \rightarrow V'$ be a linear transformation. Then the kernel $(f) = \{x \in V: f(x) = 0\}$ and the image (f)

$= \{y \in V' : \ y = f(x) \ \text{ for some } \ x \in V\}$.

Both the kernel (f) and the image (f) are linear spaces.

THEOREM 3.4. Let $f: V \to V'$ be a linear transformation with a finite-dimensional domain. Then

$$\dim(V) = \dim(\text{kernel } f) + \dim(\text{image } f).$$

PROOF. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $\{v_1, \ldots, v_m\}$, $m \le n$, be a basis for the kernel of f. Then for any vector v in V can be written

$$v = \sum_{i=1}^{m} a_i v_i + \sum_{i=m+1}^{n} a_i v_i .$$

Hence

$$f(v) = f\Big(\sum_{i=1}^{m} a_i v_i\Big) + f\Big(\sum_{i=m+1}^{n} a_i v_i\Big)$$

$$= f\Big(\sum_{i=m+1}^{n} a_i v_i\Big)$$

$$= \sum_{i=m+1}^{n} a_i \ f(v_i).$$

That is, each vector $f(v)$ can be written as the linear combination of $f(v_{m+1}), \ldots, f(v_m)$. Thus $f(v_{m+1}), \ldots, f(v_n)$ spans image (f). Suppose $\sum_{i=m+1}^{n} a_i \ f(v_i) = 0$. Then

$$f\Big(\sum_{i=m+1}^{n} a_i v_i\Big) = \sum_{i=m+1}^{n} a_i \ f(v_i) = 0.$$

Hence $\sum_{i=m+1}^{n} a_i v_i \in$ kernel (f) and there exists a set of scalars $\{b_1, \ldots, b_m\}$ such that $\sum_{i=m+1}^{n} a_i v_i = \sum_{i=1}^{m} b_i v_i$.

$$\sum_{i=1}^{m} b_i v_i - \sum_{i=m+1}^{n} a_i v_i = 0.$$

Therefore $b_1 = \ldots = b_m = a_{m+1} = \ldots = a_n = 0$ since $\{v_1, \ldots, v_n\}$ is linearly independent. Hence $\{f(v_{m+1}), \ldots, f(v_n)\}$ is a basis for image (f).

THEOREM 3.5. Let $f: V \rightarrow V'$ be a linear transformation with kernel K. Then $V/K \cong$ image (f).

PROOF. Define $F: V/K \rightarrow T$; $T =$ image (f), by

$$F(v + K) = f(v).$$

It is easy to show that F is well-defined.

Let u, v be any vectors in V and α be any scalar. Then

$$F(u + K + v + K) = F(u + v + K) = f(u + v) = f(u) + f(v)$$
$$= F(u + K) + F(v + K).$$
$$\alpha F(u + K) = \alpha f(u) = f(\alpha u) = F(\alpha u + K).$$

Thus F is a linear transformation of V/K into T.

Suppose $F(u + K) = F(v + K)$, then

$$0 = F(u - v + K) = f(u - v).$$

Then $u - v \in K$. Therefore, $u + K = v + K$ and thus F is one-to-one. For $f(u) \in T$, then $u + K \in V/K$ and $F(u + K) = f(u)$. Hence F is onto. Therefore, $V/K \cong T$.

CHAPTER 3

NORMED LINEAR SPACES

3.1. NORM

DEFINITION. A normed linear space is a linear space N in which to each vector x there corresponds a real number, denoted by $\|x\|$ and called the norm of x, such that

(i) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in N$;

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, for all $x \in N$ and $\alpha \in R$.

EXAMPLE 1.1. The linear space R^3 of all 3-tuples $x = (x_1, x_2, x_3)$ of real numbers is a normed linear space if the norm is defined by

$$\|x\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}.$$

EXAMPLE 1.2. The linear space R^n of all n-tuples $x = (x_1, \ldots, x_n)$ of real numbers is a normed linear space with the corresponding Postman's norm defined by

$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|.$$

EXAMPLE 1.3. With the same space R^n as in the last example, define the maximum norm by

$$\|x\|_\infty = \max \{ |x_1|, \ldots, |x_n| \}.$$

Then R^n is a normed linear space.

EXAMPLE 1.4. The set $C[a, b]$ of all bounded continuous real functions defined on $[a, b]$ is a normed linear space with the norm defined by

$$\|f\| = \sup|f(x)| .$$

This norm is called the uniform norm.

EXAMPLE 1.5. Let p be a real number such that $1 \leq p < \infty$. R^n is a normed linear space if the norm is defined by

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p} .$$

For instance, Figure 1 illustrates the unit sphere with respect to several different norms in the space R^3.

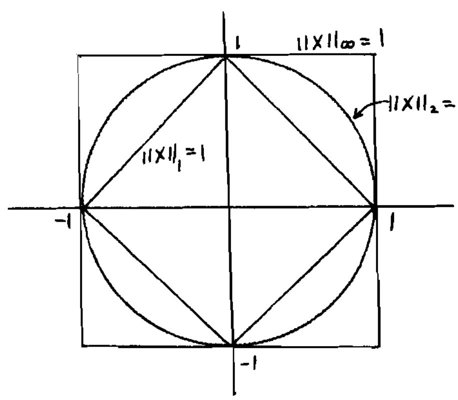

Fig. 1

EXAMPLE 1.6. Let $C_2[a, b]$ be the set of bounded continuous real functions defined on $[a, b]$. For $f \in C_2[a, b]$, define

$$\|f\| = (\int_a^b f^2 \, dx)^{1/2} .$$

Then $C_2[a, b]$ is a normed linear space. The detail will be given in Chapter 5.

THEOREM 1.1. If $p > 1$, $q > 1$ and $1/p + 1/q = 1$, then

$$a^{1/p}b^{1/q} \leq a/p + b/q ,$$

for all nonnegative real numbers a and b.

PROOF. It is trivial for $a = 0$ or $b = 0$, so assume that a and b are positive. Let $0 < k < 1$, and define

$$f(t) = k(t-1) - t^k + 1,$$

for all $t \geq 1$. Since

$$f'(t) = k - kt^{k-1} > 0.$$

Thus f is a strictly increasing function on $[1, \infty)$. Hence, for all $t \geq 1$,

$$f(t) \geq f(1) = 0.$$

Therefore,

$$t^k \leq 1 + k(t-1).$$

Suppose $a \geq b$, and let $t = a/b$ and $k = 1/p$. Then

$$(a/b)^{1/p} \leq 1 + 1/p(a/b - 1).$$

Multiply by b on both sides and recall that $1/p + 1/q = 1$, then

$$a^{1/p}b^{1/q} \leq a/p + b/q.$$

If $a < b$, let $t = b/a$, $k = 1/q$. The above inequality still holds.

Corollary 1.1. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then $\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \|y\|_q$.

PROOF. It is trivial if either $x = 0$ or $y = 0$. Assume $xy \neq 0$. Define $a_i = (|x_i| / \|x\|_p)^p$ and $b_i = (|y_i| / \|y\|_q)^q$. By the Theorem (1.1.),

$$(|x_i| / \|x\|_p)(|y_i| / \|y\|_q) \leq (|x_i| / \|x\|_p)^p/p + (|y_i| / \|y\|_q)^q/q.$$

Add the above inequalities for $i = 1, 2, \ldots, n$. Then the inequality is

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p \cdot \|y\|_q .$$

COROLLARY 1.2. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$. Then $\|x + y\|_p \leq \|x\|_p + \|y\|_p$. This is called Minkowski's inequality.

PROOF. By Holder's inequality one has

$$\|x + y\|_p{}^p = \sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i||x_i + y_i|^{p-1}$$

$$= \sum_{i=1}^{n} |x_i|\,|x_i + y_i|^{p/q} + \sum_{i=1}^{n} |y_i|\,|x_i + y_i|^{p/q}$$

$$\leq \|x\|_p \|(x + y)^{p/q}\|_q + \|y\|_p \|(x + y)^{p/q}\|_q$$

$$= (\|x\|_p + \|y\|_p)\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/q}$$

Divide both sides of this inequality by $\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1/q}$.

Then

$$\left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{1-1/q} \leq \|x\|_p + \|y\|_p .$$

Therefore,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p .$$

The Cauchy-Schwarz's inequality is a special case of Holder's inequality when $p = q = 2$.

3.2. BANACH SPACES

DEFINITION. Let X be a non-empty set. A metric on X is a function $d\colon X \times X \rightarrow R$ which satisfies the following conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in X$, and
$d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$, for all $x, y \in X$;

(iii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

A metric space is a nonempty set X together with a metric d on X.

THEOREM 2.1. A normed linear space N is a metric space if the metric d is defined by $d(x, y) = \|x - y\|$.

PROOF. Let x, y, z be any elements in N and d be defined by $d(x, y) = \|x - y\|$. It is easy to show that d is a metric on N.

(i) $d(x, y) = \|x - y\| \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

Since $d(x, y) = \|x - y\| = 0$ implies $x - y = 0$, or $x = y$;

(ii) $d(x, y) = \|x - y\| = |-1| \, \|x - y\| = \|y - x\|$
$= d(y, x)$;

(iii) $d(x, y) = \|x - y\| \leq \|x - z\| + \|z - y\|$
$= d(x, z) + d(z, y)$.

Therefore, d is a metric in N.

DEFINITION. A metric space X is complete if every Cauchy sequence in X converges. A Banach space is a normed linear space which is complete in the metric generated by its norm.

All the examples enumerated in the last section, except $C_2[a, b]$, are Banach spaces.

EXAMPLE 2.1. Let L_2 be the space of all measurable functions such that f is integrable on some domain D. Then L_2 is a normed linear space with the norm in L_2 defined by

$$\|f\| = (\int_D f^2)^{1/2}.$$

The completeness of L_2 was proved by Riesz-Fisher [10]. Some examples and theorems of measure theory are presented here and later chapters. It is out of this scope to prove them. Kolmogorov [10] and Royden [12] give some detail. A normed linear space may be complete with respect to one norm but not complete with respect to another one.

EXAMPLE 2.2. The space of all bounded continuous real functions on $[a, b]$ is complete if its norm is defined by

$$\|f\| = \sup\{|f(x)|\}.$$

But if its norm is defined by

$$\|f\| = (\int_a^b f^2 dx)^{1/2},$$

then it is not complete. This will be demonstrated in Chapter 5.

THEOREM 2.2. Let $\{x_1, \ldots, x_n\}$ be a set of n linearly independent vectors in a normed linear space N.

Then for any choice of scalars $\alpha_1, \ldots, \alpha_n$, where the α_i are not all zero, there exists a positive number δ such that

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq \delta (|\alpha_1| + \ldots + |\alpha_n|).$$

The proof of this theorem may be found in [5].

THEOREM 2.3. Every finite-dimensional normed linear space is complete.

PROOF. Let $\{x_1, \ldots, x_n\}$ be a basis for the space N, and $\{U_i\}$ be a Cauchy sequence in N. Then for each U_i

$$U_i = \alpha_{1i} x_1 + \ldots + \alpha_{ni} x_n,$$

where U_i is uniquely determined by α_{ki}, k = 1, 2, ..., n. By theorem 2.2, for each $\epsilon > 0$, there exists $N > 0$ and $\delta > 0$ such that

$$\delta \Big(\sum_{k=1}^{n} |\alpha_{ki} - \alpha_{kj}| \Big) \leq \Big\| \sum_{k=1}^{n} (\alpha_{ki} - \alpha_{kj}) x_k \Big\|$$

$$= \|U_i - U_j\| < \delta\epsilon ,$$

whenever $i\ j > N$. Hence $\{\alpha_{ki}\}$ is a Cauchy sequence in R, therefore, it converges to α_k for k = 1, 2, ..., n. Let $u = \alpha_1 x_1 + \ldots + \alpha_n x_n$. Then

$$\|u_i - u\| = \|(\alpha_{1i} - \alpha_1)x_1 + \ldots + (\alpha_{ni} - \alpha_n)x_n\|$$

$$\leq |\alpha_{1i} - \alpha_1| \, \|x_1\| + \ldots + |\alpha_{ni} - \alpha_n| \, \|x_n\|$$

$$< \epsilon/n\|x_1\| \cdot \|x_1\| + \ldots + \epsilon/n \cdot \|x_n\| \cdot \|x_n\|$$

$$= \epsilon .$$

Thus every Cauchy sequence is convergent. Therefore, a finite-dimensional normed linear space is complete.

3.3. COMPLETION

DEFINITION. Let X be a metric space and A a subset of X. A is said to be dense if $\overline{A} = X$. ($\overline{A}$ denotes the closure of A.)

EXAMPLE 3.1. The set of rational numbers is dense in the space R of real numbers with the usual metric.

EXAMPLE 3.2. The set of polynomials in the space $C_2[a, b]$ is dense in $L_2[a, b]$. This example is found in [13].

DEFINITION. Let X^* be a complete metric space and X a subspace of X^*. Then X^* is said to be the completion of X if $\overline{X} = X^*$.

THEOREM 3.1. Every metric space has a completion and all of its completions are isometric.

PROOF. Let X be any metric space. Two Cauchy sequence $\{x_n\}$ and $\{y_n\}$ of X are said to be equivalent if $\lim_{n\to\infty} d(x_n, y_n) = 0$. This relation is reflexive, symmetric, and transitive. Therefore, all Cauchy sequences which can be constructed from the elements of the space X can be partitioned into equivalent classes of sequences. Let X^* be the set of all those classes of sequences. The points in X^* are denoted by A, B, etc., and $\{x_n\}$, $\{y_n\}$ are Cauchy sequences in A, B, respectively. The distance between two classes A and B in X^* is defined by

$$d^*(A, B) = \lim_{n\to\infty} d(x_n, y_n). \qquad (3.1.)$$

Let $\epsilon > 0$ be given. Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences, then there exists $N > 0$, such that

$$\begin{aligned} |d(x_n, y_n) - d(x_m, y_m)| &\leq |d(x_n, y_n) - d(x_n, y_m)| + \\ &\quad |d(x_n, y_m) - d(x_m, y_m)| \\ &\leq d(y_n, y_m) + d(x_n, x_m) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

for all $n, m > N$. Thus $\{d(x_n, y_n)\}$ is a Cauchy sequence. But $d(x_n, y_n)$ are real numbers, hence the sequence has a limit. This proves that the limit in (3.1) does exist.

Let $\{x_n\}$, $\{x'_n\} \in A$ and $\{y_n\}$, $\{y'_n\} \in B$. Then

$$\lim_{n\to\infty} d(x_n, x'_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(y_n, y'_n) = 0$$

implies that

$$\begin{aligned} |d(x_n, y_n) - d(x'_n, y'_n)| &\leq |d(x_n, y_n) - d(x'_n, y_n)| \\ &\quad + |d(x'_n, y_n) - d(y'_n, x'_n)| \\ &\leq d(x_n, x'_n) + d(y_n, y'_n). \end{aligned}$$

That is

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n).$$

Therefore, (3.1) is well defined.

The distance defined by (3.1) is a metric in X^* since

(i) $d^*(A, B) = \lim_{n\to\infty} d(x_n, y_n) \geq 0$; $d^*(A, B) = 0$

if and only if $\lim_{n\to\infty} d(x_n, y_n) = 0$ which implies $\{x_n\}$

and $\{y_n\}$ are in the same equivalent class, therefore, $A = B$.

(ii) $d^*(A, B) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n)$

$= d^*(B, A)$;

(iii) Since

$$d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n).$$

Take the limit as n approaches infinity, then

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(z_n, y_n).$$

Hence the triangle inequality holds in X^*. Therefore X^* is a metric space.

To each point $x \in X$, there corresponds an equivalent class in X^*. Indeed, the constant sequence $\{x_n\}$, $x_n = x$ for each n, is a representative of this class. Let $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Then

$$d(x, y) = \lim_{n\to\infty} d(x_n, y_n).$$

Therefore, X is embedded isometrically in X^*. Thus, there is nothing to distinguish between X and its corresponding class in X^* and X can be considered a subset of X^*.

Let A be any class in X^*, $\{x_n\} \in A$, and $\epsilon > 0$ be given. Then there exists an N such that for all $n, m > N$ one has $d(x_n, x_m) < \epsilon$. Let $\{x_n\}$ be a constant sequence converging to x_n. Then

$$d^*(x_n, A) \leq d^*(x_n, \{x_n\}) + d^*(\{x_n\}, A)$$
$$= 0 + \lim_{m\to\infty} d(x_n, x_m)$$
$$< \epsilon \ .$$

Hence $\overline{X} = X^*$.

It remains to be proved that the space X^* is complete. Let $\{A_n\}$ be a Cauchy sequence in X^*. For given $\epsilon > 0$, there exists $N > 4/\epsilon$ such that $d^*(A_n, A_m) < \epsilon/4$ whenever $n, m > N$. Construct a sequence $\{x_n\}$ of which each point x_n in X. Let $B_n \in X^*$ be a class corresponding to x_n such that $d^*(x_n, A_n) < 1/n$.

Since
$$d(x_n, x_m) = d^*(B_n, B_m)$$
$$\leq d^*(B_n, A_n) + d^*(A_n, A_m) + d^*(A_m, B_m)$$
$$< 1/n + \epsilon/4 + 1/m$$
$$< \epsilon/4 + \epsilon/4 + \epsilon/4 = 3\epsilon/4.$$

Thus, $\{x_n\}$ is a Cauchy sequence. Let A be the class containing the sequence $\{x_n\}$. Then

$$d^*(A_n, A) \leq d^*(A_n, B_n) + d^*(B_n, A)$$
$$< 1/n + \lim_{n\to\infty} d(x_n, x_m)$$
$$< \epsilon/4 + 3\epsilon/4 = \epsilon \ .$$

Therefore, X^* is a complete metric space.

Finally, one must prove that for any two completions of X are isometric. Let X^* and X^{**} be two completions of X and X_1, x_2 be subspaces of X^*, X^{**} respectively, which are isomorphic to X. Therefore, $X_1 \cong X_2$.

Let A be any element in X^*. By the completion, there exists $\{A_n\} \in X_1$ which converges to A. Then the corresponding sequence $\{B_n\} \in X_2$ is also a Cauchy sequence, since $X_1 \cong X_2$ hence $d_1(A_m, A_n) = d_2(B_m, B_n)$, where d_1 and d_2 are metrics of X^*, X^{**} respectively. Since X^{**} is complete, it contains an element B such that

$$B = \lim_{n\to\infty} B_n.$$

Associate $B \in X^{**}$ with $A \in X^*$ which is originally chosen. Define $f: X^* \to X^{**}$ by $f(A) = B$. It is easily seen that f is one-to-one and onto. Let $x, y \in X^*$. Then there exist sequences $\{x_n\}$, $\{y_n\}$ in X converging to x, y respectively. Since f preserves convergence, $f(\{x_n\})$, $f(\{y_n\})$ converges to $f(x)$, $f(y)$ respectively. Then

$$\begin{aligned} d_2(f(x), f(y)) &= \lim_{n\to\infty} d_2(f(\{x_n\}), f(\{y_n\})) \\ &= \lim_{n\to\infty} d_1(\{x_n\}, \{y_n\}) \\ &= d_1(x, y). \end{aligned}$$

That is, f preserves distance. Hence $X^* \cong X^{**}$.

EXAMPLE 3.3. Let Q be the set of all rational numbers. Then R is the completion of Q.

EXAMPLE 3.4. Let $L_2[a, b]$ be the set of all measurable functions f such that f^2 is integrable on $[a, b]$. Then $L_2[a, b]$ is the completion of $C_2[a, b]$. This example in found in [10].

CHAPTER 4

HILBERT SPACES

4.1. INNER PRODUCT SPACES

The inner product of two vectors $u = (x_1, \ldots, x_n)$ and $v = (y_1, \ldots, y_n)$ in the n-dimensional linear space R^n with real components is given by the quantity

$$(u, v) = x_1y_1 + \ldots + x_ny_n. \qquad (4.1.)$$

DEFINITION. An inner product space S is a linear space over the field R of real numbers with a function $S \times S \to R$ denoted by (u, v). The scalar (u, v) is called the inner product of u and v which satisfies the following properties:

(i) $(u, v) = (v, u)$, for all $u, v \in S$;

(ii) $(u + v, w) = (u, w) + (v, w)$, for all $u, v, w \in S$;

(iii) $(\alpha u, v) = \alpha(u, v)$, for all $\alpha \in R$;

(iv) $(u, u) > 0$ and $(u, u) = 0$ if and only if $u = 0$.

R^n is an inner product space if the inner product is defined by (4.1).

EXAMPLE 1.1. The space $C_2[a, b]$ of all bounded continuous real functions defined on $[a, b]$ is an inner product space if its inner product is defined by

$$(f, g) = \int_a^b f(x)g(x)d\gamma(x),$$

where $\gamma(x)$ is an increasing function on $[a, b]$. This proof will be given in Chapter 5.

EXAMPLE 1.2. The linear space R^3 of all 3-tuples of real numbers in an inner product space if the inner product of two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in R^3 is defined by

$$(x, y) = (x_1 + x_2)(y_1 + y_2) + x_2y_2 + (x_2 + x_3)(y_2 + y_3).$$

THEOREM 1.1. Let x and y be two vectors in an inner product space. Then

$$|(x, y)| \leq (x, x)^{1/2} \cdot (y, y)^{1/2}.$$

This inequality is called the Schwarz inequality.

PROOF. Let $\alpha = (y, y)$ and $\beta = -(x, y)$. By the definition,

$$\begin{aligned} 0 &\leq (\alpha x + \beta y, \alpha x + \beta y) \\ &= \alpha^2(x, x) + 2\alpha\beta(x, y) + \beta^2(y, y) \\ &= (y, y)^2(x, x) - 2(x, y)^2(y, y) \\ &\quad + (x, y)^2(y, y) \\ &= (y, y)(x, x) - (x, y)^2. \end{aligned}$$

This inequality is trivial if $y = 0$. Assume $y \neq 0$. That is

$$(x, y)^2 \leq (x, y) \cdot (y, y).$$

Therefore,

$$|(x, y)| \leq (x, x)^{1/2}(y, y)^{1/2}.$$

THEOREM 1.2. Any inner product space S is a normed linear space. The norm is defined by $\|x\| = (x, x)^{1/2}$.

PROOF. For each vector x in S,

(i) The first axiom of normed linear space is satisfied trivially.

(ii) $\|x + y\|^2 = (x + y, x + y) = (x, x) + 2(x, y) + (y, y)$

$$\leq (x, x) + 2(x, x)^{1/2}(y, y)^{1/2} + (y, y)$$

$$= \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2$$

$$= (\|x\| + \|y\|)^2$$

Take the square root on both sides;

$$\|x + y\| \leq \|x\| + \|y\|.$$

(iii) $\|\alpha x\| = (\alpha x, \alpha x)^{1/2} = |\alpha|(x, x)^{1/2} = |\alpha|\|x\|.$

In the n-dimensional linear space R^n, many geometrical questions involve the length of a vector and the angle between two vectors. The inner product plays an important role in these problems.

The length $\|x\|$ of any vector x is defined to be the non-negative square root,

$$\|x\| = (x, x)^{1/2}.$$

This is possible since (x, x) is a nonnegative real number.

Let x and y be any two vectors in R^n. Then the difference $x - y$, (Fig. 4.1), of x and y is given by

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta.$$

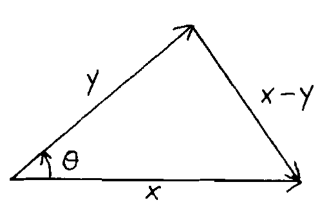

Fig. 4.1.

Since $\|x - y\|^2 = (x - y, x - y) = (x, x) - 2(x, y) + (y, y)$, one has

$$\cos\theta = (x, y)/\|x\| \cdot \|y\|.$$

Thus two vectors x and y are orthogonal if and only if $(x, y) = 0$.

4.2. HILBERT SPACES

The set of all complex numbers is a complex linear space under addition and multiplication.

DEFINITION. A Hilbert space H is a complex Banach space in which there is defined a complex function $H \times H \to C$, denoted by (x, y), with the following properties:

For all x, y, z in H,

(i) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$, for all $\alpha, \beta \in C$;

(ii) $\overline{(x, y)} = (y, x)$;

(iii) $(x, x) = \|x\|^2$.

EXAMPLE 2.1. The set of all complex functions defined on $[0, 2\pi]$ with the property $\int_{[0, 2\pi]} |f|^2 < \infty$ is a Hilbert space if the norm is defined by

$$\|f\| = \left(\int_{[0, 2\pi]} |f|^2\right)^{1/2}.$$

4.3. ORTHONORMAL SETS

DEFINITION. A set $\{e_i\}$ in a Hilbert space is called orthonormal provided

$$(e_i, e_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta symbol defined by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

DEFINITION. An orthonormal set $\{e_i\}$ is said to be complete in an inner product space S if it is not properly contained in any other orthonormal set.

THEOREM 3.1. Let $\{e_1, \ldots, e_n\}$ be an orthonormal set of a Hilbert space H. Let P by any element in H. Then

$$\sum_{i=1}^{n} |(P, e_i)|^2 \leq \|P\|^2.$$

This inequality is known as Bessel's inequality.

PROOF.

$$\begin{aligned} 0 &\leq \|P - \sum_{i=1}^{n} (P, e_i)e_i\|^2 \\ &= (P - \sum_{i=1}^{n} (P, e_i)e_i, P - \sum_{i=1}^{n} (P, e_i)e_i) \\ &= (P, P) - \sum_{i=1}^{n} (P, e_i)(P, e_i) - \\ &\quad \sum_{i=1}^{n} (P, e_i)(e_i, P) + \sum_{i=1}^{n} (P, e_i)(P, e_i) \\ &= (P, P) - \sum_{i=1}^{n} (P, e_i)(e_i, P) \\ &= \|P\|^2 - \sum_{i=1}^{n} |(P, e_i)|^2. \end{aligned}$$

Therefore,

$$\sum_{i=1}^{n} |(P, e_i)|^2 \leq \|P\|^2.$$

The geometrical interpretation of this inequality is that the sum of the squares of the components of a vector is less than or equal to the square of the length of the vector. If the orthonormal set is complete, then equality holds. This is known as Parseval's equation.

THEOREM 3.2. Let $\{e_i\}$ be an orthonormal set in a Hilbert space H. Then for any element P in H, the set $S = \{e_i : (P, e_i) \neq 0\}$ is either empty or countable.

PROOF. Suppose S is not empty. Let

$$S_n = \{e_i : |(P, e_i)|^2 > \|P\|^2/n\}.$$

for all positive integers n. By theorem 3.1,

$$\sum_{i=1}^{n} |(P, e_i)|^2 \leq \|P\|^2.$$

Therefore S_n contains at most n - 1 elements. Since $|(P, e_i)| > 0$ for all e_i in S_n. Hence $S = \bigcup_{n=1}^{\infty} S_n$ and therefore S is countable.

The next theorem is the general form of Bessel's inequality.

THEOREM 3.3. Let $\{e_i\}$ be a nonempty orthonormal set in a Hilbert space H. Then

$$\Sigma(P, e_i)^2 \leq \|P\|^2 \qquad (3.1.)$$

for all P in H.

PROOF. With the same construction as in the preceeding theorem, S is either empty or countable. If S is empty, then $\sum |(P, e_i)|^2 = 0$. In this case, the theorem is trivial. If S is countable, it means S contains finite or countably infinite number of elements. In the case of the number of elements in S being finite, theorem 3.1 is the special case. Suppose S contains countably infinitely many elements, and $S = \{e_1, \ldots, e_n, \ldots\}$. By Dirichlet's theorem, any infinite series resulting from the rearrangement of terms of an absolutely convergent series is also absolutely convergent and has the same sum as the original series. Therefore, $\sum |(P, e_i)|^2$ can be rewritten as $\sum_{n=1}^{\infty} |(P, e_n)|^2$ and (3.1.) reduces to the assertion that

$$\sum_{n=1}^{\infty} |(P, e_n)|^2 \leq \|P\|^2. \qquad (3.2.)$$

Theorem 3.1 shows that no partial sum of square of the components of a vector P can exceed $\|P\|^2$. Hence (3.2) is true.

THEOREM 3.4. Let $\{e_i\}$ be an orthonormal set in a Hilbert space H. Then $\{e_i\}$ is complete if and only if Parseval's equation holds, that is,

$$\sum |(P, e_i)|^2 = \|P\|^2,$$

for each P in H.

PROOF. Assume $\{e_i\}$ is an orthonormal set in H. Let P be any element in H and $P' = P - \sum (p, e_i)e_i$. Then, for all j,

$$\begin{aligned}(P', e_j) &= (P - \sum (P, e_i)e_i, e_j) \\ &= (P, e_j) - (\sum(P, e_i), e_j) \\ &= (P, e_j) - (P, e_j) = 0.\end{aligned}$$

That is, P' is orthogonal to all e_i. Suppose $P' \neq 0$, choose $e = P'/\|P\|$. It is clear that $\|e\| = 1$ and is orthogonal to all e_i. Then $\{e_i\} \cup \{e\}$ is an orthonormal set in H which contains $\{e_i\}$ properly.

This is a contradiction since $\{e_i\}$ is a complete orthornormal set.

Next, assume that Parseval's equation is true. Suppose $\{e_i\}$ were not complete. Then there exists a nonzero element e in H which is orthogonal to all e_i and $\|e\| = 1$ such that $\{e_i\} \cup \{e\}$ contains $\{e_i\}$ properly. Then

$$\|e\|^2 = \sum |(e,\ e_i)|^2 = 0.$$

This implies that $e = 0$, which is a contradiction.

DEFINITION. An orthonormal set $\{e_i\}$ is said to be an orthonormal basis for a normed linear space N if $\{e_i\}$ is complete in N.

EXAMPLE 3.1. Let e_i be the n-tuple in R^n such that the ith component of e_i is 1 and all the other components are zero. Then $\{e_1, \ldots, e_n\}$ is an orthonormal basis for R^n.

THEOREM 3.5. Every Hilbert space H has an orthonormal basis.

PROOF. Let E be the collection of all nonempty orthonormal subsets of H. $(E, \subset)$ is a partially ordered set. For any totally ordered subset S of E, let S' be the union of elements in S. If x and y are any elements in S', then there corresponds S_1 and S_2 in S such that $x \in S_1$ and $y \in S_2$. By the definition of a totally ordered set, either $S_1 \subset S_2$ or $S_2 \subset S_1$. Now assume that $S_1 \subset S_2$. Then x and y are both contained in S_2. Since S' is an orthonormal set, then S' is an element of E which is an upper bound of all the elements in S. By the Zorn's lemma, E contains a maximal element. That is, H contains a complete orthonormal set. Therefore, H has an orthonormal basis.

THEOREM 3.5. Let $\{e_i\}$ be an orthonormal basis for a Hilbert space H. Then for any element f in H,

$$f = \sum (f, e_i)\, e_i.$$

PROOF. $(f - \sum(f, e_i)e_i, e_i) = (f, e_i) - (\sum(f, e_i)e_i, e_i)$

$$= (f, e_i) - (f, e_i) = 0,$$

for all i. By the definition of an orthonormal basis, there is no element in H which is orthogonal to all the e_i except the zero vector. Hence $f - \sum(f, e_i)e_i = 0$. Thus, $f = \sum(f, e_i)\, e_i$ for any element f in H.

THEOREM 3.7. Let T be dense in a Hilbert space H. If S is an orthonormal basis in T, then S is also an orthonormal basis in H.

PROOF. Suppose there were an element f in H which was orthogonal to all the elements in S. Let $\bar{S}$ be the closure set of S. Then for all $g \in \bar{S}$, there exists a sequence $\{g_n\}$ of elements in S converging to g. That is, $\|hn\| \to 0$ as $n \to \infty$, if $hn = g - g_n$.

Let $\epsilon > 0$ be given. By the Schwarz inequality,

$$\begin{aligned} |(f, g) - (f, g_n)| &= |(f, g) - (f, g - h_n)| \\ &= |(f, h_n)| \\ &\leq \|f\| \cdot \|h_n\|. \end{aligned}$$

Thus $(f, g) = \lim_{n \to \infty} (f, g_n)$.

By the hypothesis, $(f, g_n) = 0$ for all $g_n \in S$. Hence $(f, g) = 0$ which implies that f is orthogonal to each element in $\overline{S}$. But the previous theorem guarantees that $\overline{S} = T$. Hence S is not a complete orthonormal set. This is a contradiction.

4.4. ORTHONOMALIZATION

THEOREM 4.1. For any orthogonal set $\{P_i\}$ in a normed linear space, there exists a corresponding orthonormal set. This is obtained by the Gram-Schmidt process.

PROOF. $P_i \neq 0$ for all $i = 0, 1, 2, \ldots$, hence $\|P_0\| > 0$. Let $e_0 = P_0/\|P_0\|$. It is evident that $\|e_0\| = 1$ and e_0 is orthogonal to P_k for $k \neq 0$. Let $P'_1 = P_1 - (P_1, e_0)e_0$. Then $P_1' \neq 0$ and

$$(P_1', e_0) = (P_1, e_0) - (P_1, e_0) = 0.$$

Hence $e_1 = P_1'/\|P_1'\|$ is orthogonal to e_0 and $\|e_1\| = 1$. Let $P_2' = P_2 - (P_2, e_0)e_0 - (P_2, e_1)e_1$. Then $P_2' \neq 0$ and

$$(P_2', e_0) = (P_2, e_0) - (P_2, e_0) = 0,$$

$$(P_2', e_1) = (P_2, e_1) - (P_2, e_1) = 0.$$

Hence $e_2 = P_2'/\|P_2'\|$ is orthogonal to e_1, e_2, and $\|e_2\| = 1$. Continuing in the same way, one obtains an orthonormal set $\{e_0, e_1, \ldots, e_n, \ldots\}$ with the required property.

EXAMPLE 4.1. Use the Gram-Schmidt process to normalize the orthogonal set $\{L_0(x), L_1(x), L_2(x), L_3(x)\}$ of polynomials in $C_2[-1, +1]$, where

$$L_0(x) = 1;$$
$$L_1(x) = x;$$
$$L_2(x) = \tfrac{1}{2}(3x^2 - 1);$$
$$L_3(x) = \tfrac{1}{2}(5x^3 - 3x).$$

SOLUTION. Let $\|L_0(x)\| = (\int_{-1}^{1} L_0^{\,2}(x)dx)^{1/2} = (\int_{-1}^{1} dx)^{1/2} = \sqrt{2}$. By the Gram-Schmidt process,

$$E_0(x) = L_0(x)/\|L_0(x)\| = 1/\sqrt{2},$$

and

$$\|E_0(x)\| = (\int_{-1}^{1} 1/2 \; dx)^{1/2} = 1.$$

$E_0(x)$ is also orthogonal to $L_k(x)$ for $k = 1, 2, 3$.

Let $L_1'(x) = L_1(x) - (L_1, E_0)E_0(x)$. Then

$$E_1(x) = L_1'(x)/\|L_1'(x)\| = \sqrt{3/2}\; x, \text{ and } \|E_1(x)\| = 1.$$

With the same process, it is easy to find

$$E_2(x) = \sqrt{5/8}(3x^2 - 1), \text{ and } \|E_2(x)\| = 1;$$
$$E_3(x) = \sqrt{7/8}(5x^3 - 3x), \text{ and } \|E_3(x)\| = 1.$$

where $E_i(x)$ is orthogonal to all E_j for $i \neq j$.

Therefore, $\{E_0(x), E_1(x), E_2(x), E_3(x)\}$ is an orthonormal set generated by $\{L_0(x), L_1(x), L_2(x), L_3(x)\}$.

CHAPTER 5

THE SPACE $C_2[a, b]$

5.1. $C_2[a, b]$

In this section, the main purpose is to prove that the set $C_2[a, b]$ of all bounded continuous real functions is a normed linear space.

First of all, $C_2[a, b]$ is not empty since it contains at least the constant functions. Define the sum and the scalar product of any two functions f and g in $C_2[a, b]$ by

$$(f + g)(x) = f(x) + g(x);$$

$$(\alpha f)(x) = \alpha(f(x)).$$

Since f and g are continuous, for given $\epsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - f(y)| < \epsilon/2 \qquad \text{whenever } |x - y| < \delta_1,$$

and

$$|g(x) - g(y)| < \epsilon/2 \qquad \text{whenever } |x - y| < \delta_2,$$

where $x, y \in [a, b]$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then

$$\begin{aligned}|(f + g)(x) - (f + g)(y)| &= |f(x) - f(y) + g(x) - g(y)| \\ &\leq |f(x) - f(y)| + |g(x) - g(y)| \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Hence $f + g \in C_2[a, b]$.

Also, for any $f, g \in C_2[a, b]$, $x \in [a, b]$, and $\alpha, \beta \in R$,

$$(\alpha f + \beta g)(x) = (\alpha f)(x) + (\beta g)(x)$$
$$= \alpha(f(x)) + \beta(g(x)).$$

Moreover, $\alpha f + \beta g \in C_2[a, b]$. Thus $C_2[a, b]$ is easily seen to be a linear space.

Define

$$(f, g) = \int_a^b f(x)g(x)d\gamma(x),$$

where $\gamma(x)$ is a strictly increasing function defined on $[a, b]$. Then

(i) $(f, g) = \int_a^b f(x)g(x)d\gamma(x) = \int_a^b g(x)f(x)d\gamma(x)$

$= (g, f)$;

(ii) $(f, g + h) = \int_a^b f(x)(g(x) + h(x))d\gamma(x)$

$= \int_a^b f(x)g(x)d\gamma(x) + \int_a^b f(x)h(x)d\gamma(x)$

$= (f, g) + (f, h)$;

(iii) $(\alpha f, g) = \int_a^b \alpha f(x)g(x)d\gamma(x) = \alpha\int_a^b f(x)g(x)d\gamma(x)$

$= \alpha(f, g)$, for all $\alpha \in R$;

(iv) $(f, f) = \int_a^b |f(x)|^2 d\gamma(x) \geq \left|\int_a^b f^2(x)d\gamma(x)\right| \geq 0.$

If $(f, f) = 0$, then $\int_a^b f^2(x)d\gamma(x) = 0$. By the First Mean Value theorem, there exists m such that

$$\int_a^b f^2(x)d\gamma(x) = m\left[\gamma(b) - \gamma(a)\right],$$

where $\inf\{f^2(x)\} \leq m \leq \sup\{f^2(x)\}$. Suppose $f(x) \neq 0$, then $m > 0$. Since $\gamma(x)$ is strictly increasing in $[a, b]$, it follows that

$$\int_a^b f^2(x)d\gamma(x) = m[\gamma(b) - \gamma(a)] > 0.$$

This contradicts the fact that $\int_a^b f^2(x)d\gamma(x) = 0$. Therefore, $C_2[a, b]$ is an inner product space.

Define the norm of f by

$$\|f\| = (f, f)^{1/2} = (\int_a^b f^2(x)d\gamma(x))^{1/2}.$$

Then

(i) $\|f\| \geq 0$, and $\|f\| = 0$ if and only if $f(x) = 0$. This follows by property (iv) of an inner product space.

(ii) Let $f, g \in C_2[a, b]$. By using Schwarz's inequality $(f + g, f) \leq \|f + g\| \cdot \|f\|$, which was proved in Chapter 4,

$$\begin{aligned}\|f + g\|^2 &= (f + g, f + g) \\ &= (f + g, f) + (f + g, g) \\ &\leq \|f + g\| \cdot \|f\| + \|f + g\| \cdot \|g\| \\ &= \|f + g\| \cdot (\|f\| + \|g\|).\end{aligned}$$

If $f + g = 0$, the triangle inequality is trivial. So assume $f + g \neq 0$ and $\|f + g\| > 0$. Divide each side by $\|f + g\|$, then

$$\|f + g\| \leq \|f\| + \|g\|.$$

(iii) For any scalar $\alpha \in R$,

$$\begin{aligned}\|\alpha f\| &= (\alpha f, \alpha f)^{1/2} \\ &= (\int_a^b \alpha^2 f^2(x)d\gamma(x))^{1/2} \\ &= (\alpha^2\int_a^b f^2(x)d\gamma(x))^{1/2} \\ &= |\alpha| \cdot \|f\|.\end{aligned}$$

Therefore, $C_2[a, b]$ is a normed linear space. But $C_2[a, b]$ fails to be a complete normed linear space as the following discussion indicates.

Let $\{f_n\}$ be a sequence of bounded continuous real functions defined by

$$f_n(x) = \begin{cases} 1 & 0 \leq x \leq 1/2, \\ -2^n(x - 1/2) + 1 & 1/2 \leq x \leq 1/2 + (1/2)^n, \\ 0 & 1/2 + (1/2)^n \leq x \leq 1. \end{cases}$$

Let $\in > 0$ be given, assume $m > n$, there exists $n = 2/\in > 0$ such that

$$\|f_n - f_m\| = (\int_{1/2}^{1/2 + (1/2)^m}((-2^n + 2^m)(x - 1/2))^2 dx)^{1/2}$$

$$+ (\int_{1/2 + (1/2)^m}^{1/2 + (1/2)^n} (-2^n(x + 1/2) + 1)^2 dx)^{1/2}$$

$$= (2^m - 2^n)(1/3 \cdot 1/2^{3m})^{1/2} + 2^n(1/3(1/2^n - 1/2^m)^3)^{1/2}$$

$$\leq 2^m(1/3 \cdot 1/2^{3m})^{1/2} = 1/\sqrt{3} \cdot (1/2^m)^{1/2}$$

$$< 1/\sqrt{3} \cdot 2/N = 1/\sqrt{3} \cdot \in$$

$$< \in \quad .$$

Hence $\{f_n\}$ is a Cauchy sequence. There exists a sequence $\{f_n\}$ which converges pointwise to $f(x) = 1$. But

$$\|f_n(x) - 1\| = (\int_{1/2 + (1/2)^n}^{1} (-1)^2 dx)^{1/2}$$

$$= (1 - 1/2 + 1/2^n)^{1/2}.$$

Then $\|f_n(x) - 1\| \to 1/\sqrt{2}$ as $n \to \infty$. Therefore, $\{f_n\}$ does not converge with respect to the given norm. This proves that $C_2[a, b]$ is not a complete normed linear space.

5.2. THE SPACE $L_2[a, b]$

It can be shown that $L_2[a, b]$ of all square measurable functions on $[a, b]$ is a normed linear space by using the analogous processes as in the preceeding section.

The proof of the completeness of $L_2[a, b]$ is given by Kolmogorov [10].

It is shown in [12], that every bounded and Riemann integrable function on $[a, b]$ is Lebesgue integrable. Since all the functions in $C_2[a, b]$ are bounded and Riemann integrable, $C_2[a, b]$ is a subspace of $L_2[a,b]$.

The concepts of measurable functions and Lebesgue integration lie beyond the scope of this paper. The interested reader is referred to [10], and [12] for a development of these topics. In order to preserve the coherence of this paper the following theorem, found in [10], is stated here without proof.

Theorem 2.1. $L_2[a, b]$ is the completion of $C_2[a, b]$.

CHAPTER 6

ORTHONORMAL POLYNOMIALS IN $C_2[a, b]$

6.1. ORTHOGONAL SETS

DEFINITION. A function $w(x)$ defined on $[a, b]$ is called weight function provided it is continuous on $[a, b]$, positive except possibly at a finite number of points, and $\int_a^b x^k w(x)dx$ exists for $k = 0, 1, 2, \ldots$.

Let $P(x)$ and $Q(x)$ be two integrable functions on $[a, b]$ and $\gamma(x)$ strictly increasing on $[a, b]$. Then the inner product of $P(x)$ and $Q(x)$ is defined by

$$(P, Q) = \int_a^b P(x)Q(x)d\gamma(x).$$

If the derivative of $\gamma(x)$ exists and is continuous, then it can be said that $P(x)$ and $Q(x)$ are orthogonal with respect to the weight function $w(x) = \gamma'(x)$.

If $\gamma(x)$ is discontinuous at most finite points $a_0, a_1, \ldots, a_n$ on $[a, b]$ and $a \leq a_0 < a_1 < \ldots < a_n \leq b$. Then

$$\int_a^b P(x)Q(x)d\gamma(x) = \int_a^{a_0} PQd\gamma + \int_{a_0}^{a_1} PQd\gamma + \ldots + \int_a^b PQd\gamma.$$

The weight function is still defined.

DEFINITION. Two functions $P(x)$ and $Q(x)$ are said to be orthogonal on $[a, b]$, if

$$(P, Q) = \int_a^b P(x)Q(x)d\gamma(x) = 0.$$

EXAMPLE 1.1. Let $P(x) = x$ and $Q(x) = x^2 - 1/3$. Then $P(x)$ and $Q(x)$ are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1$.

DEFINITION. $\{P_i(x)\}$ a set of functions is called an orthogonal set provided that the $P_i(x)$ are mutually orthogonal, and $P_i(x)$ has degree i. In other words,

(i) For each i, $P_i(x) = \alpha_i x^i$ + a polynomial of degree $< i$, with $\alpha_i \neq 0$;

(ii) $(P_i, P_j) = 0$, whenever $i \neq j$.

EXAMPLE 1.2. $\{\cos kx\}$ and $\{\sin kx\}$ for $k = 0, 1, 2, \ldots$ are orthogonal sets on the interval $[0, 2\pi]$ with respect to the weight function $w(x) = 1$.

6.2. ORTHONORMAL POLYNOMIALS

THEOREM 2.1. Let $\{P_0(x), \ldots, P_n(x)\}$ be a set of ORTHONORMAL polynomials on $[a, b]$. Then it is linearly independent.

PROOF. Let $d_0P_0(x) + \ldots + d_nP_n(x) = 0$. Since 0 is the zero vector, so $(P_i(x), 0) = 0$ for all $i = 0, 1, 2, \ldots, n$. Then

$$0 = (P_i(x), 0) = (P_i(x), d_0P_0(x) + \ldots + d_nP_n(x))$$
$$= d_i(P_i(x), P_i(x)).$$

But $P_i(x) \neq 0$, and thus $d_i = 0$ for $i = 0, 1, 2, \ldots, n$. Therefore, $\{P_0(x), \ldots, P_n(x)\}$ is linearly independent.

THEOREM 2.2. Let $\{P_0(x), \ldots, P_n(x)\}$ be an ORTHONORMAL set of polynomials defined on $[a, b]$. Then for any polynomial $P(x)$ of degree $\leq n$,

$$P(x) = d_0P_0(x) + \ldots + d_nP_n(x),$$

where $d_i = \int_a^b P_i(x)P(x)w(x)dx$ are uniquely determined by $P(x)$.

PROOF. Let $P(x)$ be a polynomial of degree n with leading coefficient a_n. Then

$$P(x) = a_n x^n + \text{a polynomial of degree} < n.$$

$P_n(x) = \alpha_n x^n$ + a polynomial of degree $< n$, where $\alpha_n \neq 0$. Let $d_n = a_n/\alpha_n$. Then

$$P(x) = d_n P_n(x) + \text{a polynomial of degree} < n.$$

Repeat the processes for $P_{n-1}(x), \ldots, P_0(x)$. Then

$$P(x) = d_0 P_0(x) + \ldots + d_n P_n(x).$$

Since

$$\begin{aligned}\int_a^b P_i(x)P(x)w(x)dx &= (P_i, P)\\ &= (P_i, d_0P_0 + \ldots + d_nP_n)\\ &= d_i,\end{aligned}$$

for $i = 0, 1, \ldots, n$. Therefore, the coefficients are uniquely determined by $P(x)$ itself.

COROLLARY 2.1. The set $\{P_0(x), \ldots, P_n(x)\}$ of ORTHONORMAL polynomials forms a basis for the n-dimensional linear space consisting of all polynomials of degree $\leq n$.

COROLLARY 2.2. If $P(x)$ is a polynomial of degree $< n$. Then $P(x)$ is ORTHONORMAL to $P_n(x)$.

THEOREM 2.3. Let $\{P_0(x), \ldots, P_n(x)\}$ be an orthonormal set of polynomials defined on $[a, b]$. Then there exists A_n, B_n, and C_n with $A_nC_n \neq 0$ such that

$$P_n(x) = (A_nx + B_n)P_{n-1}(x) + C_nP_{n-2}(x), \qquad (1.1)$$

where $P_{-1}(x) = 0$. (1.1) is called the three-term recursion formula.

PROOF. Let $P_n(x) = a_n x^n$ + a polynomial of degree n. a_{n-1} is the leading coefficient of $P_{n-1}(x)$ and $a_{n-1} \neq 0$. Hence $A_n = a_n/a_{n-1} \neq 0$. Then $P_n(x) - A_n x P_{n-1}(x)$ is a polynomial of degree $n - 1$. By theorem 1.2,

$$P_n(x) - A_n x P_{n-1}(x) = d_0 P_0(x) + \ldots + d_{n-1} P_{n-1}(x).$$

Then

$$P_n(x) = (A_n x + B_n) P_{n-1}(x) + C_n P_{n-2}(x).$$

To determine B_n, and C_n, take the inner product of (1.1) with $P_{n-1}(x)$ and $P_{n-2}(x)$ respectively. Then

$$B_n = -A_n(xP_{n-1}, P_{n-1})$$

and

$$0 = (P_n, P_{n-2}) = A_n(xP_{n-1}, P_{n-2}) + C_n. \qquad (1.2)$$

Take the inner product of $P_{n-1}(x) = (A_{n-1}x + B_{n-1})P_{n-2}(x) + C_{n-1}P_{n-3}(x)$ with $P_{n-1}(x)$. Then

$$1 = A_{n-1}(xP_{n-2}, P_{n-1})$$
$$= A_{n-1}(P_{n-2}, xP_{n-1}),$$

since $(xP_{n-2}, P_{n-1}) = \int_a^b (xP_{n-2})P_{n-1}w(x)dx$
$$= \int_a^b (P_{n-2})(xP_{n-1})w(x)dx$$
$$= (P_{n-2}, xP_{n-1}).$$

Substitute A_{n-1} into (1.2). Then

$$C_n = -(A_n/A_{n-1}).$$

Let $\{P_0(x), \ldots, P_n(x)\}$ be an orthonormal set of polynomials defined on $[a, b]$.

THEOREM 2.4. The zeros of the polynomial $P_n(x)$ defined on $[a, b]$ are all real and distinct and interior to this interval.

PROOF. Since $P_n(x)$ is a polynomial of degree $n \neq 0$, $P_n(x)$ changes sign n times in the interval $[a, b]$. $P_n(x)$ is orthogonal to a polynomial $P_0(x) = 1$ of degree 0 with respect to the weight function $w(x)$. Then

$$\int_a^b P_n(x)w(x)dx = 0.$$

Suppose $P_n(x)$ just changes signs at m points $x_1, \ldots, x_m$, $(m < n)$, in the interval $[a, b]$. Let $P(x)$ be a polynomial of degree m and

$$P(x) = (x - x_1) \ldots (x - x_m).$$

Then $P(x)$ has the same sign as $P_n(x)$ does so that the product $P_n(x)P(x)$ does not change sign in $[a, b]$. Then

$$\int_a^b P_n(x)P(x)w(x)dx \neq 0.$$

This contradicts the fact that $(P_n(x), P(x)) = 0$.

THEOREM 2.5. Let $\{P_0(x), P_1(x), \ldots, P_n(x)\}$ be an orthonormal set, then $P_n(x)$ and $P_{n-1}(x)$ have no common zeros.

PROOF. To prove this theorem by induction. P_1 and P_0 have no common zero when $n = 1$. Assume $n = k$ the theorem is true. Suppose $P_{k+1}(x)$ and $P_k(x)$ have common zero, say x_0. Then by recursion formula

$$C_k P_{k-1}(x_0) = P_{k+1}(x_0) - (A_{k+1}x_0 + B_{k+1})P_k(x_0).$$

Since $P_{k+1}(x_0) = P_k(x_0) = 0$. It implies $C_k P_{k-1}(x_0) = 0$. But $C_k \neq 0$ so that $P_{k-1}(x_0) = 0$. x_0 is also a zero of $P_{k-1}(x)$. This contradicts the induction hypothesis that $P_k(x)$ and $P_{k-1}(x)$ has no common zeros.

THEOREM 2.6. Let $\{P_0(x), \ldots, P_n(x)\}$ be an orthonormal set of polynomials defined on $[a, b]$.

If $f(x) = \sum_{i=0}^{n} a_i P_i(x)$, $g(x) = \sum_{i=0}^{n} b_i P_i(x)$, then

$$\int_a^b f(x)g(x)dx = \sum_{i=0}^{n} a_i b_i.$$

PROOF.

$$\int_a^b f(x)g(x)dx = \int_a^b \left(\sum_{i=0}^{n} a_i P_i(x)\right)\left(\sum_{j=0}^{n} b_j P_j(x)\right)dx$$

$$= \sum_{i=0}^{n} a_i \sum_{j=0}^{n} b_j \int_a^b P_i(x)P_j(x)dx$$

$$= \sum_{i=0}^{n} a_i b_i.$$

CHAPTER 7

SPECIAL ORTHOGONAL POLYNOMIALS

7.1. SOME EXAMPLES OF ORTHOGONAL SETS

Orthogonal sets are very important in approximating continuous functions. It is quite efficient to use an orthogonal set in approximating continuous functions. Besides that, sets of orthogonal polynomials also play an important role in physics. The Hermite polynomials are used in connection with a form of the Schrodinger wave equation in quantum mechanics and the Laguerre polynomials are used in connection with the wave equation of the hydrogen atom.

Table 7.1 illustrates several kinds of orthogonal polynomials and their weight functions. Legendre polynomials and Chebyshev polynomials are two special cases of Jacobi's polynomials. Jacobi polynomials reduce to Legendre's polynomials if $\alpha = \beta = 0$, and to Chebyshev's polynomials when $\alpha = \beta = -1/2$.

Some orthogonal sets have a very interesting phenomena. That is, those polynomials $P_n(x)$ contain only even powers of x or only odd powers of x according to whether n is even or odd. Let $P_n(x)$ be defined on a symmetric interval $[-a, a]$, with an even weight function $w(x)$.

$P_n(x)$	$w(x)$	$[a, b]$
Jacobi	$(1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha > -1, \beta > -1)$	$[-1, 1]$
Legendre	1	$[-1, 1]$
Chebyshev	$(1-x^2)^{-1/2}$	$[-1, 1]$
Laguerre	$x^{\alpha} e^{-x}$ $(\alpha > -1)$	$[0, \infty)$
Hermite	e^{-x^2}	$(-\infty, \infty)$

Table 7.1.

To prove the above assertion, let $Q(x)$ be any polynomial of degree less than n. Then

$$\int_{-a}^{a} P_n(x)Q(x)w(x)dx = 0.$$

Changing variable by $-x$,

$$0 = -\int_{a}^{-a} P_n(-x)Q(-x)w(-x)dx$$

$$= \int_{-a}^{a} P_n(-x)Q(-x)w(x)dx.$$

Since $Q(x)$ is a polynomial of degree less than n, so is $Q(-x)$. Thus, $P_n(-x)$ is a orthogonal set. $P_n(-x)$ has the same sign on the even powers of x and opposite sign on the odd powers of x as $P_n(x)$. Therefore,

$$P_n(x) = (-1)^n P_n(-x).$$

This equation shows that $P_n(x)$ contains only even powers of x or only odd powers of x, according as n is even or odd.

7.2. THE DEVELOPMENT OF THE CHEBYSHEV POLYNOMIALS

THEOREM 2.1. Let $T_n^*(x) = \cos n\theta$ defined on $[0, 1]$ and $x = (1 - \cos\theta)/2$, where $0 \leq \theta \leq \pi$. Then T_n^* is a polynomial of degree n. This polynomial is said to be a shifted Chebyshev polynomial.

PROOF. This theorem may be proved by induction. The relation $x = (1 - \cos\theta)/2$ can be written as $\cos\theta = 1 - 2x$, which in turn says that $\cos\theta$ is transformed into $T_1^*(x)$. That is, $T_1^*(x) = \cos\theta$. Suppose it is true when $n = k$. Since

$$\cos(k + 1)\theta = \cos\theta\cos k\theta - \sin\theta\sin k\theta,$$

and
$$\cos(k - 1)\theta = \cos\theta\cos k\theta + \sin\theta\sin k\theta.$$

Adding them one has,

$$\begin{aligned}\cos(k + 1)\theta &= 2\cos\theta\cos k\theta - \cos(k - 1)\theta \\ &= 2(1 - 2x)T_k^*(x) - T_{k-1}^*(x).\end{aligned}$$

This is a polynomial of degree $k + 1$. Thus $T_{k+1}^*(x) = \cos(k + 1)\theta$.

Use the trigonometric integral,

$$\int_0^{\pi}\cos n\theta\cos m\theta\, d\theta = C_n\delta_{mn},$$

where $C_0 = \pi$ and $C_n = \pi/2$ $(n \neq 0)$. With the change of variable $x = (1 - \cos\theta)/2$,

$$\int_0^1 \frac{T_n^*(x)T_m^*(x)}{\sqrt{x(1-x)}}\, dx = C_n\delta_{mn}.$$

Therefore, the set of shifted Chebyshev polynomials is orthogonal over $[0,1]$ with respect to the weight function $(x(1-x))^{-1/2}$. With the aid of three-term recurrence formula, the first four shifted Chebyshev polynomials are

$$T_0^*(x) = 1,$$

$$T_1^*(x) = 1 - 2x,$$

$$T_2^*(x) = 1 - 8x + 8x^2,$$

$$T_3^*(x) = 1 - 18x + 48x^2 - 32x^3.$$

These polynomials defined on $[0, 1]$ are graphed in Figure 7.1.

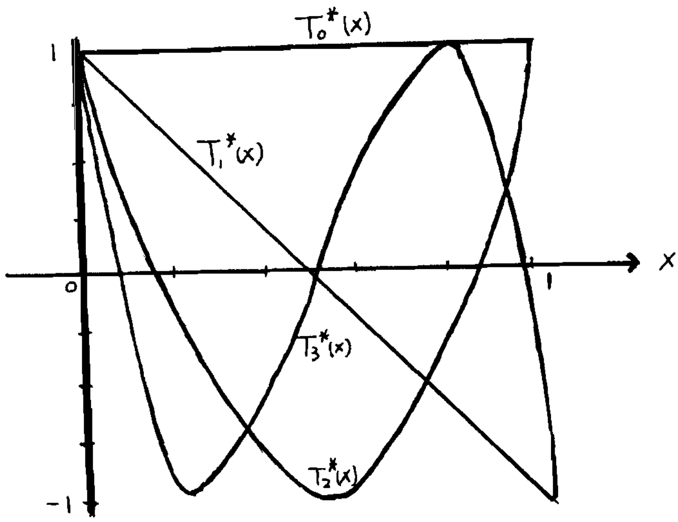

Fig. 7.1.

The Chebyshev polynomials $T_n(x)$ are defined in terms of the shifted Chebyshev polynomials $T_n^*(x)$ by the relation

$$T_n(x) = T_n^*((1-x)/2).$$

$T_n(x)$ is defined on $[-1, 1]$, since

$$T_n(x) = T_n^*((1-x)/2)$$
$$= \cos n\theta,$$

where $0 \leq \Theta \leq \pi$.

According to the trigonometric identity,

$$\cos n\theta \cos m\theta = \frac{1}{2} \cos(n + m)\theta + \cos (n - m)\theta \quad ,$$

it can be rewritten as

$$T_n(x)T_m(x) = \frac{1}{2}\left[T_{n+m}(x) + T_{n-m}(x)\right].$$

Setting $m = 1$,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

This formula is called the recurrence relation of the Chebyshev polynomials. This formula is quite useful in approximation problems because it is self-starting and efficient in terms of computation time and storage space.

With the aid of the recurrence relation, the first four polynomials are listed below:

$T_0(x) = 1,$

$T_1(x) = x,$

$T_2(x) = 2x^2 - 1,$

$T_3(x) = 4x^3 - 3x.$

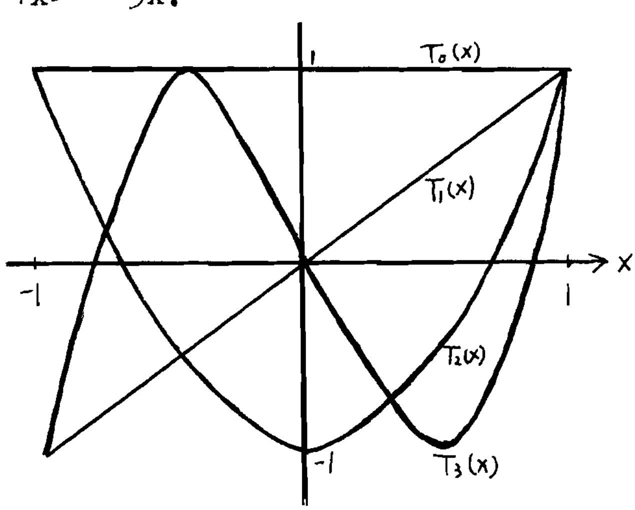

Fig. 7.2.

It is not a loss of generalization to consider only the interval $[-1, 1]$, since by a change of variable one can handle an interval $[a, b]$.

The orthogonality of the Chebyshev polynomials over $[-1, 1]$ with respect to the weight function $(1 - x^2)^{-1/2}$ can be proved in the same manner as in the case of the shifted Chebyshev polynomials.

From the relation

$$T_n(x) = \cos nx,$$

it is true that $|T_n(x)| \leq 1$ for all $x \in [-1, 1]$. This property is very important in the least-square approximation by Chebyshev polynomials.

Theorem 2.4 in Chapter 6 guaranteed that the zeros of $T_n(x)$ are all real, distinct, and lie on the interval $[-1, 1]$.

EXAMPLE 7.1. Find the roots of $T_3(x)$.

SOLUTION.

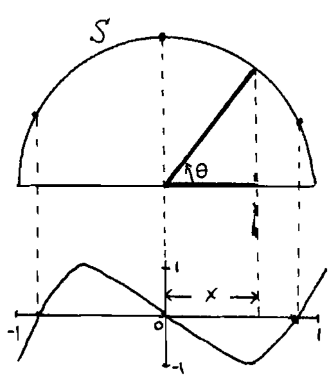

Fig. 7.3.

In Figure 7.3, S is a semicircle with radius 1. the relation between x and θ is

$$x = \cos \theta, \qquad 0 \leq \theta \leq \pi.$$

Let $T_3(x) = 0$. Then

$$T_3(x) = \cos 3\theta = 0.$$

Hence $3\theta = n\pi + \pi/2$, $n = 0, 1, 2$. Thus the roots of $T_3(x)$ will be at $\theta = \pi/6$, $\pi/2$, and $5\pi/6$. Then since $x = \cos\theta$, $x = \sqrt{3}/2$, 0, and $-\sqrt{3}/2$. They are real, distinct, and lie in $[-1, 1]$.

In general, the zeros of Chebyshev polynomials of degree n are

$$x_k = \cos^{(2k-1)\pi}/2n\ , \qquad k = 1, 2, \ldots, n.$$

The zeros are symmetric with respect to the origin. Moreover, the roots accumulate around the end points -1 and 1 for large values of n.

CHAPTER 8

APPLICATION OF ORTHOGONAL POLYNOMIALS

8.1. LEAST-SQUARE APPROXIMATION TO A CONTINUOUS FUNCTION

A set $G = \{P_i(x)\}$ of polynomials distinguishes points of $[a, b]$ if the $P_i(x)$ are defined on $[a, b]$ and for each pair of points $x_1 \neq x_2$ of $[a, b]$ there is a function $P_k(x) \in G$ with $P_k(x_1) \neq P_k(x_2)$.

Let $F(x)$ be a continuous real function defined on $[a, b]$. The least squares technique is a method to find a set $\{d_1, \ldots, d_k\}$ of scalars and

$$P(x) = d_1P_1(x) + \ldots + d_kP_k(x),$$

such that

$$E(d_1, \ldots, d_k) = \int_a^b (F(x) - P(x))^2dx$$

is minimized.

The condition for a minimum is $\partial E/\partial d_i = 0$ for $i = 1, \ldots, k$. Since $F(x)$ is not a function of the d_i, one has that $\partial F/\partial d_i = 0$. Then by Leibnitz's rule

$$\begin{aligned} 0 &= \partial E/\partial d_i \\ &= \int_a^b \partial/\partial d_i(F(x) - P(x))^2dx \\ &= (-2) \int_a^b (F(x) - P(x)) \partial P(x)/\partial d_i \, dx \\ &= (-2)(\int_a^b F(x)P_i(x)dx - \int_a^b P(x)P_i(x)dx. \end{aligned}$$

Thus

$$d_1 \int_a^b P_1(x)P_i(x)dx + \ldots + d_k \int_a^b P_k(x)P_i(x)dx = \int_a^b F(x)P_i(x)dx$$

$$i = 1, \ldots, k. \qquad (1.1)$$

This is a system of k linear equations in the k unknowns. The coefficient matrix of this system is quite often ill-conditioned, and thus the application of the Gauss elimination procedure may produce unreliable results.

But every linear equation of this system can be reduced to a very simple equation if a continuous function may be approximable by an orthogonal set. By the Weierstress theorem, every continuous real function $F(x)$ on $[a, b]$ can be approximated by a polynomial. Theorem 2.2 in Chapter 6 showed that every polynomial can be represented by the linear combination of orthonormal polynomials. Therefore, it is possible to approximate a continuous real function by an orthonormal set.

Let $P_1(x), \ldots, P_n(x)$ be an orthonormal set. Then, (1.1) reduces to

$$d_i = \int_a^b F(x)P_i(x)dx$$

By using this equation, the equations can be solved directly without round-off problems. Also, there is only one operation for solving for each d_i. Therefore, least-square approximation by orthogonal polynomials is not only less likely to produce large errors but is also more efficient.

8.2. THE BEST APPROXIMATION BY POLYNOMIALS OVER $[a, b]$

Let F be a continuous function on $[a, b]$, and P

be a polynomial of degree n. Then P is called the best approximation to F, with respect to the norm $\| \ \|$, if $\|F - P\| \leq \|F - P_n\|$ for all polynomials P_n of degree n.

THEOREM 2.1. Let F be a continuous function on $[a, b]$. Then there exists a polynomial P of degree n which is the best approximation to F.

PROOF. Let $\{P_i : 1 \leq i \leq n\}$ be a set of linearly independent polynomials, where each P_i has degree of i, and $\sum_{i=1}^{n} P_i = M$. Let $P = \sum_{i=1}^{n} a_i P_i$ and $E(a_1, \ldots, a_n) = \|F - P\|$. Then $E(a_1, \ldots, a_n)$ is continuous. Since for given $\epsilon > 0$,

$$\begin{aligned} |E(a_1, \ldots, a_n) - E(b_1, \ldots, b_n)| &= \big| \|F - P\| - \|F - P'\| \big| \\ &\leq \|F - P - F + P'\| \\ &= \sum_{i=1}^{n} |b_i - a_i| \, \|P_i\| \\ &\leq (\max_i |b_i - a_i|) \cdot \sum_{i=1}^{n} \|P_i\|. \end{aligned}$$

Then $|E(a_1, \ldots, a_n) - E(b_1, \ldots, b_n)| < \epsilon$ whenever $\max_i |b_i - a_i| < \epsilon/M$.

Let $S = \left\{(a_1, \ldots, a_n) : \sum_{k=1}^{n} a_k^2 = 1\right\}$. $\|\sum_{i=1}^{n} a_i P_i\|$ is also continuous on S. S is a closed subset of R^n and bounded. Therefore $\|\sum_{i=1}^{n} a_i P_i\|$ attains its minimum value, say m, on S. The polynomials P_i are assumed to be linearly independent and the a_i are not all zeros so that $m > 0$.

$E(a_1, \ldots, a_n)$ is also bounded since $\|\sum_{i=1}^{n} a_i P_i\|$ is bounded on S. By completeness, $E(a_1, \ldots, a_n)$ has a greatest lower bound m'. If $E(a_1, \ldots, a_n)$ attains m' in some region, then the best approximation exists in that region. Let

$$R = (m' + 1 + F)/m.$$

Suppose $\sum_{i=1}^{n} a_i^2 > R^2$ for $(a_1, \ldots, a_n) \in R^n$. Then

$$\|\sum_{i=1}^{n} a_i P_i\| = (\sum_{j=1}^{n} a_j^2)^{1/2} \cdot \|(\sum_{j=1}^{n} a_j^2)^{-1/2} \cdot \sum_{i=1}^{n} a_i P_i\|$$

$$= (\sum_{j=1}^{n} a_j^2)^{1/2} \cdot \|\sum_{i=1}^{n} (\sum_{j=1}^{n} a_j^2)^{-1/2} a_i P_i\|,$$

and $\sum_{i=1}^{n} ((\sum_{j=1}^{n} a_j^2)^{-1/2} a_i)^2 = 1$. Then, since these coefficients sum to one, it follows that

$$\|\sum_{i=1}^{n} a_i P_i\| \geq (\sum_{i=1}^{n} a_i^2)^{1/2} \cdot m.$$

Thus

$$\|F - \sum_{i=1}^{n} a_i P_i\| \geq \|\sum_{i=1}^{n} a_i P_i\| - \|F\|$$

$$\geq (\sum_{i=1}^{n} a_i^2)^{1/2} m - \|F\|$$

$$> mR - \|F\|$$

$$= m' + 1.$$

But this is a contradiction, since m' is the greatest lower bound of $E(a_1, \ldots, a_n)$. Therefore $\sum_{i=1}^{n} a_i^2 \leq R^2$. This region is closed and bounded. Thus the continuous function $E(a_1, \ldots, a_n)$ attains its minimum value in this region. Hence the theorem is proved.

THEOREM 2.2. Let F be a continuous function defined on $[a, b]$. Then for each n, there exists a unique polynomial of degree n that is the best approximation to F.

PROOF. Let $P = a_1P_1 + \ldots + a_nP_n$ and $P' = b_1P_1 + \ldots + b_nP_n$ be two best approximations of a continuous function F with the same order, where the P_i are linearly independent and each P_i has degree i. Then $\|F - P\| = \|F - P'\| = E$. Let
$G = (\alpha a_1 + (1-\alpha)b_1)P_1 + \ldots + (\alpha a_n + (1-\alpha)b_n)P_n$,
for $0 < \alpha < 1$. Thus

$$\begin{aligned} \|F - G\| &= \|F - \sum_{i=1}^{n} (\alpha a_i + (1-\alpha)b_i)P_i\| \\ &= \|\alpha(F - P) + (1-\alpha)(F - P')\| \\ &\leq \alpha\|F - P\| + (1-\alpha)\|F - P'\| \\ &= E. \end{aligned}$$

Since P and P' are already the best approximations of F, G cannot give a better approximation, hence equality must hold. Therefore, either

$$\|F - P\| = 0 \qquad \text{or } \|F - P'\| = 0,$$

or for some scalar β

$$F - P = \beta(F - P').$$

In the last case,

$$(1 - \beta)F = (1 - \beta)(b_1P_1 + \ldots + b_nP_n)$$
$$= (a_1 - \beta b_1)P_1 + \ldots + (a_n - \beta b_n)P_n.$$

That is,

$$((1 - \beta)b_1 - (a_1 - \beta b_1))P_1 + \ldots + ((1 - \beta)b_n - (a_n - \beta b_n))P_n = 0.$$

Since P_i are linear independent, this implies that $(1 - \beta)b_i - (a_i - \beta b_i) = 0$, for $i = 1, \ldots, n$. Hence $a_i = b_i$, for $i = 1, \ldots, n$. Thus $\beta = 1$. Thus the approximation is unique in either case.

Theorem 2.2. can be extended to the L_p spaces. This development may be found in [10].

THEOREM 2.3. Let $\{P_1, \ldots, P_n\}$ be a set of polynomials such that $\deg(P_i) = i$. If for each continuous function on $[a, b]$, the best polynomial of degree less than or equal to n is a linear combination of the P_i, then $P_1, \ldots, P_n$ are Chebyshev polynomials. This famous theorem was proved by A. Harr.

8.3. MIN-MAX APPROXIMATION BY POLYNOMIALS OVER $[a, b]$

The min-max approximation to a continuous function F on $[a, b]$ is the best approximation to F in the L_∞ norm.

In general, a function $F(x)$ has a Taylor's series expansion if its derivatives exist and are continuous on $[a, b]$. If $F(x)$ is expanded by a Maclaurin series, then

$F(x) = \sum_{r=0}^{\infty} a_r x^r$. This series converges uniformly to $F(x)$. This means that for each $\epsilon > 0$, there exists $n(\epsilon) > 0$ such that

$$\left| F(x) - \sum_{r=0}^{n} a_r x^r \right| < \epsilon ,$$

for all $x \in [a, b]$. Then $P_n(x) = \sum_{r=0}^{n} a_r x^r$ is an approximating polynomial for $f(x)$.

For example, let $F(x) = e^x = \sum_{n=0}^{\infty} x^n/n!$. Then $n = 12$ is required to approximate e^x on $[0, 1]$ with an error 10^{-8}. In this case, the Maclaurin series of e^x converges very fast. But some series, for instance, $\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n/n$ converges around $x = 1$ so slowly that 10^8 terms are needed to guarantee an error less than 10^{-8}.

The primary objective now is to find a polynomial of lower degree for the approximation but having the same accuracy. The Chebyshev economization technique is used for this purpose.

Let $F(x)$ be a continuous function on $[-1, +1]$. Let $\epsilon > 0$ be given. By the Stone-Weierstrass theorem there exists a polynomial $p(x)$ such that

$$|F(x) - p(x)| < \epsilon$$

for each $x \in [-1, +1]$.

Since the collection $\{T_0, T_1, \ldots\}$, of Chebyshev polynomials is a basis for $C_2[-1, +1]$ it follows that

$p(x)$ can be expressed as a linear combination of a finite subset of $\{T_0, T_1, \ldots\}$. Then there exists a set of Chebyshev polynomials, $\{T_0, T_1, \ldots, T_n\}$ such that

$$|F(x) - P_n(x)| < \epsilon ,$$

where $P_n(x) = d_0T_0 + \ldots + d_nT_n$. Then

$$\begin{aligned} E_n &= \| F - P_n \| \\ &= \sup \{|F(x) - P_n(x)| : x \in [-1, +1]\} \\ &< \epsilon . \end{aligned}$$

For the same reason, $P_n(x)$ has its best approximation $P_{n-1}^*(x)$ in $\{T_0(x), \ldots, T_{n-1}(x)\}$ of Chebyshev polynomials.

Let $E_{n-1} = \sup |P_n(x) - P_{n-1}^*(x)|$. Then

$$\begin{aligned} \sup |F(x) - P_{n-1}^*(x)| &\leq \sup|F(x) - P_n(x)| + \sup|P_n(x)-P_{n-1}^*(x)| \\ &= E_n + E_{n-1}. \end{aligned}$$

If $E_n + E_{n-1}$ is still less than ϵ , one may repeat the process to get the best approximation $P_{n-2}^*(x)$ of $P_{n-1}^*(x)$; and continue in this manner until,

$E_n + E_{n-1} + \ldots + E_j \leq \epsilon < E_n + E_{n-1} + \ldots + E_j + E_{j-1}$.

Then $P_j^*(x)$ is the lowest-degree approximating polynomial to $F(x)$ within the error allowance ϵ .

THEOREM 3.1. Let $P_n(x)$ be a polynomial of degree n defined on $[-1, +1]$. If $P_n(x) = a_nx^n + Q(x)$, where $Q(x)$ is a polynomial of degree $< n$, then the min-max approximation $P_{n-1}^*(x)$ to $P_n(x)$ by polynomials of degree less than n is given by $P_{n-1}^*(x) = P_n(x) - a_n2^{1-n}T_n(x)$.

PROOF. It can be shown by induction that

$T_n(x) = 2^{n-1}x^n + Q'(x)$, where $Q'(x)$ is a polynomial of degree $< n$. Then

$$P_{n-1}^*(x) = P_n(x) - a_n 2^{1-n} T_n(x)$$
$$= a_n x^n + Q(x) - a_n 2^{1-n}(2^{n-1}x^n + Q'(x))$$
$$= Q(x) - a_n 2^{1-n} Q'(x),$$

is a polynomial of degree $< n$. Suppose the theorem were false. Then there exists a $P_{n-1}(x)$ such that

$$\sup|P_n(x) - P_{n-1}^*(x)| > \sup|P_n(x) - P_{n-1}(x)|.$$

Let $e(x) = P_{n-1}^*(x) - P_{n-1}(x)$ be nonzero polynomials of degree $< n$ and

$$P_n(x) - P_{n-1}(x) = P_n(x) - P_{n-1}^*(x) + e(x).$$

Since $P_n(x) - P_{n-1}^*(x) = a_n 2^{1-n} T_n(x)$,

$$\sup|a_n 2^{1-n} T_n(x)| > \sup|a_n 2^{1-n} T_n(x) + e(x)|.$$

But $|T_n(x)| \leq 1$ for $x \in [-1, 1]$, therefore,

$$|a_n| 2^{1-n} > \sup|a_n 2^{1-n} T_n(x) + e(x)|$$
$$> |a_n{}^{1-n} T_n(x) + e(x)|$$
$$> a_n 2^{1-n} T_n(x) + e(x)$$
$$> -|a_n| 2^{1-n},$$

for all $x \in [-1, 1]$. Set $x_k = \cos k\pi/n$, $k = 0, 1, \ldots, n$. Then

$$|a_n| 2^{1-n} > a_n 2^{1-n} T_n(\cos k\pi/n) + e(x_k)$$
$$= a_n 2^{1-n} \cos k\pi + e(x_k)$$
$$= a_n 2^{1-n}(-1)^k + e(x_k)$$
$$> -|a_n| 2^{1-n}.$$

Assuming $a_n > 0$,

$$0 > e(x_k) \quad \text{if} \quad k \quad \text{is even,}$$

and

$$0 < e(x_k) \quad \text{if} \quad k \quad \text{is odd.}$$

If $a_n < 0$, the opposite direction applies for these inequalities. Since $x_0 > x_1 > \ldots > x_k$, in either case $a_n > 0$ or $a_n < 0$, these inequalities imply that $e(x)$ changes sign in $[-1, 1]$ at least n times. Therefore $e(x)$ has at least n zeros. This contradicts the fact that $e(x)$ is a polynomial of degree $\leq n-1$.

The following table which expresses $T_n(x)$ in powers of x is useful in the application of the Chebyshev economization process.

$1 = T_0$

$x = T_1$

$x^2 = 2^{-1}(T_0 + T_2)$

$x^3 = 2^{-2}(3T_1 + T_3)$

$x^4 = 2^{-3}(3T_0 + 4T_2 + T_4)$

$x^5 = 2^{-4}(10T_1 + 5T_3 + T_5)$

$x^6 = 2^{-5}(10T_0 + 15T_2 + 6T_4 + T_6)$

EXAMPLE. The approximation

$$\ln(1 + x) \approx x - \frac{1}{2} x^2 + \frac{1}{3} x^3 - \frac{1}{4} x^4$$

has truncation error less than 0.12 at $x = 1$. With the aid of the preceeding table,

$$\ln(1 + x) \approx 1/192(141\ T_0 + 24\ T_1 - 52\ T_2 + 8\ T_3 - T_4).$$

The advantage of Chebyshev economization is that the truncation error is still less than 0.12 after the omission of the terms T_3 and T_4.

CHAPTER 9

SUMMARY

In Chapter 3, theorem 3.1 proved that every metric space has a unique completion. This theorem is quite important in approximation theory.

The next problem is to find a set of polynomials to approximate a continuous function in $C_2[a, b]$. Orthogonal polynomials are the answer, and a complete orthonormal set is shown to be a basis for $C_2[a, b]$.

For each continuous function $f(x)$ on $[a, b]$, there may be several approximating polynomials. Among the approximating polynomials, which is the best one of degree n? The theorems in Chapter 8 proved that the best approximating polynomial of a given degree does exist and is unique.

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