AN ABSTRACT OF THE THESIS OF

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The development of a mathematical system must follow a rigid set of rules. There is, however, one rule or process that may be included at different times in the development of the system. That is the introduction of a model for the system. In most cases, the system is developed first and then a model is constructed. In this paper a few axioms and theorems are introduced and then the system is expanded after the examination of two isomorphic models. This process is used to examine a geometry of 25 points.

When considering a geometry, it is a common process to compare the system to Euclidean geometry. Any discussion of Euclidean geometry leads to a consideration of Euclids fifth postulate or one of several other statements equivalent to it. The two statements which are discussed in detail in this paper are Playfair's axiom and the Pythagorean Theorem.

The thesis then consists of a partial development of a 25 point

geometry, considerable discussion in chapters two, three, and five of models for the geometry, and a comparison of the 25-point geometry with Euclidean geometry.

TWENTY-FIVE POINT GEOMETRY

A Thesis

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by

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Chapter 1

INTRODUCTION

The use of models for axiomatic systems is usually for the purpose of testing the consistency of the system. Thus, one usually has a set of axioms first and creates a model from this set of axioms. Yet, in the history of the development of Euclidean geometry it is evident that the model - namely, man's conception of his physical environment - came first and that the creation of a satisfactory set of axioms for a geometry represented by this model came afterward.

This paper attempts to follow this second procedure. Beginning with a very minimum of statements to be accepted as true and with a model satisfying these statements as well as many other facts not originally taken as axioms, the author hopes to ascertain which of these many other facts are basic enough to be included among the axioms. By then considering another model possibly isomorphic to the first, the author feels that the task of deciding upon the necessary axioms may be simplified.

There is no expectation that the set of axioms which finally evolve in this paper will be complete, independent, and categorical as is usually required for a mathematical system. Rather, the expectation is that this work will be a first step in the development of such a system. In the meanwhile, many interesting relations and many interesting ideas are found in the process of studying the models themselves.

It should also be noted that this approach was chosen for a very specific reason. In all of the articles on the 25-point geometry that the author read, none of the articles attempted to develop a set of axioms independent of the model that will be presented later. In every case, the authors of those articles used the model as if it were the first axiom of the system. In other words, in the opinion of this author, each of those articles assumed the following axiom.

> Axiom: There are exactly 25 points arranged in three given arrays.

The arrays used by these authors will be the same arrays used in this paper and since they are presented in the next chapter, they will not be introduced at this time.

Hence, this paper will attempt, with the models in mind, to begin development of a 25-point geometry.

Chapter 2

A MODEL OF A 25-POINT GEOMETRY

Of the numerous articles that have been written about finite geometries, very few are particularly concerned in detail with a geometry of 25 points. Beginning with the undefined terms "point", "line", and "on", consider the following axioms.

AXIOM 1: There are exactly 25 distinct points.

AXIOM 2: Every line contains exactly five distinct points.

AXIOM 3: Through any two distinct points there is exactly one line.

With these axioms the following theorems can be proven.

THEOREM 1: There are exactly 30 lines.

Proof: From axiom 1, there are exactly 25 points. Hence, there must be 25 X 24 = 600 pairs of different points. Now, assuming that AB and BA are the same point pair, there will be 600 / 2 = 300 unique point pairs. From axiom 2, every line contains exactly 5 points. These five points of line 1 contain (5 X 4) / 2 point pairs. Hence, there are 300 / 10 = 30 unique lines.

THEOREM 2: Given any line, there exists a point not on the given line.

Proof: Given line 1. From axiom 2, 1 contains exactly 5 distinct points. But by axiom 1 there are 25 distinct points and hence, given any line there exists a point not on the given line.

THEOREM 3: Every point is on exactly six lines.

Proof: Given any arbitrary point A. From axiom 1, there exists a point other than A. Call this point B. From axiom 3, there is a line 1 containing A and B. Now, according to axiom 2, there are three other points on this line 1. From theorem 2, there exists a point F not on 1 and again by axiom 3 there is a line m through A and F which contains no point of 1 other than A or that would contradict axiom 3. This line m contains three points other than A and F. Now, the two lines 1 and m have used nine distinct points. Continuing the process of choosing points until all 25 points have been used allows the construction of six distinct lines. Now, since A was arbitrary, every point is on exactly six lines.

Prior to the next theorem, the first definition of the system is needed.

DEFINITION 1: Two lines intersect if they have a point in common.

THEOREM 4: Two distinct lines which intersect, intersect in exactly one point.

Proof: Given two distinct lines 1 and m which intersect. By definition of intersect, they have a point in common.

> Therefore, it must be shown that there is only one point of intersection. An indirect proof will show this. Suppose l and m intersect in more than one point, $p_1, p_2, \ldots p_n$. Therefore, there are two distinct lines through points p_1 and p_2. This contradicts axiom 3 and hence, two distinct lines which intersect, intersect in exactly one point.

At this point a discussion about a model for the geometry is necessary. Consider the following array of 25 letters.

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

Let each letter denote a point and each row or column denote a line. This array then satisfies the first two axioms but not the third. For example, there is not a line on A and G. In fact this array exhibits only 10 of the 30 lines. Consider then the original array together with the two following arrays.

A	I	L	T	W
S	V	E	H	K
G	O	R	U	D
Y	C	F	N	Q
M	P	X	B	J

A H O Q X

N P W E G

V D F M T

J L S U C

R Y B I K

These three arrays were introduced in the book The Education of T. C. Mits[1] and have been used as the model of the 25-point geometry by the subsequent authors. Hence, these arrays will serve as a model in this paper.

A close examination of the three arrays reveals that all of the axioms and theorems are exhibited by this model. Additional discussion about the relationship of the three arrays will be found in chapter five. There is no claim here that this model is isomorphic to all other models for a geometry satisfying the given axioms. This particular model is of interest in itself and this paper is concerned as much with the model as with a geometry represented by the model.

Using the model as a basis, the question arises as to what other axioms are needed for the geometry represented. So far, these interpretations have been introduced.

A point is any of the 25 letters of the arrays.

A line is any row or column in any of the arrays.

These lines are often called row-lines or column-lines.

For example, ABCDE, AFKPU, and SVEHK are lines in the geometry.

In order to simplify notation, each of the basic arrays of the 25-point model is a matrix and hence, one may refer to the arrays

[1] Lieber and Lieber, The Education of T.C. Mits, p. 153.

using common matrix notation.

Now, the following definitions are introduced.

DEFINITION 2: Three points are collinear if they lie on the same line.

DEFINITION 3: Two lines are parallel if they are any two rows or any two columns in the same array.

DEFINITION 4: Two lines are perpendicular if they are any row and any column in the same array.

Some words of caution are necessary at this time. Notice that in the definitions of parallel and perpendicular that these are defined only on rows and columns of the same array. For example, ABCDE and AFKPU are perpendicular but no such relationship exists between ABCDE and ASGYM.

Examining the model, the following observations are noted.

OBSERVATION 1: Through a point not on a line there is exactly one line parallel to the given line.

Proof: Part 1 - Given line l. By theorem 2, there exists a point a_{ij} not on l. If l is a row-line then row i is parallel to l. If l is a column-line then column j is parallel to l. Hence, through a point not on a line there exists a line parallel to the given line.

Part 2 - An indirect proof will be used to show that there is not more than one line through a_{ij} parallel to l. Suppose there is more than one line through a_{ij} parallel to l. Call them m, n, ... z.

Now, a_{ij} appears in more than one row or column of one of the arrays. Therefore, the 25 points of this array are not distinct and this contradicts axiom 1. Our assumption is false and there is not more than one line through a_{ij} parallel to 1. Now, from parts 1 and 2, through a point not on a line there is exactly one line parallel to the given line.

OBSERVATION 2: Two distinct lines parallel to a third line are parallel to each other.

Proof: Given lines k, 1, and m such that k || m and 1 || m. By definition of parallel, k, 1, and m must all be rows or all columns of the same array. Hence, by definition of parallel, 1 || k.

OBSERVATION 3: Through a point not on a line there is exactly one line perpendicular to the given line.

Proof: Part 1 - Given line 1. By theorem 2, there exists a point a_{ij} not on 1. If 1 is a row-line then column j is perpendicular to 1 by definition of perpendicular. If 1 is a column-line then row i is perpendicular to 1 by definition. Hence, Through a point not on a line there exists a line perpendicular to the given line.

Part 2 - An indirect proof will show that there is not more than one line through a_{ij} perpendicular to 1. Suppose that there is more than one line through a_{ij} perpendicular to 1. Call them m, n, ... z. Now, a_{ij} appears in more than one row or column of one of the arrays. Therefore, the 25 points of this array are not distinct and this contradicts axiom 1. Our assumption is false and there is not more than one line through a_{ij} perpendicular to 1.

Now, from parts 1 and 2, through a point not on a line there is exactly one line perpendicular to the given line.

OBSERVATION 4: Two distinct lines perpendicular to the same line are parallel.

Proof: Given lines 1, m, and n such that $1 \perp n$ and $m \perp n$. If n is a row-line then by definition of perpendicular 1 and m are columns in the same array as n and $1 \parallel m$ by definition of parallel. Likewise, if n is a column-line then 1 and m are rows in the same array as n and $1 \parallel m$. Hence, two distinct lines perpendicular to the same line are parallel.

The four observations regarding the model can not be logically derived from the original three axioms since they are based on the concepts of parallel and perpendicular which were both defined in terms of the model and not in terms of the primitive terms of the axioms.

Therefore, let observations 1 and 3 be taken as axioms, in which case the other two observations will follow as theorems.

The axiomatic system now consists of five primitive terms; point, line, on, parallel, and perpendicular, and of the five axioms that follow.

A 1: There are exactly 25 points.

A 2: Every line contains exactly 5 points.

A 3: Through any two points there is exactly one line.

A 4: Through a point not on a line there is exactly one line parallel to the given line.

A 5: Through a point not on a line there is exactly one line perpendicular to the given line.

Note that the model under study does satisfy all five axioms, but again, there is no assumption that this is the only model satisfying the axioms. In this paper, any conclusion drawn from the model will be labeled as an observation to distinguish it from those conclusions derived from the axioms. With the inclusion of axioms four and five, all conclusions made so far can be derived from the axioms.

DEFINITION 5: A line segment is any set of consecutive points of a line.

For example, AB, ABC, and AFKP are line segments while ABD and AED are not line segments. In addition, ABCDE and AFKPU and so on will also be considered as line segments as well as lines.

Secondly, the common notation $\overline{AC}$ will be used to represent a segment such as ABC and $\overline{AC}$ and $\overline{CA}$ will be considered as denoting the same line segment.

THEOREM 5: Any line contains exactly four line segments with a fixed endpoint.

Proof: Given any arbitrary line 1. By axiom 2, 1 contains exactly 5 distinct points, say ABCDE. Choosing any point, C, one can combine it with consecutive points of the line terminating with any of the other four points. Hence, there exist segments $\overline{CA}$, $\overline{CB}$, $\overline{CD}$, and $\overline{CE}$. Now, since 1 and C were arbitrary, any line contains exactly four line segments with a fixed endpoint.

THEOREM 6: Every point is on exactly three row-lines.

Proof: By theorem 3, every point is on exactly six lines. By axiom 1, every point appears exactly once in each of the three arrays. Now, on every point in an array there is exactly one row-line or axiom 1 would be contradicted. Therefore, since there are three arrays, every point is on exactly three row-lines.

THEOREM 7: There are exactly twelve row-segments containing any point.

Proof: Given any arbitrary point A. From theorem 6, there are three row-lines containing A. Combining this with theorem 5, there are exactly 3 X 4 = 12 row-segments containing A. Now since A was arbitrary there are exactly twelve row-segments containing any point.

THEOREM 8: Through a point on a line there is exactly one line perpendicular to the given line.

Proof: Part 1 - Given line 1. On line 1 choose a point a_{ij}. If 1 is a row-line then column j is perpendicular to 1 by definition of perpendicular. Likewise, if 1 is a column-line then row i is perpendicular to 1. Hence, through a point on a line there exists a line perpendicular to the given line.

Part 2 - An indirect proof will be used to show that there is not more than one line through a_{ij} perpendicular to 1. Suppose that there is more than one line through a_{ij} perpendicular to 1. Call them m, n, ... z. Now, a_{ij} is in more than one row or column of one of the arrays. Therefore, the 25 points of this array are not distinct. This contradicts axiom 1. Our assumption is false and there is not more than one line through a_{ij} perpendicular to 1.

Now, from parts 1 and 2, through a point on a line there is exactly one line perpendicular to the given line.

THEOREM 9: There are exactly eight line segments perpendicular to a given line segment at its endpoints.

Proof: Given an arbitrary line segment AB. From theorem 8, there is exactly one line through A perpendicular to $\overline{AB}$ and there is exactly one line through B perpendicular to $\overline{AB}$. Each of these two lines contains four line segments by theorem 5 and now there are 4 X 2 = 8 line segments perpendicular to $\overline{AB}$ through A and B.

Now, since AB was arbitrary, there are exactly eight line segments perpendicular to a given line segment at its endpoints.

DEFINITION 6: A Triangle is the union of the line segments connecting 3 non-collinear points.

DEFINITION 7: A right triangle is a triangle which has two sides perpendicular.

For example, triangle ABF is a right triangle.

Examination of the triangles of the 25-point geometry reveals an interesting phenomenon. The sides of a triangle do not appear in any single array. Consider as an example the three points H, L, and R. The side HR is in the first array, side LR is in the second array, and side HL is in the third array.

The discussion of triangles in this chapter will conclude with the introduction of the following theorems.

THEOREM 10: There are exactly 2000 triangles.

Proof: From axiom 1, there are 25 points in the system and again, there are 600 point pairs. Now, the three points that form a triangle must not be collinear, so each of the point pairs may be combined with any of 20 distinct points. Hence, there are 600 X 20 = 12000 point triples. But there are 3! permutations of a triple. Thus, 3! of these 12000 triangles are the same triangle. Therefore, there are 12000 / 6 = 2000 distinct triangles.

THEOREM 11: There are exactly 1200 distinct right triangles.

Proof: From axiom 1, there are 25 distinct points and combining this with theorem 7, there are 25 X 12 = 300 row-segments. Since AB and BA are the same segment, there will be 300 / 2 = 150 distinct row-segments. Now, using theorem 9, there must be 150 X 8 = 1200 pairs of perpendicular segments. Axiom 3 justifies a line and hence a segment through the non-joined endpoints of the perpendicular segments. Therefore, these each form a right triangle by definition of right triangle. Hence, there are exactly 1200 distinct right triangles.

Note that all theorems hold on the basis of the axioms, not merely on inspection of the model. However, the next concept to be introduced - the concept of length - will be introduced in this chapter only on the model. It is hoped that one can determine from a study of this model what further axioms, if any, are needed for the geometry in order that this concept be realized there.

Consider the three basic arrays. In them, regard the lines as being closed and the points as being cyclically permutable. In other words, in the first array the point A immediately "follows" the point E. Hence, the following definitions.

DEFINITION 8: The distance between two row points is the least number of steps separating the two points. The distance between two column points is $\sqrt{2}$ times the least number of steps separating them.

For example, the distance between A and B is 1 while the distance between A and F is $\sqrt{2}$. The distance between A and D is 2 while the distance between A and P is $2\sqrt{2}$.

DEFINITION 9: The length of segment AB shall be the distance between A and B.

Notice that when discussing distance or length, row-wise distance and column-wise distance are regarded as incommensurable.

Secondly, notice that distance is defined on a line and hence, in the first array, d(A,G) is undefined. To find d(A,G), the second array is used and $d(A,G) = 2\sqrt{2}$.

The rationale for these definitions on the model and the necessary extensions to the axiomatic system for the geometry will be discussed in a later chapter. Before that, it will be necessary to look at another model satisfying the five axioms.

Chapter 3

A SECOND MODEL OF A 25-POINT GEOMETRY

Consider the set of all ordered pairs, (x,y), of elements in the field Z_5. There are exactly twenty five such pairs. If this set is taken to be the set of points in a geometry, then axiom 1 is satisfied.

Let a line be any set of ordered pairs, (x,y), of elements in Z_5 which satisfy the equation $ax + by = c$ where a, b, and c are in Z_5 and where a and b are not both zero. Then for any given a, b, and c if $b \neq 0$, a value $y = b^{-1}(c - ax)$ is determined as x ranges over Z_5. Hence, in any case where $b \neq 0$ there are five ordered pairs satisfying the equation. If $b = 0$, then a value $x = a^{-1}c$ is determined as y ranges over Z_5. In this case, there are also five ordered pairs satisfying the equation. Thus, there are exactly five points on every line and axiom 2 is satisfied.

Let (x_1,y_1) and (x_2,y_2) be any two distinct ordered pairs. Is there a line through these two points? That is, are there elements a, b, and c in Z_5 such that $ax + by = c$ is satisfied by the coordinates of both points?

Case 1 - If there is $k \neq 0$ such that $x_1 = kx_2$ and $y_1 = ky_2 \pmod 5$ then $akx_2 + bky_2$ must equal c and $ax_2 + by_2$ must equal c. Thus, choose $c = 0$. There necessarily exist a and b in Z_5, not both equal 0, such that $ax_2 + by_2 = 0$. For these values it is clear that $akx_2 + bky_2$ will also be 0. In short, there exist a, b, and c for which both pairs (x_1,y_1) and (x_2, y_2) satisfy the equation $ax + by = c$.

Case 2 - If there is $k \neq 0$ such that $x_1 = ky_1$ and $x_2 = ky_2 \pmod 5$) again choose $c = 0$. Then $aky_1 + by_1 = 0$ and $aky_2 + by_2 = 0$ when $ak + b = 0$. In Z_5 there always exist elements a and b, not both 0, such that $ak + b = 0$ for any given $k \neq 0$. Hence, again a, b, and c exist such that both pairs (x_1,y_1) and (x_2,y_2) satisfy the equation.

Case 3 - If neither of the above cases occur, then $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0$. A system of two linear equations with non-zero determinent of coefficients always has a non-zero solution when all elements are in a field and $c \neq 0$. Thus again there exist a, b, and c satisfying the system.

In other words, given any two points (x_1,y_1) and (x_2,y_2), there exist a, b, and c such that $ax + by = c$ is satisfied by the coordinates of both points. Thus through any two distinct points there is a line.

To insure uniqueness of lines one considers all linear equations which are equivalent mod 5 as representing the same line.

Before considering the other axioms, it is enlightening to compare this model with the one of the previous chapter. The aim is to see if the two are isomorphic. To accomplish this, regard the first array of letters as the key array and associate with each point an ordered pair of numbers from the set of residue classes (mod 5).

4	A	B	C	D	E
3	F	G	H	I	J
2	K	L	M	N	O
1	P	Q	R	S	T
0	U	V	W	X	Y
	0	1	2	3	4

Hence, the coordinates of U are (0,0) and the coordinates of S are (3,1).

It is now possible to associate with each of the thirty lines an algebraic expression.

DEFINITION 10: The slope of the line through the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is found by the following formula.

$$m \equiv \frac{y_2 - y_1}{x_2 - x_1} \pmod 5$$

and the equation of the line by the formula

$$y - y_1 \equiv m(x - x_1) \pmod 5$$

For example, consider points L(1,2) and W(2,0).

$$y - 0 \equiv m(x - 2)$$

$$\text{but } m \equiv \frac{2 - 0}{1 - 2} \equiv \frac{2}{-1} \equiv -2 \equiv 3 \pmod 5$$

$$\text{so, } y - 0 \equiv 3(x - 2)$$

$$y \equiv 3x + 4 \pmod 5$$

Using the same method, the equation of the line GORUD is $y \equiv 3x \pmod 5$.

To verify that these are indeed the equations of the lines AILTW and GORUD one may substitute the points into the equations and see if they yield a true statement. For example, A(0,4) is on the line AILTW

$$4 \equiv 3(0) + 4 \pmod 5$$

At this stage, the equations of all 30 lines are produced.

$y \equiv 0 \pmod 5$	(0,0),(1,0),(2,0),(3,0),(4,0)	U V W X Y
$y \equiv 1 \pmod 5$	(0,1),(1,1),(2,1),(3,1),(4,1)	P Q R S T
$y \equiv 2 \pmod 5$	(0,2),(1,2),(2,2),(3,2),(4,2)	K L M N O
$y \equiv 3 \pmod 5$	(0,3),(1,3),(2,3),(3,3),(4,3)	F G H I J
$y \equiv 4 \pmod 5$	(0,4),(1,4),(2,4),(3,4),(4,4)	A B C D E
$y \equiv x \pmod 5$	(0,0),(1,1),(2,2),(3,3),(4,4)	U Q M I E
$y \equiv x + 1 \pmod 5$	(0,1),(1,2),(2,3),(3,4),(4,0)	P L M D Y
$y \equiv x + 2 \pmod 5$	(0,2),(1,3),(2,4),(3,0),(4,1)	K G C X T
$y \equiv x + 3 \pmod 5$	(0,3),(1,4),(2,0),(3,1),(4,2)	F B W S O
$y \equiv x + 4 \pmod 5$	(0,4),(1,0),(2,1),(3,2),(4,3)	A V R N J
$y \equiv 2x \pmod 5$	(0,0),(1,2),(2,4),(3,1),(4,3)	U L C S J
$y \equiv 2x + 1 \pmod 5$	(0,1),(1,3),(2,0),(3,2),(4,4)	P G W N E
$y \equiv 2x + 2 \pmod 5$	(0,2),(1,4),(2,1),(3,3),(4,0)	K B R I Y
$y \equiv 2x + 3 \pmod 5$	(0,3),(1,0),(2,2),(3,4),(4,1)	F V M D T
$y \equiv 2x + 4 \pmod 5$	(0,4),(1,1),(2,3),(3,0),(4,2)	A Q H X O
$y \equiv 3x \pmod 5$	(0,0),(1,3),(2,1),(3,4),(4,2)	U G R D O
$y \equiv 3x + 1 \pmod 5$	(0,1),(1,4),(2,2),(3,0),(4,3)	P B M X J
$y \equiv 3x + 2 \pmod 5$	(0,2),(1,0),(2,3),(3,1),(4,4)	K V H S E
$y \equiv 3x + 3 \pmod 5$	(0,3),(1,1),(2,4),(3,2),(4,0)	F Q C N Y
$y \equiv 3x + 4 \pmod 5$	(0,4),(1,2),(2,0),(3,3),(4,1)	A L W I T
$y \equiv 4x \pmod 5$	(0,0),(1,4),(2,3),(3,2),(4,1)	U B H N T
$y \equiv 4x + 1 \pmod 5$	(0,1),(1,0),(2,4),(3,3),(4,2)	P V C I O
$y \equiv 4x + 2 \pmod 5$	(0,2),(1,1),(2,0),(3,4),(4,3)	K Q W D J
$y \equiv 4x + 3 \pmod 5$	(0,3),(1,2),(2,1),(3,0),(4,4)	F L R X E
$y \equiv 4x + 4 \pmod 5$	(0,4),(1,3),(2,2),(3,1),(4,0)	A G M S Y

$x \equiv 0 \pmod 5$	(0,1),(0,1),(0,2),(0,3),(0,4)	U	P	K	F	A
$x \equiv 1 \pmod 5$	(1,0),(1,1),(1,2),(1,3),(1,4)	V	Q	L	G	B
$x \equiv 2 \pmod 5$	(2,0),(2,1),(2,2),(2,3),(2,4)	W	R	M	H	C
$x \equiv 3 \pmod 5$	(3,0),(3,1),(3,2),(3,3),(3,4)	X	S	N	I	D
$x \equiv 4 \pmod 5$	(4,0),(4,1),(4,2),(4,3),(4,4)	Y	T	O	J	E

Going back to the original definitions, recall that AILTW and GORUD are parallel lines. An examination of the equations of these two lines and of other parallel lines leads to the following interpretations.

Two lines are parallel if they have the same slope.

Now, if attention is turned to the relationship between perpendicular lines and to comparing the slopes of perpendicular lines, one is led to the following conclusion.

Two lines are perpendicular if the slope of exactly one is twice the slope of the other (mod 5) or if one line has no slope while the other is of slope 0.

For example, the lines AILTW and IVOCP were originally defined to be perpendicular. Notice that the slope of AILTW is 3 while the slope of IVOCP is 4 and $2(4) \equiv 3 \pmod 5$. Also notice that in array one the slopes of the row-lines are 0 while the slopes of the column-lines are undefined.

Now to examine some of the familiar statements about parallel and perpendicular lines. Many were considered in chapter two but the approach to the proofs is changed in this chapter.

OBSERVATION 5: Two distinct lines parallel to the same line are parallel to each other.

Proof: Suppose l, m, and n are distinct lines such that $l \parallel n$ and $m \parallel n$. The equations of the lines are $l: y \equiv a_1x + b_1$, $m: y \equiv a_2x + b_2$, $n: y \equiv a_3x + b_3$. Since $l \parallel n$, $a_1 \equiv a_3$ and $b_1 \not\equiv b_3$. and since $m \parallel n$, $a_2 \equiv a_3$ and $b_2 \not\equiv b_3$. Hence, $a_1 \equiv a_2$. It must now be shown that $b_1 \not\equiv b_2$. Suppose that $b_1 \equiv b_2$. Then the equation of l is $y \equiv a_1x + b_1$ and the equation of m is $y \equiv a_1x + b_1$. Hence, l and m are not distinct. This contradicts the given and $b_1 \not\equiv b_2$. Therefore, l and m are parallel.

OBSERVATION 6: Two distinct lines perpendicular to the same line are parallel to each other.

Proof: Suppose l, m, and n are distinct lines such that $l \perp n$ and $m \perp n$. The equations of the lines are $l: y \equiv a_1x + b_1$, $m: y \equiv a_2x + b_2$, $n: y \equiv a_3x + b_3$. Now since $l \perp n$ $a_1 \equiv 2a_3$ and since $m \perp n$, $a_2 \equiv 2a_3$. Hence, $a_1 \equiv a_2$ and $l \parallel m$.

OBSERVATION 7: A line perpendicular to one of two parallel lines is perpendicular to the other.

Proof: Suppose l, m, and n are distinct lines such that $l \perp m$ and $m \parallel n$. The equations of the lines are $l: y \equiv a_1x + b_1$, $m: y \equiv a_2x + b_2$, $n: y \equiv a_3x + b_3$. Now, since $l \perp m$, $a_1 \equiv 2a_2$ and since $m \parallel n$, $a_2 \equiv a_3$. Hence, $a_1 \equiv 2a_3$ and therefore, $l \perp n$.

OBSERVATION 8: Through a point not on a line there exists a line parallel to the given line.

Proof: Given $P(x_1,y_1)$ and line l: $y \equiv ax + b$. Since Z_5 is closed for addition, there exists b' such that $b' \neq b$. Hence, $y \equiv ax + b'$ is an equation of the required line.

OBSERVATION 9: Through a point not on a line there exists a line perpendicular to the given line.

Proof: Given point $P(x_1,y_1)$ and line l: $y \equiv ax + b$. Since Z_5 is closed for multiplication, there exists an a' such that $a' = 2a$. Hence, $y - y_1 \equiv 2a(x - x_1)$ is an equation of the required line.

OBSERVATION 10: Through a point not on a line there exists exactly one line parallel to the given line.

Proof: Observation 8 justifies the existence of a line. It is necessary to show that there is only one line. An indirect proof will be used to show this. Suppose there exist two distinct lines l and m through (x,y) such that $l \parallel n$ and $m \parallel n$. The equations of the lines are l: $y \equiv a_1x + b_1$, m: $y \equiv a_2x + b_2$, n: $y \equiv a_3x + b_3$. Since $l \parallel n$, $a_1 \equiv a_3$ and since $m \parallel n$, $a_2 \equiv a_3$. Therefore, $a_2 \equiv a_1$. Hence, the equations of l and m are l: $y_1 \equiv a_1x_1 + b_1$ and m: $y_1 \equiv a_1x_1 + b_2$ when the point (x_1,y_1) is substituted in the equations. Therefore, $b_1 \equiv b_2$ and l and m are not distinct. This contradicts the given. Our assumption is false and there is not more than one line through (x,y) parallel to n.

Now from observation 3 and this proof, through a point not on a line there exists exactly one line parallel to the given line.

OBSERVATION 11: Through a point on a line there exists exactly one line perpendicular to the given line.

Proof: The proof of this observation is similar to the proof of observation 10 and hence the author will not reproduce it.

This second model of a geometry of 25 points has been shown to satisfy the five axioms satisfied by the first model. In both cases the undefined terms were point, line, on, parallel, and perpendicular. The axioms are the same five as has been mentioned before.

The fact that there is an isomorphism between these two models does not suggest that the set of axioms is categorical. One objective of this paper is to study the two models themselves in order to find out what other axioms are needed to come up with a set which may be categorical. At this stage, a major concept - that of length - has been established for the first model but not included in the axiomatic system. Before including it, it may worthwhile to interpret the concept in the second model.

Remember that distance between two points of an array could only be measured horizontally or vertically by following a row or column. This definition results in only two non-zero units of length, 1 or 2, for row distance and two non-zero units of length, $\sqrt{2}$ or $2\sqrt{2}$, for column distance. Thus, the following distances from the first model.

$$d_r(A,B) = d_r(A,E) = 1$$

$$d_r(A,C) = d_r(A,D) = 2$$

$$d_c(A,F) = d_c A,U) = \sqrt{2}$$

$$d_c(A,K) = d_c(A,P) = 2\sqrt{2}$$

To maintain this idea, define distance between $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ by the following formula.

$$d \equiv \sqrt{(x_1 - x_2)^2 + 2(y_1 - y_2)^2} \pmod 5$$

An examination of this formula shows that it always yields positive answers which is desirable for distance and it distinguishes between row and column distance. For example,

$$\begin{aligned} d(A,B) &\equiv \sqrt{(0 - 1)^2 + 2(4 - 4)^2} \pmod 5 \\ &\equiv \sqrt{(1)^2 + 2(0)^2} \pmod 5 \\ &\equiv \sqrt{1} \pmod 5 \\ &\equiv 1 \pmod 5 \end{aligned}$$

and

$$\begin{aligned} d(A,F) &\equiv \sqrt{(0 - 0)^2 + 2(4 - 3)^2} \pmod 5 \\ &\equiv \sqrt{(0)^2 + 2(1)^2} \pmod 5 \\ &\equiv \sqrt{2} \pmod 5 \end{aligned}$$

Likewise, $d(A,C) \equiv 2 \pmod 5$ and $d(A,K) \equiv 2\sqrt{2} \pmod 5$. This exhibits the fact that the non-zero row lengths remain 1 and 2 while the non-zero column lengths remain $\sqrt{2}$ and $2\sqrt{2}$.

The content of this chapter may be summarized as follows: Coordinates have been assigned to points by interpreting a point to be an ordered pair of numbers in the field Z_5. The number of distinct points is thus the number of distinct ordered pairs (x,y) over Z_5 which is clearly 25. A line is then represented as a set of five points related by a single linear equation in two unknowns (mod 5). All the other axioms of the original non-coordinate geometry were then imposed and a few of the observations about the new model were discussed.

It is worth particular notice that in both models and in the axiomatic geometry, through a point not on a line there exists exactly one line parallel to the given line. Thus, the question arises as to what other important theorems from Euclidean geometry are true in the two models. This study could obviously be an unending process; however, one of these involves a concept already introduced to both models but not yet axiomatized. This important concept will be the focal point of chapter four.

Chapter 4

THE PYTHAGOREAN THEOREM

In chapter three, as was discussed in its conclusion, the discovery that through a point not on a line there is a unique parallel to the given line gives rise to the question as to what other theorems of Euclidean geometry are also true in the two 25-point models. This chapter will be devoted to a discussion revolving around the pythagorean theorem and whether or not it is valid in the 25-point models.

Theorem 11 from chapter two limits the number of right triangles in this geometry and in the models to 1200. Now since the three original arrays are symmetric, there will be 400 right triangles in each array. Therefore, the discussion of right triangles will center around a discussion of the 400 right triangles in the first array. And since there are only 400 of them to consider, the author will adopt a method of exhausting cases rather than discussing the general right triangle.

For reference, the first array will be reproduced at this time.

A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

From the definition of right triangle, and from the definition of perpendicular lines, one of the legs of a right triangle must be a row-segment while the other leg must be a column-segment. Combining this idea with the fact that row distances are 1 and 2 and column distances are $\sqrt{2}$ and $2\sqrt{2}$, it is obvious that there are only four possible pairs of lengths for the legs of a right triangle. These pairs are 1 and $\sqrt{2}$, 1 and $2\sqrt{2}$, 2 and $\sqrt{2}$ and 2 and $2\sqrt{2}$. Now, each of these pairs may be combined with any of the four possible lengths as the length of the hypotenuse and so there are only 16 possible distinct right triangles in our system. To classify the 400 right triangles of array one into these possibilities, the author examined all four hundred of them using the following method.

First, examine all possible right triangles with legs of length 1 and $\sqrt{2}$. These fall into a number of definite patterns. One group consists of all right triangles formed as follows: choose any arbitrary point, say D. From that point proceed one unit to the right to point E and then down one unit to point J. These three points form right triangle DEJ. Now, there are 25 points from which to begin and hence, there are 25 right triangles formed by this method. Examination of all of these right triangles shows that the hypotenuse is found in the second array and is found to always be of length $2\sqrt{2}$.

A second group of right triangles is found by proceeding from the beginning point, say D again, one unit to the left to point C and then down one unit to point H. Therefore, triangle DCH is formed. Again there will be 25 triangles formed by this method. In each of these cases, the hypotenuse is found in the third array and is found to be of length $2\sqrt{2}$.

The third group is found by proceeding from the beginning point, D again, down one unit to the point I and then one unit to the right to point J and hence triangle DIJ. Once more there are 25 of these right triangles and in each of these cases when searching for the hypotenuse, a different situation results. The hypotenuse of triangle DIJ is the line segment DJ which is found in the second array. But DJ is also the hypotenuse for right triangle DEJ which was already considered in group one. In fact, each hypotenuse of the right triangles in group three is found to be one of the twenty five possibilities already considered in group one. Hence, the length of each hypotenuse is $2\sqrt{2}$.

Finally, in group four, proceed from the beginning point D down one unit to point I and then left one unit to point H. Right triangle DIH is one of the 25 possibilities in this group. Each hypotenuse of the right triangles in this group is one of the 25 possibilities already considered in group two and again the hypotenuse is of length $2\sqrt{2}$.

Now, from these four cases a statement can be made about right triangles with legs of length 1 and $\sqrt{2}$. This statement could be called a theorem or observation since the author has proved it by exhausting all possible cases but since a formal proof of the statement has not been discovered, it shall be presented as a conjecture.

CONJECTURE 1: Any right triangle with legs of length 1 and $\sqrt{2}$ has a hypotenuse of length $2\sqrt{2}$.

Secondly, examine all right triangles with legs of length 1 and $2\sqrt{2}$. Using the method just outlined, each of the four groups yields right triangles such as triangle ABL, triangle BAK, triangle AKL or triangle BLK. The hypotenuse of right triangle ABL is found in array two as is the hypotenuse of each of the twenty five right triangles in this group and it is found to be of length 2. The hypotenuse of right triangle BAK is found in the third array and is of length 2 as is all of this group. The hypotenuse of right triangle AKL and each hypotenuse in group three is one of the possibilities from group one as they were in the first case. The hypotenuse of right triangle BLK and each hypotenuse in group four is one of the possibilities from group two. Having checked all of these possibilities led to the second conjecture.

CONJECTURE 2: Any right triangle with legs of length 1 and $2\sqrt{2}$ has a hypotenuse of length 2.

Thirdly, examine all right triangles with legs of length 2 and $\sqrt{2}$. Using the same method as before, the right triangles of this case will be such as triangle ACH, triangle CAF, triangle AFH, or triangle CHF. When examining each hypotenuse, the same pattern exists as existed in the first two cases. These one hundred triangles yield the following conjecture.

CONJECTURE 3: Any right triangle with legs of length 2 and $\sqrt{2}$ has a hypotenuse of length 1.

Finally, examine all right triangles with legs of length 2 and $2\sqrt{2}$. These are right triangles such as triangle ACM, triangle CAK, triangle AKM, or triangle CMK. The pattern holds true for these one hundred right triangles and leads to the fourth conjecture.

CONJECTURE 4: Any right triangle with legs of length 2 and $2\sqrt{2}$ has a hypotenuse of length $\sqrt{2}$.

Now, the process used considered one hundred right triangles in each group examined which obviously yields a total of four hundred right triangles. By previous discussion, this exhausts all the possibilities for the first array. The process could be used to examine the right triangles in arrays two and three but this is equivilant to examining the first array a second and a third time. Three examinations of the same array is obviously not necessary. Therefore, organizing the four conjectures into one statement yields the following conclusion.

CONJECTURE 5: Any right triangle of the 25-point model has sides whose lengths are one of the four following possibilities: legs of length 1 and $\sqrt{2}$ and hypotenuse of length $2\sqrt{2}$, legs of length 1 and $2\sqrt{2}$ and hypotenuse of length 2, legs of length 2 and $\sqrt{2}$ and hypotenuse of length 1, or legs of length 2 and $2\sqrt{2}$ and hypotenuse of length $\sqrt{2}$.

Therefore, a discussion of right triangles and specifically a discussion of the Pythagorean theorem involves consideration of only four possibilities.

OBSERVATION 12: In any right triangles, the sum of the squares of the lengths of the legs (mod 5) is congruent to the square of the length of the hypotenuse(mod 5).

Proof: Case 1 - Consider the right triangles with legs of length 1 and $\sqrt{2}$ and hypotenuse of length $2\sqrt{2}$.

$1^2 \equiv 1 \pmod 5$, $(\sqrt{2})^2 \equiv 2 \pmod 5$, $(2\sqrt{2})^2 \equiv 3 \pmod 5$ and therefore, $1^2 + (\sqrt{2})^2 \equiv (2\sqrt{2})^2 \pmod 5$. Hence, the theorem is true in this case.

Case 2 - Consider the right triangles with legs of length 1 and $2\sqrt{2}$ and hypotenuse of length 2. $1^2 \equiv 1 \pmod 5$, $(2\sqrt{2})^2 \equiv 3 \pmod 5$, $2^2 \equiv 4 \pmod 5$ and therefore, $1^2 + (2\sqrt{2})^2 \equiv 2^2 \pmod 5$. Hence, the theorem is true in this case.

Case 3 - Consider the right triangles with legs of length 2 and $2\sqrt{2}$ and hypotenuse of length $\sqrt{2}$. $2^2 \equiv 4 \pmod 5$, $(2\sqrt{2})^2 \equiv 3 \pmod 5$, $(\sqrt{2})^2 \equiv 2 \pmod 5$ $4 + 3 \equiv 2 \pmod 5$ and therefore, $2^2 + (2\sqrt{2})^2 \equiv (\sqrt{2})^2 \pmod 5$. Hence, the theorem is true in this case.

Case 4 - Consider the right triangles with legs of length 2 and $\sqrt{2}$ and hypotenuse of length 1. $2^2 \equiv 4 \pmod 5$, $(\sqrt{2})^2 \equiv 2 \pmod 5$, $1^2 \equiv 1 \pmod 5$, $2 + 4 \equiv 1 \pmod 5$ and therefore, $2^2 + (\sqrt{2})^2 \equiv 1^2 \pmod 5$. Hence, the theorem is true in this case.

Now, from the four cases, in any right triangle, the sum of the squares of the lengths of the legs is congruent to the square of the length of the hypotenuse (mod 5).

In conclusion, chapter four has exhibited the fact that the study of right triangles of which there are 1200, may be reduced to the study of only four specific cases.

Chapter 5

ISOMETRIES OF THE 25-POINT PLANE

The preceding chapters consist of conclusions arrived at by the author through his own original methods, using only the axioms and definitions presented. In this chapter, the author makes use of a suggestion made by H. Martyn Cundy in a paper entitled "25-point Geometry".[2] The suggestion involves the use of group theory.

In his article, Mr. Cundy noted how the three original arrays of the 25-point geometry were related to each other. If p is the operator transforming the first array into the second array then p applied to the second array yields the third array. For example, C is the third letter in the first row of array one and it is removed to the second position of row four by the operator p. Now, if the second array is considered, L is the point that is in the third position of the first row and it is also removed to the second position of row four by the operator p. Graphically, the result of the operation p on all 25 points accomplishes the following.

$$\begin{matrix} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{matrix} \quad \xrightarrow{p} \quad \begin{matrix} A & I & L & T & W \\ S & V & E & H & K \\ G & O & R & U & D \\ Y & C & F & N & Q \\ M & P & X & B & J \end{matrix} \quad \xrightarrow{p} \quad \begin{matrix} A & H & O & Q & X \\ N & P & W & E & G \\ V & D & F & M & T \\ J & L & S & U & C \\ R & Y & B & I & K \end{matrix}$$

The question immediately arises as to what happens if the operator p is applied to the third array. The discussion of this will be left until later in the chapter, after introducing additional definitions

[2] Cundy, "25-Point Geometry", <u>Mathematical Gazette</u>, XXXVI (September 1952), 158-166.

DEFINITION 11: "1" is the operator which leaves any array unchanged.

DEFINITION 12: "i" is the operator which reverses the cyclic order in the rows of the arrays.

DEFINITION 13: "-1" is the operator which reverses the cyclic order in both the rows and columns of the arrays.

Now, $\{-1, i, p\}$ generate a group of order twelve. The elements of the group and their relationships are exhibited in the following table. Examination of the table will reveal that the properties of a group - closure, inverse, identity, and associativity - are satisfied by this system.

	-1	1	i	$-i$	p	$-p$	pi	$-pi$	p^2	ip	$-ip$	$-p^2$
-1	1	-1	$-i$	i	$-p$	p	$-pi$	pi	$-p^2$	$-ip$	ip	p^2
1	-1	1	i	$-i$	p	$-p$	pi	$-pi$	p^2	ip	$-ip$	$-p^2$
i	$-i$	i	1	-1	ip	$-ip$	$-p^2$	p^2	$-pi$	p	$-p$	pi
$-i$	i	$-i$	-1	1	$-ip$	ip	p^2	$-p^2$	pi	$-p$	p	$-pi$
p	$-p$	p	pi	$-pi$	p^2	$-p^2$	$-ip$	ip	-1	i	$-i$	1
$-p$	p	$-p$	$-pi$	pi	$-p^2$	p^2	ip	$-ip$	1	$-i$	i	-1
pi	$-pi$	pi	p	$-p$	i	$-i$	1	-1	ip	p^2	$-p^2$	$-ip$
$-pi$	pi	$-pi$	$-p$	p	$-i$	i	-1	1	$-ip$	$-p^2$	p^2	ip
p^2	$-p^2$	p^2	$-ip$	ip	-1	1	$-i$	i	$-p$	pi	$-pi$	p
ip	$-ip$	ip	$-p^2$	p^2	$-pi$	pi	$-p$	p	$-i$	1	-1	i
$-ip$	ip	$-ip$	p^2	$-p^2$	pi	$-pi$	p	$-p$	i	-1	1	$-i$
$-p^2$	p^2	$-p^2$	ip	$-ip$	1	-1	i	$-i$	p	$-pi$	pi	$-p$

The twelve operations on the first array will yield the following.

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{1} \begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{i} \begin{array}{ccccc} A & E & D & C & B \\ F & J & I & H & G \\ K & O & N & M & L \\ P & T & S & R & Q \\ U & Y & X & W & V \end{array}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{-i} \begin{array}{ccccc} A & B & C & D & E \\ U & V & W & X & Y \\ P & Q & R & S & T \\ K & L & M & N & O \\ F & G & H & I & J \end{array}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{p} \begin{array}{ccccc} A & I & L & T & W \\ S & V & E & H & K \\ G & O & R & U & D \\ Y & C & F & N & Q \\ M & P & X & B & J \end{array}$$

$$\begin{matrix} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{matrix} \xrightarrow{-p} \begin{matrix} A & W & T & L & I \\ M & J & B & X & P \\ Y & Q & N & F & C \\ G & D & U & R & O \\ S & K & H & E & V \end{matrix}$$

$$\begin{matrix} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{matrix} \xrightarrow{-1} \begin{matrix} A & E & D & C & B \\ U & Y & X & W & V \\ P & T & S & R & Q \\ K & O & N & M & L \\ F & J & I & H & G \end{matrix}$$

$$\begin{matrix} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{matrix} \xrightarrow{pi} \begin{matrix} A & W & T & L & I \\ S & K & H & E & V \\ G & D & U & R & O \\ Y & Q & N & F & C \\ M & J & B & X & P \end{matrix}$$

$$\begin{matrix} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{matrix} \xrightarrow{-pi} \begin{matrix} A & I & L & T & W \\ M & P & X & B & J \\ Y & C & F & N & Q \\ G & O & R & U & D \\ S & V & E & H & K \end{matrix}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{ip} \begin{array}{ccccc} A & H & O & Q & X \\ R & Y & B & I & K \\ J & L & S & U & C \\ V & D & F & M & T \\ N & P & W & E & G \end{array}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{-ip} \begin{array}{ccccc} A & X & Q & O & H \\ N & G & E & W & P \\ V & T & M & F & D \\ J & C & U & S & L \\ R & K & I & B & Y \end{array}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{p^2} \begin{array}{ccccc} A & H & O & Q & X \\ N & P & W & E & G \\ V & D & F & M & T \\ J & L & S & U & C \\ R & Y & B & I & K \end{array}$$

$$\begin{array}{ccccc} A & B & C & D & E \\ F & G & H & I & J \\ K & L & M & N & O \\ P & Q & R & S & T \\ U & V & W & X & Y \end{array} \xrightarrow{-p^2} \begin{array}{ccccc} A & X & Q & O & H \\ R & K & I & B & Y \\ J & C & U & S & L \\ V & T & M & F & D \\ N & G & E & W & P \end{array}$$

Returning to the original definitions of the operators, since distance is defined in the three original arrays, distance is obviously preserved under the operations "p" and "p^2".

The operator "i" reverses the cyclic order in the rows but leaves the order in the columns unchanged. Now, reversing the cyclic order simply interchanges the points in positions 2 and 5 and in positions 3 and 4. With the definition of distance used in chapter two, "i" also preserves distance. By the same argument, the operator "-1" which changes the cyclic order in both the rows and columns must also preserve distance.

Finally along this line, the operator "-i" simply reverses the cyclic order within the columns and leaves the order in the rows unchanged and hence, it will also preserve distance.

Now, "-p", "pi", "ip", "-pi", "-ip", and "$-p^2$" will also preserve distance since they use either the second or third array and do one of the "order exchanging" operations just discussed. Hence, the entire group preserve distance.

Returning to chapter four and the study of right triangles can be simplified now because of these new operators. These isometries of the 25-point plane lead us to the following facts. There are now twelve arrays instead of three. Therefore, since there are 1200 right triangles, each of the twelve arrays must represent one hundred triangles. It is still true that every triangle has base-height length of $1 - \sqrt{2}$, $1 - 2\sqrt{2}$, $2 - \sqrt{2}$, or $2 - 2\sqrt{2}$. At each of the 25 points in any array consider a right triangle of each of the four types. Hence, instead of checking 400 right triangles as was necessary in the previous method, it is now necessary to check only one hundred right triangles and in essence all 1200 will then have been considered.

Hence, it has been shown in this chapter that when considering the group generated by $\{-1, i, p\}$ the examination of right triangles can be reduced to an examination of only one hundred cases in any one of the arrays generated by the group and therefore it is not impossible to use the method of exhausting cases when considering theorems about right triangles.

Chapter 6

CONCLUSION

As is quite often the case when dealing with a somewhat unexplored concept, this study of the 25-point geometry has raised nearly as many questions as it has presented answers. For example, what would the geometry become if two parallel lines were simply lines which have no points in common and were not restricted to being in the same array in the model? Some of the questions raised in the previous chapters deserve some additional comment, and that will be the intent of this chapter.

One point which possibly needs clarification deals with perpendicular lines in the coordinate chapter. For the purpose of discussion, the definition of perpendicular is repeated here.

> Two lines are perpendicular if the slope of exactly one is twice the slope of the second.

Notice that contrary to Euclidean geometry, with this definition and the slopes of the lines, A perpendicular to B does not imply that B is perpendicular to A. Hence, we have a deviation from Euclidean geometry.

A second concept from Euclidean geometry that is not used in the 25-point geometry is the concept of betweeness. The lack of this concept in the 25-point geometry makes the definition of line segment divorced from the same concept in Euclidean geometry. The definition of betweeness from Euclidean geometry is:

> Point B is between points A and C if all are distinct and $\overline{AB} + \overline{BC} = \overline{AC}$.

If this definition is used in the 25-point geometry, two problems immediately arise. First, notice that point E would be between points A and D since AE = 1, ED = 1, AD = 2, and hence, AE + ED = AD. Secondly, since AB = 1, BD = 2, and AD = 2 and hence AB + BD ≠ AD, point B does not lie between A and D even though it "appears" to.

Another problem also arises when considering betweeness. Hilbert's second axiom of order states that given any two points on a line there always exists a point between them. Obviously, this axiom would not be true if the two given points were A and B. The lack of satisfying this axiom if betweeness were defined in the 25-point geometry would probably not be a major factor but the contradictions raised in the previous paragraph and a lack of discovery of a suitable substitute led the author to omission of the concept of betweeness in the 25-point geometry.

Closely related to the just completed discussion is the concept of line segment. When making the definitions, the definition of line segment is one of the most difficult. Remember that in the definition of line segment, the lines are not considered to be closed. Hence, EAD is not a line segment. One of the reasons for this is that the author wished to preserve the fact that BE and EB would represent the same segment. If the lines were considered closed then would EB represent the segment consisting of points E, A, and B or the segment consisting of points E, D, C, and B? For the author's geometry, EB should represent the segment consisting of the points E, D, C, and B and then EB and BE will represent the same segment. The reader might be interested in examining what course the geometry would take if the lines were considered closed. Obviously, the discussion of right triangles in this paper would drastically be altered by the change in the definition of line segment.

Another concept that was deleted by the author is the concept of angle in the general sense. Right angle can be discussed since perpendicular lines are defined but no suitable definition for angle was found. It is possible that some type of definition related to triangles with the same lengths of sides could be introduced but this does not seem to offer any particular insight into the study of the 25-point geometry and hence, it was omitted. This omission then eliminated two concepts equivilent to the fifth postulate of Euclid. They are that the sum of the angles of a triangle is equal to two right angles and the existence of similar non-congruent triangles, that is of non-congruent triangles which have all three angles of one congruent to all three angles of the other.

The paper also suggests that possibly there are concepts from Euclidean geometry that are also true in the 25-point geometry but have not been examined in this paper. For example, is it possible that an in-depth study of group theory would simplify the system even more than the group of order 12 which was presented? What properties of triangles in general are true in this system? Do the conic sections exist in the 25-point geometry and if so, what properties of them are true? These are just a few of the many concepts that could be examined and the reader has possibly found others for which he has a special interest.

Finally, some concluding remarks about the Pythagorean theorem are in order here. The study of the models has indicated that the sum of the squares of the two legs of a right triangle is equal to the square on the hypotenuse. The question is to determine whether or not another axiom for the geometry of 25 points should be this fact itself or whether it should be some other more basic statement from which this fact could be logically derived.

Consider the six sets of axioms used by Hilbert for Euclidean geometry. These are the axioms of connection, order, continuity, congruence, parallels, and completeness. It has just been noted why the concept of order has not been desirable for the 25-point geometry suggested by the models studied. For similar reasons, the notion of continuity does not fit this geometry. Regarding the axioms of connection, axiom 2 and 3 for the 25-point geometry are precisely that type of axiom, and the 4th axiom listed for the 25-point geometry is a "parallels" axiom. The axiom on perpendiculars listed for the 25-point geometry does not correspond to any of Hilbert's axioms. It was needed in our geometry but not in Euclid's. The completeness axiom by Hilbert states that it is not possible to add (to the system of points of a line) points such that the extended system shall form a new geometry for which all the other axioms hold. Clearly, the axiom 1 for the 25-point geometry regarding the existence of exactly 25 points and the axiom fixing the number of points on a line make the addition of new points impossible. There remains therefore, only the axioms of congruence.

The author suggests that it is possible that Hilbert's axioms of congruence could be replaced in the 25-point geometry by axioms of length. This results from the fact that the author has found no suitable way to consider congruence for anything other than segments.

Now, Hilbert's axioms of congruence of segments are as follows:

a) If A,B are points on a line L and if A' is on a line L', there is exactly one point B' on a given side of A' on L' such that $\overline{AB}$ is congruent to $\overline{A'B'}$.

b) Every segment is congruent to itself.

c) If $\overline{AB}$ is congruent to segment $\overline{A'B'}$ and if $\overline{AB}$ is congruent to segment $\overline{A''B''}$, then $\overline{A'B'}$ is congruent to $\overline{A''B''}$.

d) If AB and BC are segments on L with only B in common, and if A'B' and B'C' are segments on L' with only B' in common, and if $\overline{AB}$ is congruent to $\overline{A'B'}$ and $\overline{BC}$ is congruent to $\overline{B'C'}$ then segment $\overline{AC}$ is congruent to segment $\overline{A'C'}$.

Now, it is immediately evident that these four axioms cannot be used in the 25-point geometry. With the definition of distance used in this geometry, Hilbert's axiom which has been labeled a) is not true. For example, if $\overline{AB}$ is of length 1, there are two segments with endpoint A' of length 1. Clearly more investigation along this line is needed but this investigation has not been completed in time for inclusion in this paper. It is hoped, however, that enough has been included to make the effort worthwhile and to stimulate the reader to also examine the 25-point geometry in more detail.

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