A STUDY OF THE QUARTIC CURVES WHICH ARE THE INVERSES OF THE CONICS WITH RESPECT TO A CIRCLE

A THESIS

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By

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CHAPTER I

INTRODUCTION

1. Purpose of the Study. The purpose of this study is to determine the forms of those quartic curves which are the inverses of conics with respect to a circle.

2. Definition of Inversion. If two points P and P' lie on a straight line through the center of a given circle and such that $OP \cdot OP' = r^2$, where r is the radius of the circle, then P' is the inverse of P. By virtue of this definition each of the points P, P' is the inverse of the other. The circle is known as the circle of inversion and its center as the center of inversion.

By this definition each point in the plane except the center of the circle of inversion has a unique inverse. With each point exterior to the circle is associated an interior point. Each point on the circle of inversion is its own inverse, and every self-inverse point is on this circle.

As a matter of convenience we extend the definition so that the inverse of the center of inversion is taken to be the line at infinity, and the inverse of any point at infinity is defined to be the center of inversion.

3. Fundamental Properties. To any figure in the plane corresponds a second figure such that corresponding points of the two figures are mutually inverse. Every circle or straight line is transformed by an inversion into a circle or straight line. A few elementary but important

properties associated with inversion are the following.[1]

1. A straight line passing through the center of inversion is inverse to itself.

2. The inverse of any straight line not passing through the center of inversion is a circle through that center, and conversely.

3. The inverse of a circle not passing through the center of inversion is a circle.

4. The angles between two curves or arcs equals the angles oppositely directed at the inverse point between the corresponding inverse lines or arcs.

It will sometimes be convenient to include inversion with respect to a circle with center at infinity. Such an inversion is essentially a reflection with respect to a line.

Variation of the radius of the circle of inversion has no effect on the form of an inverse curve. In a study of this character, where the form and not the size, of the inverse curve is of interest, it is generally necessary to indicate only the center of inversion. The radius will be varied to suit our convenience.

The inverse of a geometric figure may be constructed mechanically by means of any one of several linkwork devices. The best known of these and one of the simplest, is the "Peaucellier Cell," invented by A. Peaucellier in 1874.[2]

4. History and Development. F. Butzberger has recently pointed

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[1] Nathan Altshiller-Court, College Geometry (Richmond: Johnson Publishing Company, 1925), pp. 201-210; see also Roger A. Johnson, Modern Geometry (Chicago: Houghton Mifflin Company, 1929), pp. 42-67.

[2] Frank Morley and F. V. Morley, Inversive Geometry (Chicago: Ginn and Company, 1933), p. 42; see also Thomas F. Holgate, Projective Pure Geometry (New York: The Macmillan Company, 1930), p. 264.

out that in an unpublished manuscript Steiner disclosed a knowledge of the principle of inversion as early as 1824. In 1847 Liouville called it the "transformation by reciprocal radii." After Steiner this transformation was found independently by J. Bellavitis in 1836, J. W. Stubbs and J. R. Ingram in 1842 and 1843, and by William Thompson (Lord Kelvin) in 1845.[3]

The inversion of conics is given considerable attention by Smith and Gale in their, Elements of Analytic Geometry.[4] Several of the cubic and quartic curves met in elementary applications are actually the inverses of certain conic sections. The most outstanding of these, which are derived by Smith and Gale and which will be among those derived in Chapter Six of this study, are the strophoid, the lemniscate of Bernoulli, the cardioid, the cissoid and the family of limacons of Pascal.

The derivation of the properties of these special curves through the inversion of conics has been studied in considerable detail by Querry.[5]

5. Statement of Problem. In the present study we shall make a survey of the forms of those curves which are the inverses of the conic sections with respect to a circle. The procedure will be to move the center of inversion along a given line generally an axis, and observe the corresponding variation in the forms of the inverse quartic.

In the available sources, except for the special curves mentioned in paragraph 4, no such study appears to have been made up to the present.

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[3] Florian Cajori, A History of Mathematics (New York: The Macmillan Company, 1922), p. 292.

[4] Percy F. Smith and Arthur Sullivan Gale, The Elements of Analytic Geometry (Chicago: Ginn and Company, 1904), pp. 297-309.

[5] John W. Querry, Generalization by Inversion (unpublished Master's thesis, University of Iowa, Iowa City, Iowa, 1931), p. 108.

CHAPTER II

ANALYTIC SOLUTION OF PROBLEM

6. Inversion as a Quartic Transformation. While inversion of straight lines and circles may be easily handled geometrically, it is generally necessary to treat the inversion of other figures, such as conic sections, by algebraic means. It may be readily established that if the cartesian coordinates of a point, P, are (x, y) and those of its inverse, P', are (x', y'), these coordinates are connected by the relations[6]

(1) $$x' = r^2x/(x^2+y^2), \quad y' = r^2y/(x^2+y^2);$$

or

(1.1) $$x = r^2x'/(x'^2+y'^2), \quad y = r^2y'/(x'^2+y'^2),$$

where r is the radius of the circle of inversion, the center of inversion being at the origin.

7. Equation of the Inverse of a Conic. The most general equation of the second degree in x and y is of the form,

(2) $$Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$$

This represents a conic if $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0$. This conic is an ellipse if $H^2 - AB < 0$, a parabola if $H^2 - AB = 0$, and a hyperbola if $H^2 - AB > 0$.

When the equation (2) is subjected to the quadratic transformation (1) of the preceding paragraph the equation of the associated inverse curve is

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[6] William C. Graustein, Introduction to Higher Geometry (New York: The Macmillan Company, 1933), p. 387.

$$(3) \qquad C(x^2+y^2)^2+2r^2(Gx+Fy)(x^2+y^2)+r^4(Ax^2+2Hxy+By^2)=0.$$

8. __Derivatives.__ In a discussion of the curves (3) it is desirable to have available first and second derivatives of x with respect to y (or of y with respect to x).

$\frac{dx}{dy}$ is given implicitly by the equation

$$(4) \qquad 2C(x^2+y^2)\left(x\frac{dx}{dy}+y\right)+2r^2(Gx+Fy)\left(x\frac{dx}{dy}+y\right)+r^2(x^2+y^2)\left(G\frac{dx}{dy}+F\right)$$
$$+r^4\left(Ax\frac{dx}{dy}+Hy\frac{dx}{dy}+Hx+By\right)=0.$$

or by

$$(4') \qquad \left\{2Cx(x^2+y^2)+2r^2x(Gx+Fy)+r^2G(x^2+y^2)+r^4Ax+r^4Hy\right\}\frac{dx}{dy}$$
$$+\left\{2Cy(x^2+y^2)+2r^2y(Gx+Fy)+r^2F(x^2+y^2)+r^4Hx+r^4By\right\}=0.$$

$\frac{d^2x}{dy^2}$ is given implicitly by the equation

$$(5) \qquad \left\{2Cx(x^2+y^2)+2r^2x(Gx+Fy)+r^2G(x^2+y^2)+r^4Ax+r^4Hy\right\}\frac{d^2x}{dy^2}$$
$$+\left\{2C(x^2+y^2)\frac{dx}{dy}+4Cx\left(x\frac{dx}{dy}+y\right)+2r^2(Gx+Fy)\frac{dx}{dy}\right.$$
$$\left.+2r^2x\left(G\frac{dx}{dy}+F\right)+2r^2G\left(x\frac{dx}{dy}+y\right)+r^4A\frac{dx}{dy}+r^4H\right\}\frac{dx}{dy}+\left\{2C(x^2+y^2)\right.$$
$$+4Cy\left(x\frac{dx}{dy}+y\right)+2r^2(Gx+Fy)+2r^2y\left(G\frac{dx}{dy}+F\right)$$
$$\left.+2r^2F\left(x\frac{dx}{dy}+y\right)+r^4H\frac{dx}{dy}+r^4B\right\}=0.$$

9. __The Origin as a Singular Point.__ The point (0,0) is a singular point of the quartic (3), and the tangents at the origin are given by

$$Ax^2+2Hxy+By^2=0.$$

If the conic (3) is an ellipse, the origin is an isolated point. If the conic (3) is a parabola, the origin is a cusp. If the conic (3) is a hyperbola, the origin is a node.

10. <u>Other Intersections with the Axis.</u> The remaining intersections with the axis are

(6) $$y = 0, \qquad x = \frac{r^2}{C}\left(-G \pm \sqrt{G^2 - AC}\right),$$

and

(7) $$x = 0, \qquad y = \frac{r^2}{C}\left(-F \pm \sqrt{F^2 - BC}\right).$$

It will be noticed that one of these can coincide with the origin if, and only if, A or B equals zero; that is, if the associated conic is a parabola.

11. <u>Center of Inversion on the Conic.</u> If $C = 0$, that is, if the given conic passes through the origin; the inverse curve is composite, consisting of the line at infinity and a cubic curve given by

(8) $$2(Gx+Fy)(x^2+y^2) + r^2(Ax^2+2Hxy+By^2) = 0.$$

12. <u>Concavity; Rays.</u> If the associated conic is symmetrical with respect to the x-axis, ($F = 0$, $h = 0$), then at the intersections of the inverse curve with the axis, namely,

$$N',\quad \left(\frac{r^2}{C}\left[-G + \sqrt{G^2 - AC}\right], \quad 0\right),$$

and

$$N'',\quad \left(\frac{r^2}{C}\left[-G - \sqrt{G^2 - AC}\right], \quad 0\right),$$

we have $\frac{dx}{dy} = 0$ and $\frac{d^2x}{dy^2} = K$, where K denotes the curvature. If $\frac{d^2x}{dy^2} < 0$, the curve is concave to the left; and if $\frac{d^2x}{dy^2} > 0$, the curve is concave to the right.

The curvatures at the mentioned points, M' and N', depend upon the radius of the circle of inversion as well as upon the position of the center of inversion. However, as the center of inversion moves along an axis of symmetry of a given conic, there will be found at least one position, K, of the center of inversion for which the curvature at M ' (or N ') is reversed. If the inverse curve is a true quartic, a "bay" is acquired or lost at M ' (or N ') as the center of inversion, O, passes through K along the axis.

This situation is of sufficient significance to warrant closer study, and it will be illustrated in Figure 1, page 11, as well as in several sequences in the drawings at the end of this study.

The conditions which must be satisfied at point M ' (or N ') at which the tangent is vertical and the curvature is zero are

(9) $$x = \frac{r^2}{C}\left(-G \pm \sqrt{G^2 - AC}\right),$$

$$y = 0,$$

$$\frac{dx}{dy} = 0,$$

$$\frac{d^2x}{dy^2} = 0.$$

Equation (8) then becomes after reduction

(10) $$G(4ABG + 8AG^2 - 4BG^2 - 4A^2C - B^2C) = 0.$$

The equation

(11) $$4G^2(A - B) = 2(A - B)C,$$

or

(12) $$4(G^2 - AC)A - B = B^2C,$$

expresses the condition under which the inverse quartic (that is C 0) has concavity neither to left nor right at its intersections with the x -axis.

In the case of an ellipse with center at (-h,0) and the semi-axes a and b, the points of intersection with the x -axis are

$$\left(-\frac{r^2}{a-h}, \quad 0\right) \text{and} \left(\frac{r^2}{a-h}, \quad 0\right).$$

If the ellipse passes through the origin ($h=a$ or $h=-a$), one of these points will be at infinity. In the case of such a quartic, equation (12) reduces to

(13) $$a^2h^2=(2b^2+a^2)^2,$$

or

(14) $$h^2=\left(a-\frac{2b^2}{a}\right)^2.$$

The significance of this relation is immediately apparent since $\frac{2b^2}{a}$ is the diameter of the circle osculating the conic at the vertex H. Let K denote the other extremity of this diameter. Then K is a critical position of the center of inversion, in the sense that as the center of inversion passes through K, the concavity of the quartic changes from right to left.

In the case of a hyperbola with center at $(-h,0)$, transverse semi-axis a, and conjugate semi-axis b, we find that the critical positions of the center of inversion on the transverse axis are given by

(15) $$h^2=\left(a+\frac{2b^2}{a}\right)^2,$$

the interpolation of which is the same as that for the ellipse. Then the center of inversion is on the conjugate axis, there are no intercepts on that axis except zero.

When $A=H=F=0$, equation (2) is that of a parabola with x-axis as a line of symmetry. The equation (6) gives as the third finite intersection of the associated quartic with the x-axis the point

(16) $$\left(-\frac{2Gr^2}{C},\quad 0\right).$$

The curvature of the quartic at this point is zero if equation (10) holds. Since $B\neq 0$ this relation reduces to

(17) $$4G^2=-BC,$$

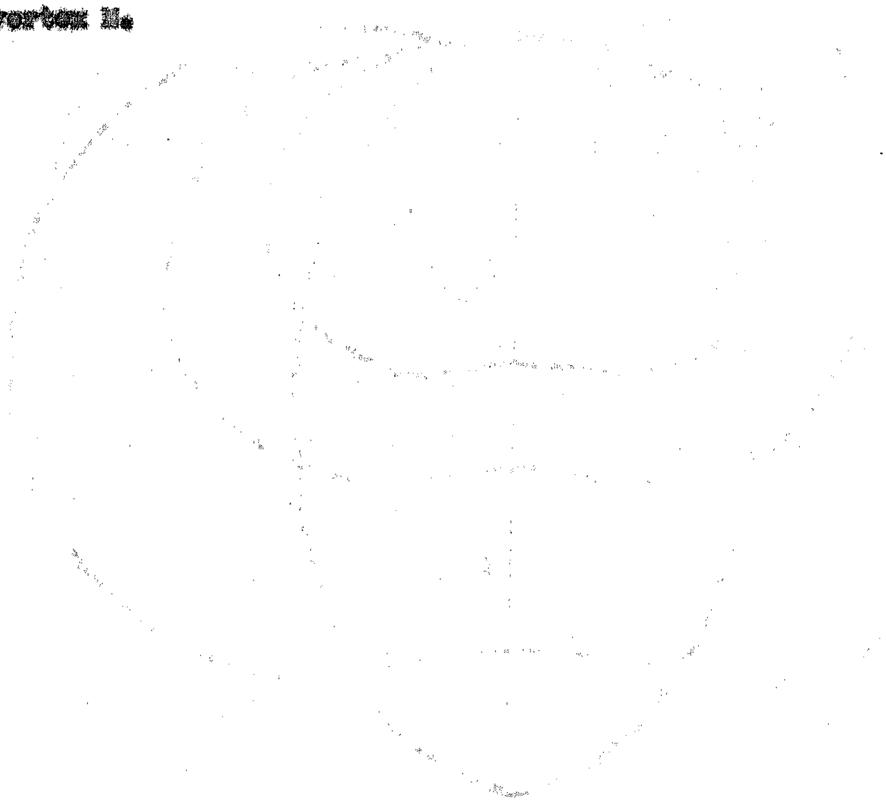
or

(17 ') $$\frac{2G}{B}=\frac{-C}{2G}.$$

But $\frac{2G}{B}$ equals the length of the latus rectum and $-\frac{C}{2G}$ equals the distance from the origin to the vertex. Hence, as in the case of the ellipse and hyperbola, the critical position of the center of inversion on the axis is at a distance from the vertex equal to the length of latus rectum (on the concave side), or it is the intersection of the axis with the circle osculating the parabola at the vertex.

The appearance and disappearance of bays is illustrated in Figure 1, page 11. A conic, C, and accompanying circle of curvature at the vertex N are indicated; K is the point of this circle on the axis of the conic. KN equals the latus rectum. O_1 and O_2 are any points separated by K, O_1 being on the N side of K. Q, Q_1, Q_2 are inverses of C with respect to the circles having centers at K, O_1, O_2 respectively. For the sake of easier comparison the radii of these circles have been selected so that the inverse curves all pass through N. The curvatures of Q_1, Q, Q_2, at N are respectively negative, zero, and positive. Q_2, it is observed, has a bay at N. As the center of inversion moves through N along the axis, the inverse quartic acquires (or loses) a bay at N.

Each of the quartics illustrated has a bay at the left. The bay will disappear when the center of inversion reaches and passes a point L at a distance equal to the latus rectum to the right of this other vertex M.



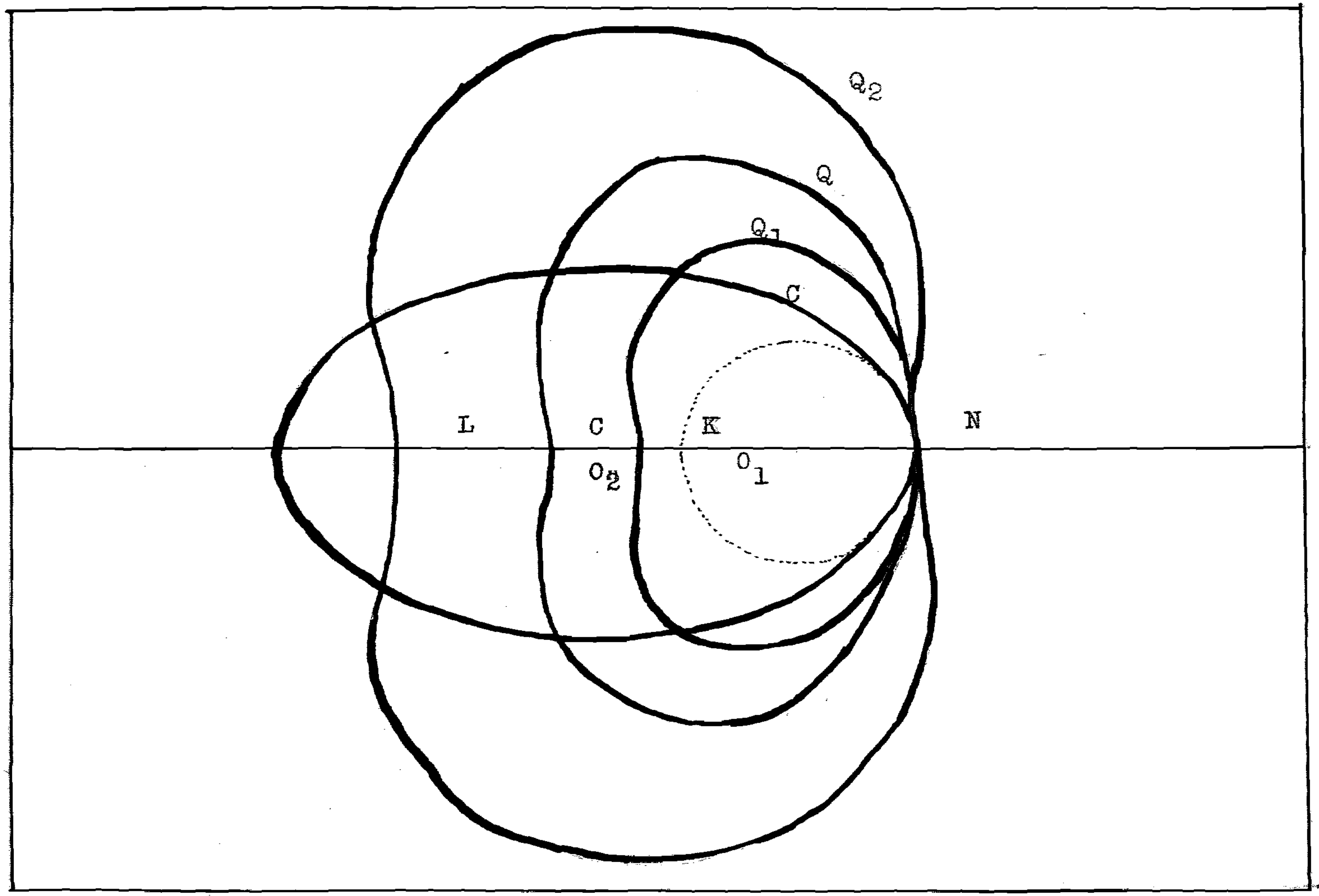


FIGURE 1

CRITICAL POSITION OF CENTER OF INVERSION

CHAPTER III

THE INVERSES OF THE PARABOLA

13. <u>The Equation of the Inverse of a Parabola when the Center of Inversion is on the Axis.</u> The equation of the parabola is

(18) $$y^2 + 4px - 4ph = 0,$$

where $h = ON$ and $4p = NK$, and where N is the vertex and K is the critical position for the center of inversion. (See Plate 1).

The equation of the inverse curve is

(19) $$4ph(x^2 + y^2) - 4pr^2(x^2 + y^2) - r^4y^2 = 0.$$

One such curve is shown in Plate 1, O being the center of the circle of inversion and N ' the inverse of N.

This curve has a cusp at the origin, with the axis as the cuspidal tangent. The point N ', the inverse of $N(h, 0)$, has coordinates $(r^2/h, \quad 0)$. Since $dx/dy = 0$ at N ', the curvature d^2x/dy^2, of the curve at N is readily found to be

$$\frac{d^2x}{dy^2} = \frac{(h - 4p)}{2pr^2}.$$

14. <u>The Forms of the Inverses of a Parabola when the Center of Inversion is on the Axis.</u> The critical values of h are 0 and 4p. In the first case, h 0, the curve is the cissoid with $x = -r^2/4p$ as the asymptotes. In the latter case, if h 4p, the curvature at N ' is zero, and the curve is concave neither to left nor right at that point.

If h is very large and positive, as in Plates 1 and 2, the inverse quartic is approximated by two circles; and in the limiting case, $h = \infty$, the quartic is composite, consisting of two circles.

If h is greater than 4p, the quartic always has a bay at N' with negative curvature. For large values of h the bay is very prominent.

If $h = 4p$, the quartic loses the bay at N', and at this point the curvature is zero. This is shown in Plate 3.

In Plate 4, $0 < h < 4p$, the curvature has changed from positive to negative.

If $h = p$, the quartic is the cardioid. Plate 5.

If O and N coincide, the quartic is the cissoid. Plate 6.

In Plates 7 and 8, $h > 0$.

If h increases indefinitely, the circle of inversion approaches a straight line and the inverse curve approaches a parabola which is a reflection of the conic with respect to this line. Plate 9.

15. <u>The Equation of the Inverse of a Parabola when the Center of Inversion is on the Tangent at the Vertex.</u> If we use

$$(x+h)^2 = 4py \tag{20}$$

as the equation of the parabola, the center of inversion being on the tangent at the vertex, the equation of the inverse curve is

$$h^2(x^2+y^2)^2 + 2r^2(hx+2py)(x^2+y^2) + r^4x^2 = 0. \tag{21}$$

16. <u>The Forms of the Inverses of a Parabola when the Center is on the Tangent at the Vertex.</u> As h varies from 0 to ∞ along the x-axis, the inverse quartic curves are shown in Plates 10 to 16 inclusive. In each case there is a cusp at the center of inversion with the cuspidal tangent

parallel to the y-axis. The x-axis is tangent to the curve at H. The inverse quartic shown in Plate 10 is the same as that of Plate 6 except for position. As h/p varies from 0 to ∞ with a finite center of inversion; the quartic changes in form from the cissoid, Plate 10, through approximations of double semi-circles, Plates 13, 14, and 15 to the straight line, Plate 16.

TABLE I

TYPES OF INVERSES OF A PARABOLA

	h	Curvature at P' $\frac{d^2x}{dy}$	
$y^2+4ph-4ph=0$	$h > 4p$	$+$	Plate 1 and 2.
	$h = 4p$	0	Plate 3.
	$h < 4p$	$-$	Plate 4.
	$h = p$	$-$	Plate 5.
	$h = 0$	0	Plate 6.
	$h > 0$	$+$	Plate 7 and 8
	$h = \infty$	$+\ \infty$	Plate 9.
$(x+h)^2 = 4py$	$h = 0$	$+$	Plate 10.
	$h = 8p$	$+$	Plate 11.
	$h = 4p$	$+$	Plate 12.
	$h = 16p$	$+$	Plate 13.
	$h = 64p$	$+$	Plate 14.
	$h = 256p$	$+$	Plate 15.
	$\frac{h}{p} = \infty$	$+$	Plate 16.

CHAPTER IV

THE INVERSE OF AN ELLIPSE

17. <u>The Equation of the Inverse of an Ellipse when the Center of Inversion is on an Axis of the Ellipse.</u> Let $h = CO$, where C is the center of the conic. The equation of the ellipse is

$$\frac{(x+h)^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{32}$$

The equation of the inverse curve of this ellipse is

$$b^2(h^2+a^2)\,(x^2+y^2)^2 + 2hb^2r^2x(x^2+y^2) + r^4(b^2x^2+a^2y^2) = 0. \tag{33}$$

This is generally a quartic, but it will be composite, consisting of a cubic and the line at infinity if $h = a$.

Let N and M be the vertices on the x-axis; then N' and M' are the inverses of the points N and M, respectively. It is a simple matter to determine the curvatures, d^2x/dy^2, at N' and M', by virtue of the fact that at these points $y = 0$ and $dx/dy = 0$.

(34) N': $\quad x = \dfrac{r^2}{a-h}$; $\quad \dfrac{d^2x}{dy^2} = \dfrac{a(h+a)}{b^2r^2}\left\{h - \left(a - \dfrac{2b^2}{a}\right)\right\}$.

(35) M': $\quad x = \dfrac{-r^2}{a+h}$; $\quad \dfrac{d^2x}{dy^2} = \dfrac{a(h-a)}{b^2r^2}\left\{h - \left(a - \dfrac{2b^2}{a}\right)\right\}$.

18. <u>The Critical Points in the Variation of the Center of Inversion.</u> The diameter of the osculating circle of curvature at M or N is $\dfrac{2b^2}{a}$. Let K be the point at the end of this diameter on the x-axis opposite to the vertex N. This point is a critical position for the center of inversion, for as the center passes through K the curvature of the inverse quartic at N' changes sign.

The form of the inverse curve will naturally depend upon both the ratio of b to a and the position of the center, O, of inversion. For an ellipse whose semi-axis a and b bear to each other a given ratio, the forms of the inverse curve can be traced as h varies continuously from 0 to ∞. All possible types will be discovered as the ratio is made to vary from 0 to ∞. The critical values of this ratio are $1/\sqrt{2}$, 1, and $\sqrt{2}$; and the effect of varying the center of inversion will be considered for each of the seven ranges:

Case I. $\sqrt{2}\, b < a$.

Case II. $\sqrt{2}\, b = a$.

Case III. $\frac{a}{\sqrt{2}} < b < a$.

Case IV. $b = a$.

Case V. $\frac{b}{\sqrt{2}} < a < b$.

Case VI. $\sqrt{2}\, a = b$.

Case VII. $\sqrt{2}\, a < b$.

The critical values of h are 0, $\left| a - \frac{2b^2}{a} \right|$, and a; and the variation of the inverse curve will be traced for these values and the several ranges determined by them.

10. Case I. In this case $\sqrt{2}\, b < a$, h, the distance of the center of inversion from the center of the ellipse, will be allowed to vary from 0 to ∞ . We will note the curvatures, d^2x/dy^2, at E' and E", and we will call attention to the positions under which one of these curvatures changes sign.

If $h = 0$, the center of inversion is at C, since $h = CO$; and the quartic curve is shown in Plate 17. This curve has two bays and has positive curvature at N' and negative curvature at N". In the extreme case, as $a/b \to \infty$, the inverse curve approaches two circles. This is shown in Plate 17.

As the ratio a/b increases indefinitely the quartic conforms more and more to two circles.

If O lies between C and K, where K is the intersection of the axis with the N-osculating circle, the quartic still has two bays, although, one is more prominent than the other. Plate 18.

In Plate 19, the center of inversion is at K and the quartic has zero curvature at N'.

As the center of inversion passes through point K, the curvature at N' changes from positive to negative. If the center of inversion lies between K and N, the quartic has negative curvatures at both N' and N". Plate 20.

If $h = a$, the quartic curve is composite, consisting of a cubic curve and the line of infinity. Plate 21. If h is greater than a, the curve is egg-shaped and is without bays, as in Plate 22. The curvature is positive at N' and negative at N".

If h is much greater than a, the quartic is still similar to the quartic in Plate 22.

If $h = \infty$, the quartic is a reflection of the ellipse with respect to a straight line. Plate 23. The curvature is $+a/b^2$ at N' and $-a/b^2$ at N".

20. Case II. If the ellipse has semi-axes a and b such that $a = \sqrt{2}\ b$, and if $h = 0$, the quartic curve has zero curvature at both N' and N'. Plate 24.

If h lies in the range, 0 to a, the quartic has a bay at N' with negative curvatures at both N' and N'. Plate 25.

If $h = a$, the quartic is composite, consisting of a cubic and the line at infinity. Plate 26.

If $h > a$, the quartic is egg-shaped with positive curvature at N' and negative curvature at N'. Plate 27.

If $h = \infty$, the quartic is a reflection of the conic with respect to a straight line. Plate 28.

21. Case III. If $h = 0$, the quartic looks somewhat like an ellipse. The curvatures at N' and N' are positive and negative respectively. Plate 29.

If O lies between C and K, the quartic curve is of the type shown in Plates 29 and 30.

If O is at K, the quartic has zero curvature at N' and negative curvature at N'. Plate 31.

If O lies between K and N, the quartic has a bay at N'. Plate 32.

If $h = a$, the quartic is composite, consisting of a cubic and the line at infinity. Plate 33.

If $h > a$, the quartic has positive curvature at N' and negative curvature at N'. Plates 34 and 35.

22. Case IV. If the axes of the ellipse are equal, the quartic is a special case of the ellipse, which is the circle. If $h = 0$, the inverse curve is the circle. Plate 36.

If $h < a$, the quartic is a circle. Plate 37.

If $h = a$, the quartic is a straight line. Plate 38.

If $h = \infty$, the quartic is a reflection of the circle. Plate 40.

23. <u>Case V.</u> If $h = 0$, the quartic is very much like an ellipse, having positive curvature at M' and negative curvature at N'. Plate 41.

If h lies between 0 and a, the quartic has negative curvature at M' and positive curvature at N'. Plate 42.

If $h = a$, the quartic is composite, consisting of a cubic and the line at infinity as shown in Plate 43.

If h is greater than a, the quartic has a bay at M' and the curvature is positive at M' and N'. Plate 44.

If O coincides with M, the quartic has zero curvature at M' and positive curvature at N'. Plate 45.

If O is to the right of M, the quartic has positive curvature at N' and negative at M' as in Plate 46.

If $h = \infty$, the quartic is a reflection of the conic with respect to a line. Plate 47.

24. <u>Case VI.</u> If $h = 0$, the quartic is very much like an ellipse. The curve has negative curvature at N' and positive curvature at M' as shown in Plate 48.

If $0 < h < a$, the quartic has negative curvature at N' and positive curvature at M'; and the quartic also has two bays close to M'. Plate 49.

If $h = a$, the quartic is composite, consisting of a cubic and the line at infinity. Plate 50.

If $a < h < 2\ b^2/a - a$, the quartic has a bay at M', and the curvature is positive at both N' and N'', as shown in Plate 51.

If the center of inversion coincides with K, the critical position of the center of inversion, the quartic has zero curvature at N' and positive curvature at N'', as shown in Plate 52.

If $h > 2\,b^2/a - a$, the quartic is again shaped somewhat like an ellipse with positive curvature at N' and negative curvature at N''. Plate 53.

If $h = \infty$, the quartic is a reflection of the ellipse. Plate 54.

55. <u>Case VII.</u> If $h=0$, the quartic has negative curvature at N' and positive curvature at N''. The quartic also has two bays on the b-axis, as shown in Plate 55.

If $0 < h < a$, the quartic has negative curvature at N' and positive curvature N''. The quartic also has two bays close to N'', as shown in Plate 56.

When $h=a$, the quartic is composite, consisting of a cubic and the line at infinity. The curvature is positive at N''. Plate 57.

If $0 < h < \frac{2b^2}{a} - a$, then the quartic has a bay at N' and the curvature at N' and N'' is positive. Plate 58.

If $h = K$, the curvature at N' is zero, and has positive curvature at N''. Plate 59.

If $h > 2b^2/a - a$, the quartic is somewhat like the quartic in Plate 59, except that it has negative curvature at N'. Plate 60.

When $h = \infty$, the quartic is degenerate, being the limiting reflection of the ellipse. Plate 61.

TABLE II

TYPES OF INVERSES OF AN ELLIPSE

	h	Curvature, $\frac{d^2x}{dy^2}$ at N'	M'	
$a > \sqrt{2}\,b$	$h=0$	$+$	$-$	Plate 17.
	$0<h<a-\frac{2b^2}{a}$	$+$	$-$	Plate 18.
	$h=a-\frac{2b^2}{a}$	0	$-$	Plate 19.
	$0<h<a$	$+$	$-$	Plate 20.
	$h=a$	$[0]$	$-$	Plate 21.
	$h>a$	$+$	$-$	Plate 22.
	$h=\infty$	$+\frac{a}{b^2}$	$-\frac{a}{b^2}$	Plate 23.
$a=\sqrt{2}\,b$	$h=0$	0	0	Plate 24.
	$0<h<a$	$-$	$-$	Plate 25.
	$h=a$	$[0]$	$-\frac{4}{ay^2}$	Plate 26.
	$h>a$	$+$	$-$	Plate 27.
	$h=\infty$	$+2/a$	$-2/a$	Plate 28.
$\frac{a}{\sqrt{2}}<b<a$	$h=0$	$-$	$+$	Plate 29.
	$0<h<\frac{2b^2}{a}-a$	$-$	$+$	Plate 30.
	$h=\frac{2b^2}{a}-a$	$-$	0	Plate 31.
	$0<h<a$	$-$	$-$	Plate 32.
	$h=a$	$[0]$	$-$	Plate 33.
	$h>a$	$+$	$-$	Plate 34.
	$h=\infty$	$+\frac{a}{b^2}$	$-\frac{a}{b^2}$	Plate 35.

TABLE II (continued)

TYPES OF INVERSES OF AN ELLIPSE

	h	Curvature, $\frac{d^2x}{dy^2}$, at		
		E′	M′	
$a < \frac{b}{\sqrt{2}}$	$h = 0$	$-$	$+$	Plate 55.
	$0 < h < a$	$-$	$+$	Plate 56.
	$h = a$	[0]	$+$	Plate 57.
	$0 < h < \frac{2b^2}{a} - a$	$+$	$+$	Plate 58.
	$h = \frac{2b^2}{a} - a$	$+$	0	Plate 59.
	$h > \frac{2b^2}{a} - a$	$+$	$-$	Plate 60.
	$h = \infty$	$+\frac{a}{b^2}$	$-\frac{a}{b^2}$	Plate 61.

CHAPTER V

THE INVERSE OF THE HYPERBOLA

26. <u>The Equation of the Inverse of a Hyperbola when the Center of Inversion is on an Axis of the Hyperbola.</u> If $h = OO'$, where O is the center of the conic, then the equation of the hyperbola is

(26) $$\frac{(x+h)^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The equation of the inverse curve of the hyperbola is

(27) $$b^2(h^2+a^2)(x^2-y^2)^2 + 2hb^2r^2x(x^2+y^2) + r^4(b^2x^2-a^2y^2) = 0.$$

The intercepts, ON' and OM', on the x-axis are $\frac{r^2}{a-h}$ and $-\frac{r^2}{a+h}$; and the curvatures, $\frac{d^2x}{dy^2}$, at the points N' and M' are

(28) $$N': \frac{d^2x}{dy^2} = -\frac{a(h-a)}{b^2r^2}\left[h - \left(a + \frac{2b^2}{a}\right)\right].$$

and

(29) $$M': \frac{d^2x}{dy^2} = \frac{a(h+a)}{b^2r^2}\left[h + \left(a + \frac{2b^2}{a}\right)\right].$$

Critical values of h are 0, a, and $a + \frac{2b^2}{a}$; and the variation of the quartic (2) is traced through these values of h from $h = 0$ to $h + \infty$.

27. <u>The Forms of the Inverses of a Hyperbola when the Center of Inversion is on the Axis of the Hyperbola.</u> If $h = 0$, the quartic is very much like a figure-eight. This curve is symmetrical with respect to both axes. Plate 62.

If h lies between 0 and a, one of the loops of this quartic is larger than the other. Plate 63.

If $h = a$, the quartic is composite, consisting of a cubic and the line at infinity. Plate 64.

If $a = b$, this cubic is the strophoid, which will be discussed in the next chapter.

If h lies between 0 and $a + \frac{2h^2}{a}$, the quartic is the Limacon, which is shown in Plate 65. This curve will also be discussed in the next chapter.

28. <u>The Equation of the Inverse of a Hyperbola when the Center of Inversion is on the Conjugate Axis.</u> If the center of inversion is on the conjugate axis of the hyperbola at the distance h from the center, it is convenient to use the equation

$$\text{(30)} \qquad -\frac{(x+h)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

In this case the equation of the inverse quartic is

$$\text{(31)} \qquad b^2(h^2+a^2)(x^2+y^2) + 2hb^2r^2x(x^2+y^2) + r^2(b^2x^2-a^2y^2) = 0.$$

There exists a value α such that when $h = \alpha$ the curve has a horizontal inflectional tangent. To determine the position of the center of inversion for which this quartic has a horizontal inflectional tangent eliminate x and y between the three equations: $f(x,y) = 0$; $\frac{dx}{dy} = 0$; and $\frac{d^2x}{dy^2} = 0$, where $f(x,y)$ denotes the left member of equation (31).

These three conditions are equivalent to

$$b^2(h^2+a^2)(x^2+y^2) + 2h^3b^2x(x^2+y^2) + h(b^2x^2-a^2y^2) = 0,$$

$$2(h^2+a^2)(x^2+y^2)x + 3h^3x^2 + h^3y^2 + h^4x = 0,$$

$$2(h^2+a^2)(3x^2+y^2) + 6h^3x + h^4 = 0.$$

These equations will be consistent if

(32) $$h = \frac{(2a^2+b^2)^{\frac{3}{2}}}{a},$$

and for this position of the center of inversion the quartic has a horizontal inflectional tangent at

(33) $$h_0 = \frac{(2a^2+b^2)^{\frac{7}{2}}}{2a^2(3a^4+3a^2b^2+b^4)} .$$

If $b = \frac{a}{2}$, h is equal to 27a and $x_0 = -\frac{2187a}{976}$; if $b = a$, then $h = 3\sqrt{3}\,a$ and $x_0 = \frac{27\sqrt{3}}{14}\,a$; and if $b = \sqrt{2}\,a$, then $h = 8a$ and $x_0 = -\frac{64}{13}\,a$.

28. <u>The Forms of the Inverses of a Hyperbola when the Center of Inversion is on the Conjugate Axis.</u> If $h = 0$, the quartic is like a figure-eight. Plate 70.

In Plate 71 the center of the quartic is to the right.

In Plate 72 the horizontal inflectional tangent, when $h = \frac{27a}{8}$, is $x_0 \approx -\frac{2187}{976}\,a$.

In Plates 73 and 74 the quartic is approaching two circles as the ratio h/a increases indefinitely. The two circles are the inverses of the asymptotes.

If $a = b$ and $h = 0$, the quartic is the Leminscate of Bernoulli, another well known curve. Plate 75. It will be discussed in the next chapter.

If $h = 3\sqrt{3}\,a$, the horizontal inflectional tangent is at $-\frac{27\sqrt{3}}{14}\,a$. Plate 77.

Plates 78 and 79 show that the inverse curves are approaching two circles as h/a approaches $+\infty$.

If $b = \sqrt{2}a$ and if $h = 8a$, there is a horizontal inflectional tangent at $-\frac{64}{13}a$. Plate 32.

As h/a approaches ∞, the quartic curve approaches the two limiting circles which are the inverse of the asymptotes.

CHAPTER VI

SPECIAL CURVES

30. Introduction. As was pointed out in the first chapter, several of the inverses of conics are well known curves of historical interest and mechanical usefulness.

The straight line and circle were encountered as the inverses of certain limiting cases of conics, as is indicated in Plates 38 and 36, 37, 39, 40.

When the center of inversion is taken at the center of the conic, or at the focus, or at any extremity of an axis, certain special types of inverse curves are obtained, most of which are familiar to us under special names. These include the Lemniscate of Bernoulli, the Limacons of Pascal (including the well known Cardioid), the Cissoid of Diocles, and the Strophoid. These will be identified in the next three paragraphs.

31. Center of Inversion at Center of Conic. When the center of inversion is at the center of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{34}$$

the associated quartic is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=r^2(x^2+y^2)^2, \tag{35}$$

an elliptic Lemniscate of Booth.

These curves are shown in Plates 17, 24, 29, 41, 40, 58.

When the center of inversion is the center of the hyperbola

(36) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

the associated quartic is

(37) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = r^2(x^2 + y^2)^2,$$

a hyperbolic Lemniscate of Booth.

Each of these is a "figure-eight curve," as is indicated in Plates 62, 70, 75, 80. When a = b, that is, when the hyperbola is rectangular, the equation may be written

(38) $$(x^2 + y^2)^2 = \frac{1}{r^2a^2}(x^2 - y^2),$$

which is recognized as the equation of a Lemniscate of Bernoulli. This quartic is shown in Plate 75.

32. Center of Inversion at Focus. The equation of a conic in standard polar coordinate position with focus at the origin is

(39) $$\rho = \frac{\ell}{1 - e \cos \theta} .$$

The inverse of this conic with respect to the origin is the curve

(40) $$\rho = \frac{r^2}{\ell}(1 - e \cos \theta).$$

These curves are known as the Limacons of Pascal.

When $e < 1$, the associated conic is an ellipse and the limacons are acnodal. These are shown in Plates 20, 25, 32.

When $e = 1$, the associated conic is a parabola. In this case the limacon is the cuspidal type called the Cardioid. This is shown in Plate 6.

When $e > 1$, the associated conic is a hyperbola, and the limaçons are of the nodal type shown in Plate 65.

38. <u>Center of Inversion at a Vertex</u>. If the center of inversion is at a vertex of the conic, the inverse quartic is composite, consisting of the line at infinity and the cubic curve. (See paragraph 10). Inversion of the ellipse

(41) $$\frac{(x+a)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

yields the cubic

(42) $$(x^2+y^2)\,(x+\frac{ar}{b^2} = r\ \frac{a^2-b^2}{2ab}\ \cdot\ x^2,$$

a curve which bears a striking resemblance to part of a Conchoid of Nichomedes. Forms of this curve when $a > b$ are shown in Plates 21, 26, 33; $a = b$, Plate 38; $a < b$, Plates 43, 50, 57.

Inversion of the parabola

(43) $$y^2 + 4px = 0\ .$$

yields the cubic

(44) $$y^2 = \frac{-x^3}{x + \frac{1}{4pr^2}}\ .$$

This is the well known <u>Cissoid of Diocles</u>. It is shown in Plate 8.

Inversion of the hyperbola

(45) $$\frac{x+a^2}{a^2} - \frac{y^2}{b^2} = 1,$$

yields the cubic

(46) $$y^2 = x^2\ \frac{\frac{r^2}{2a} + x}{\frac{ar^2}{2b^2} - x}\ .$$

The form of this curve is shown in Plate 64, for $a = \sqrt{2}\, b$. For varying values of the ratio a/b the loop is relatively larger or smaller. When $a \neq b$, corresponding to a rectangular hyperbola, the equation is

$$(47) \qquad y^2 = x^2 \, \frac{\frac{a^2}{2x} + x}{\frac{b^2}{2x} - x},$$

and the inverse curve is a Strophoid.

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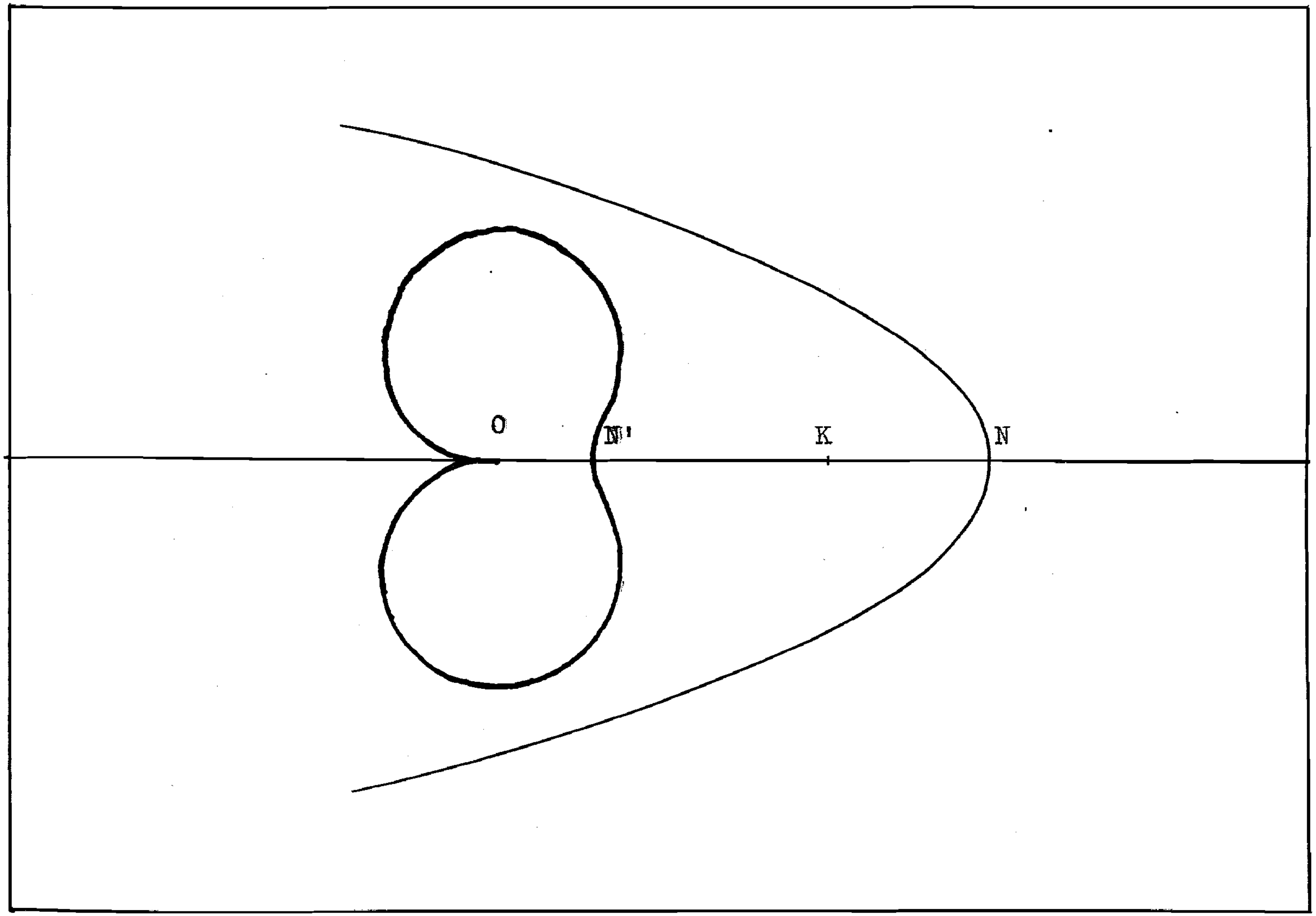
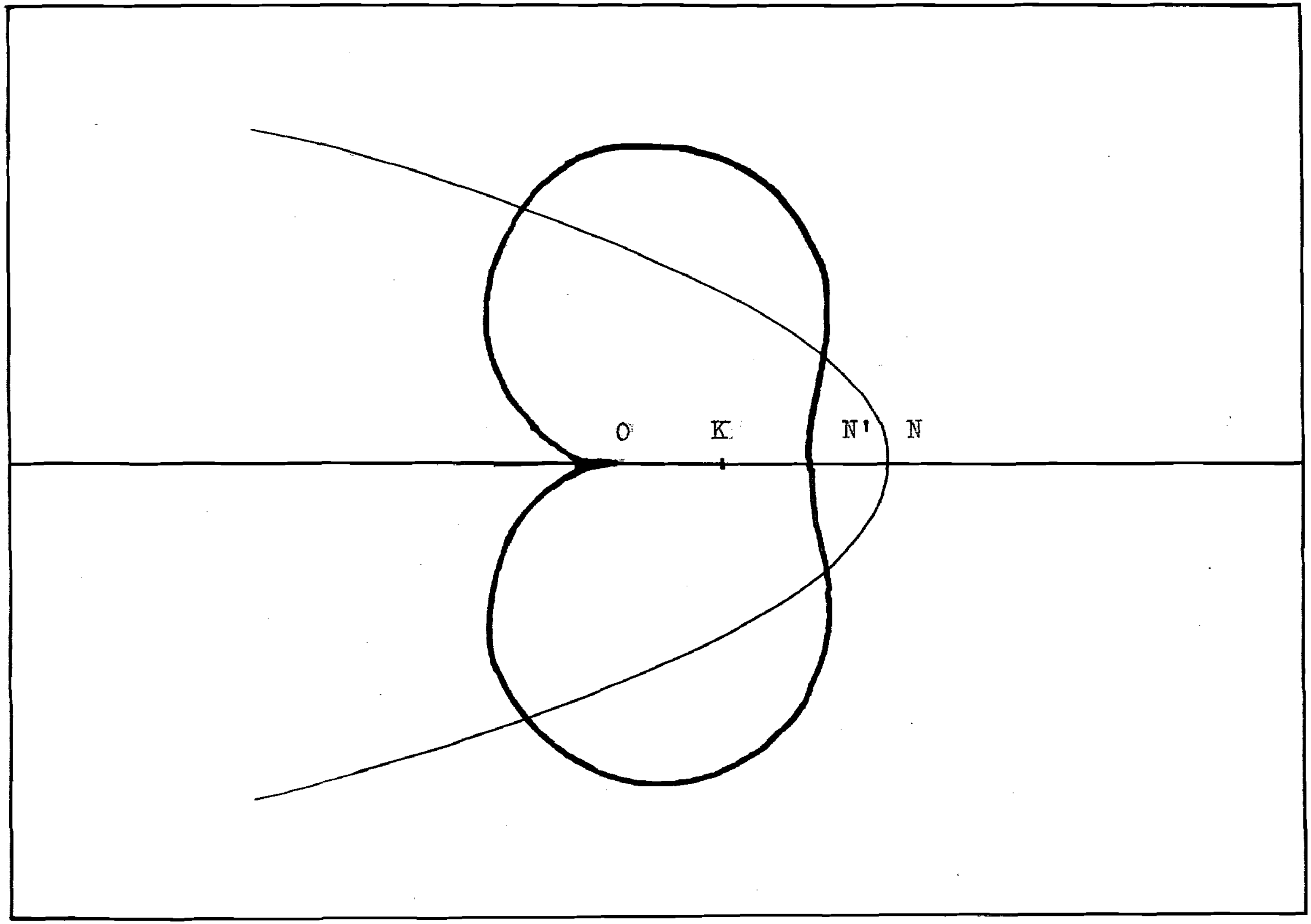


PLATE 1.



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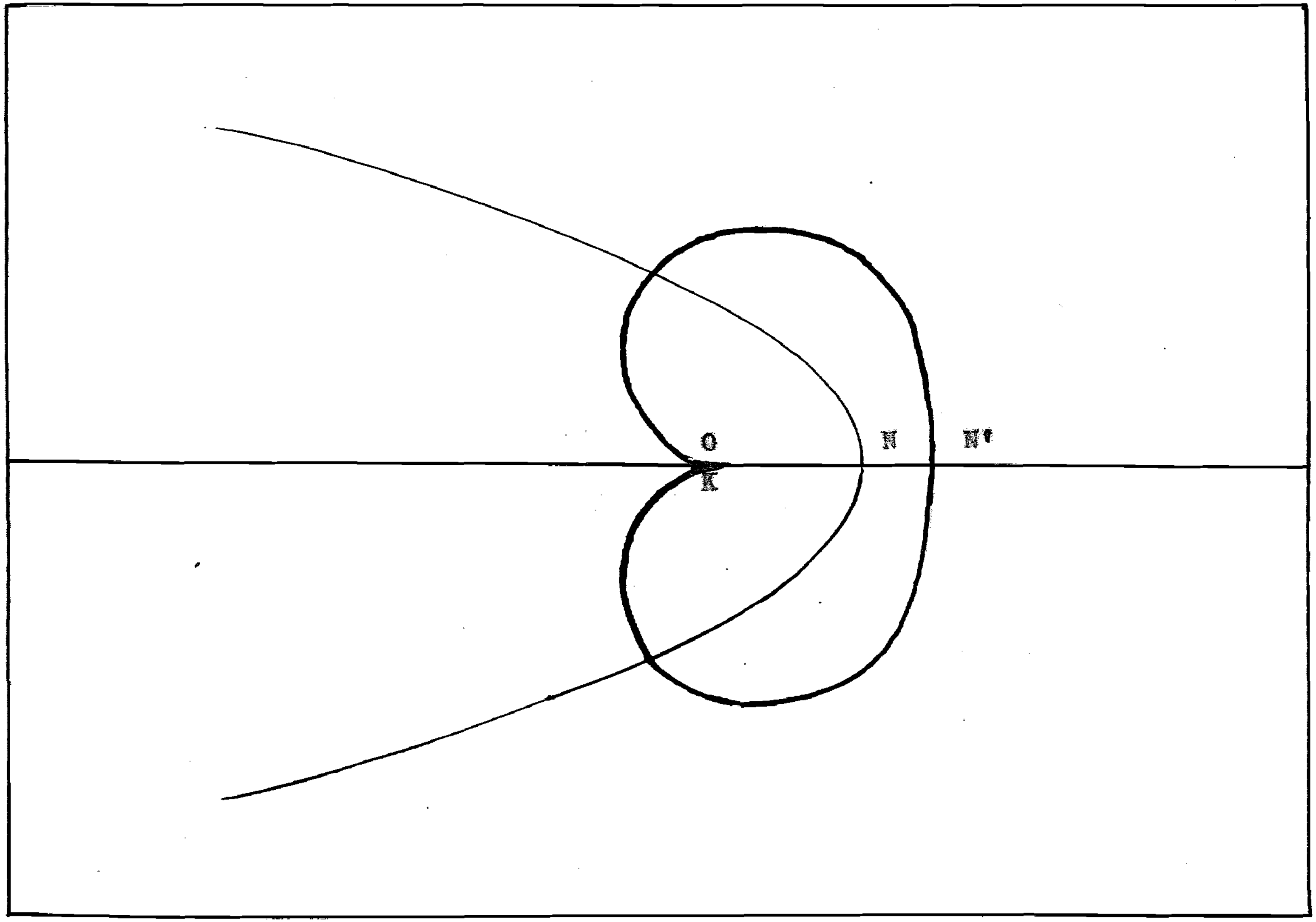

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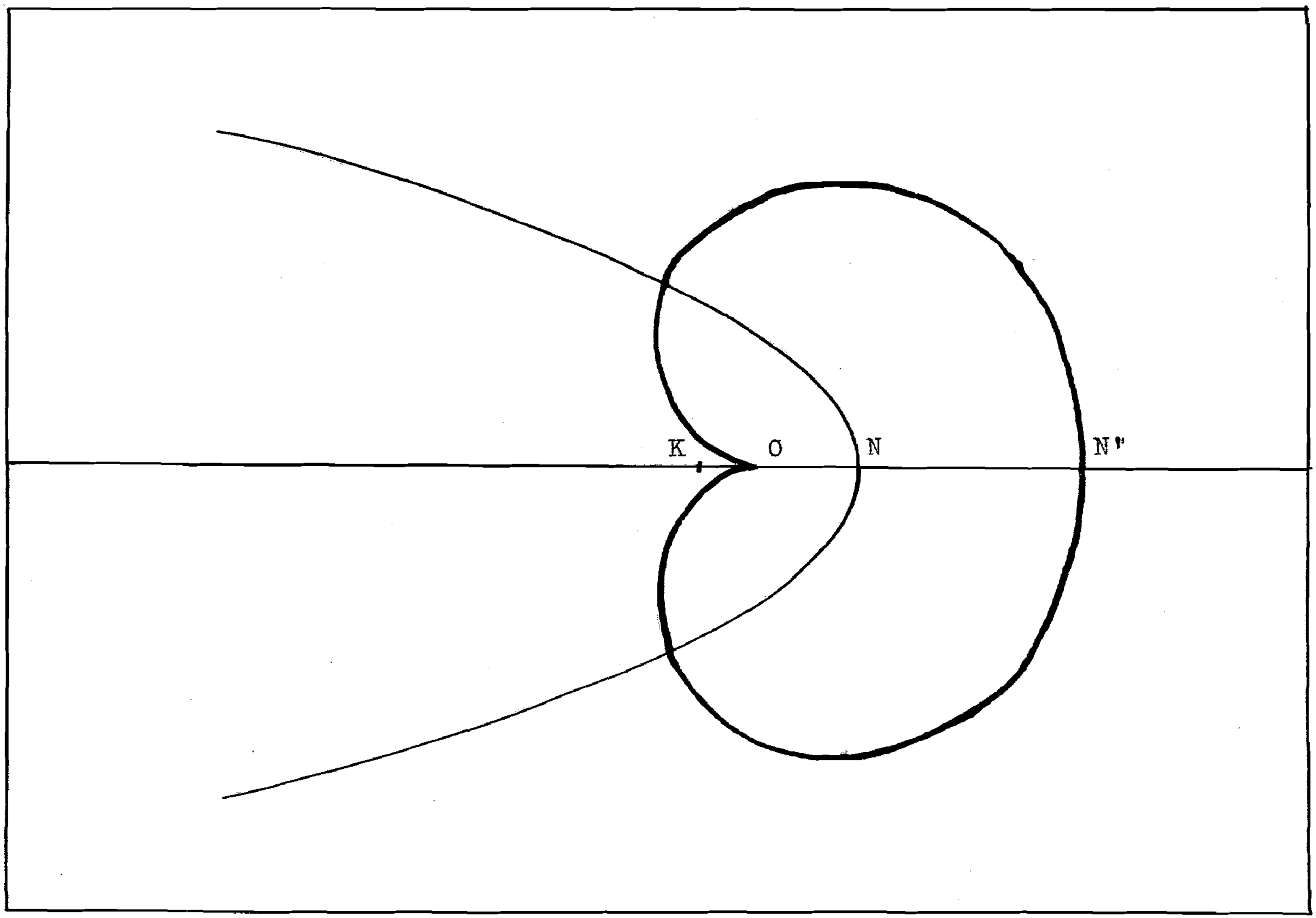

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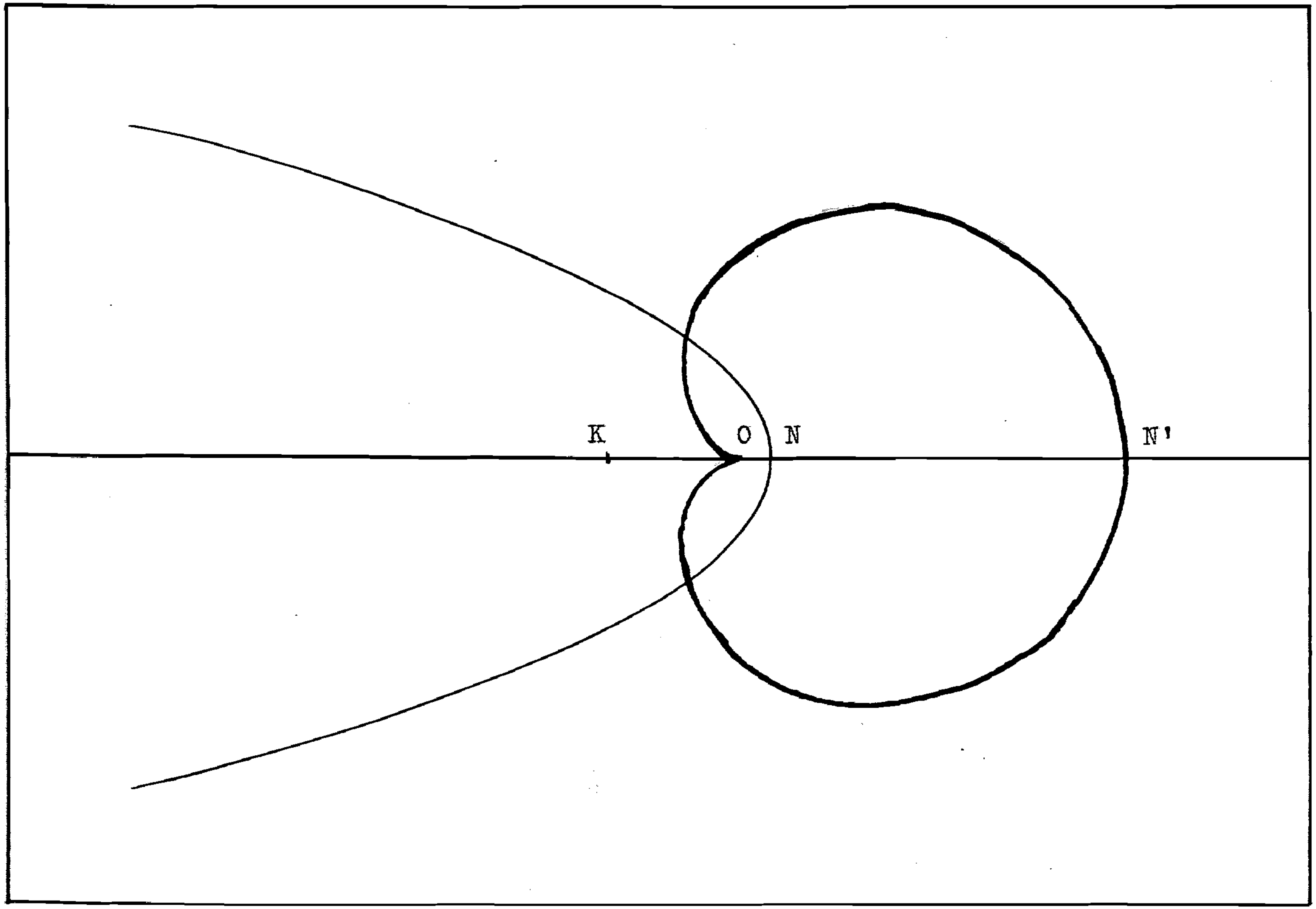

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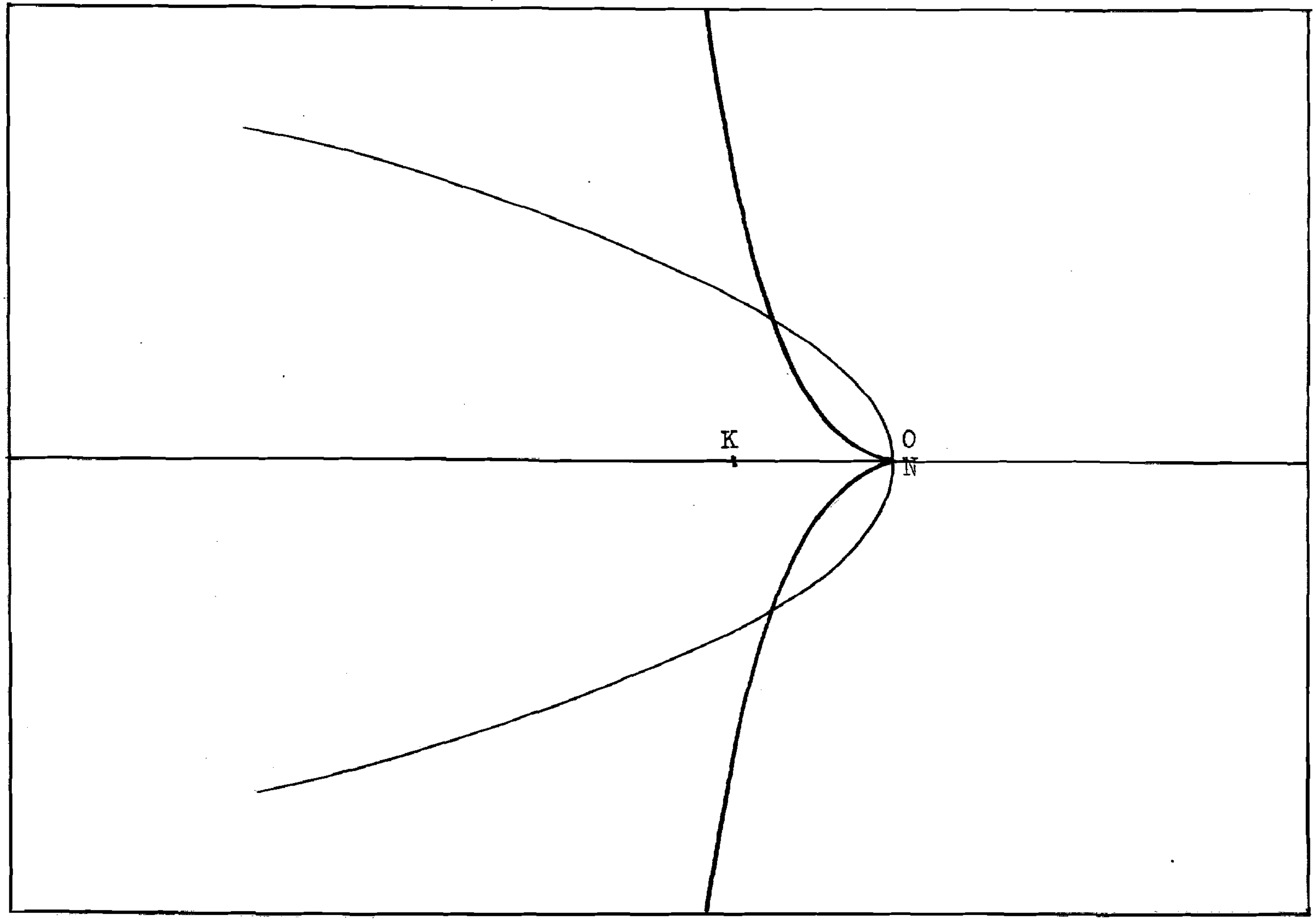

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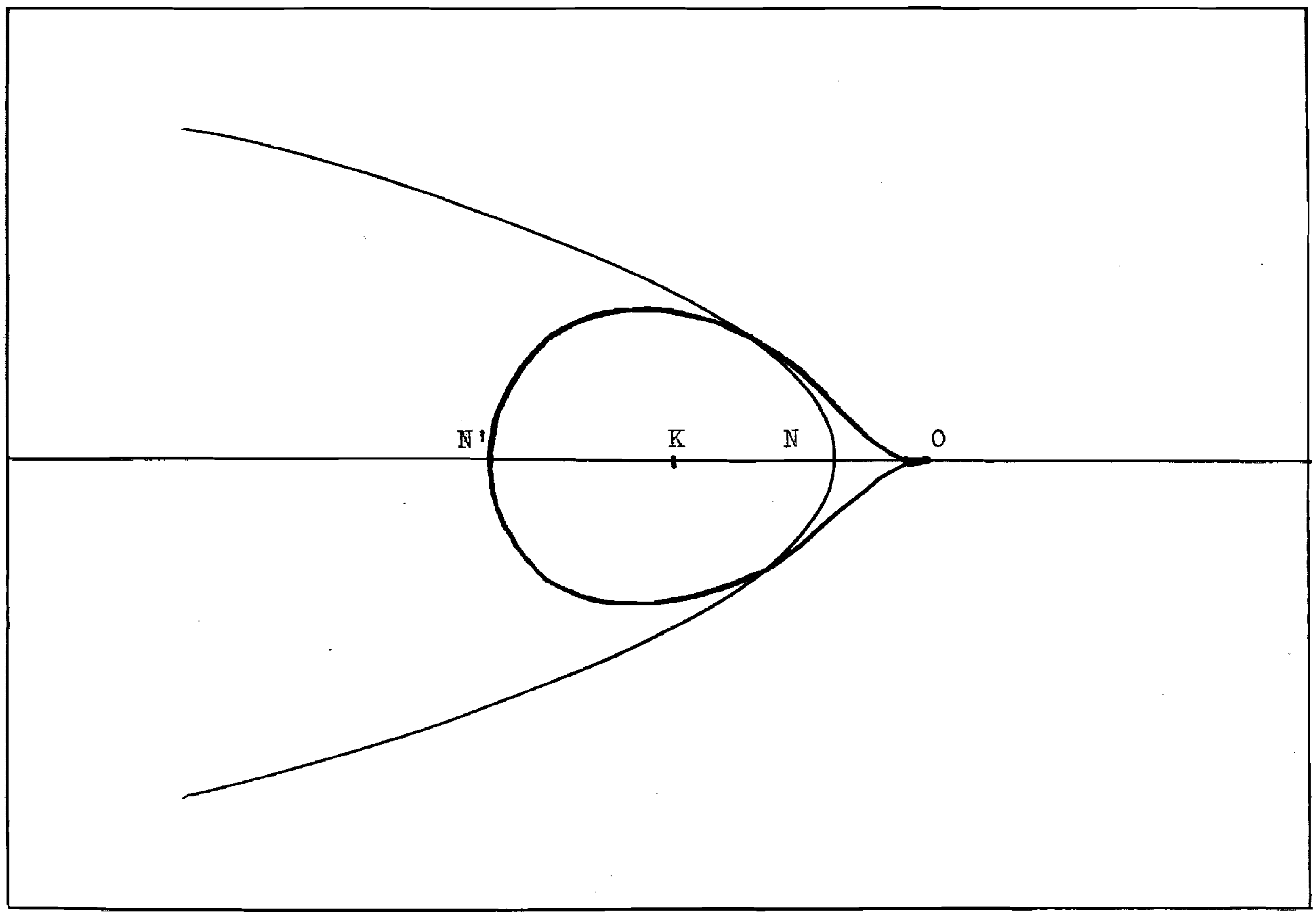

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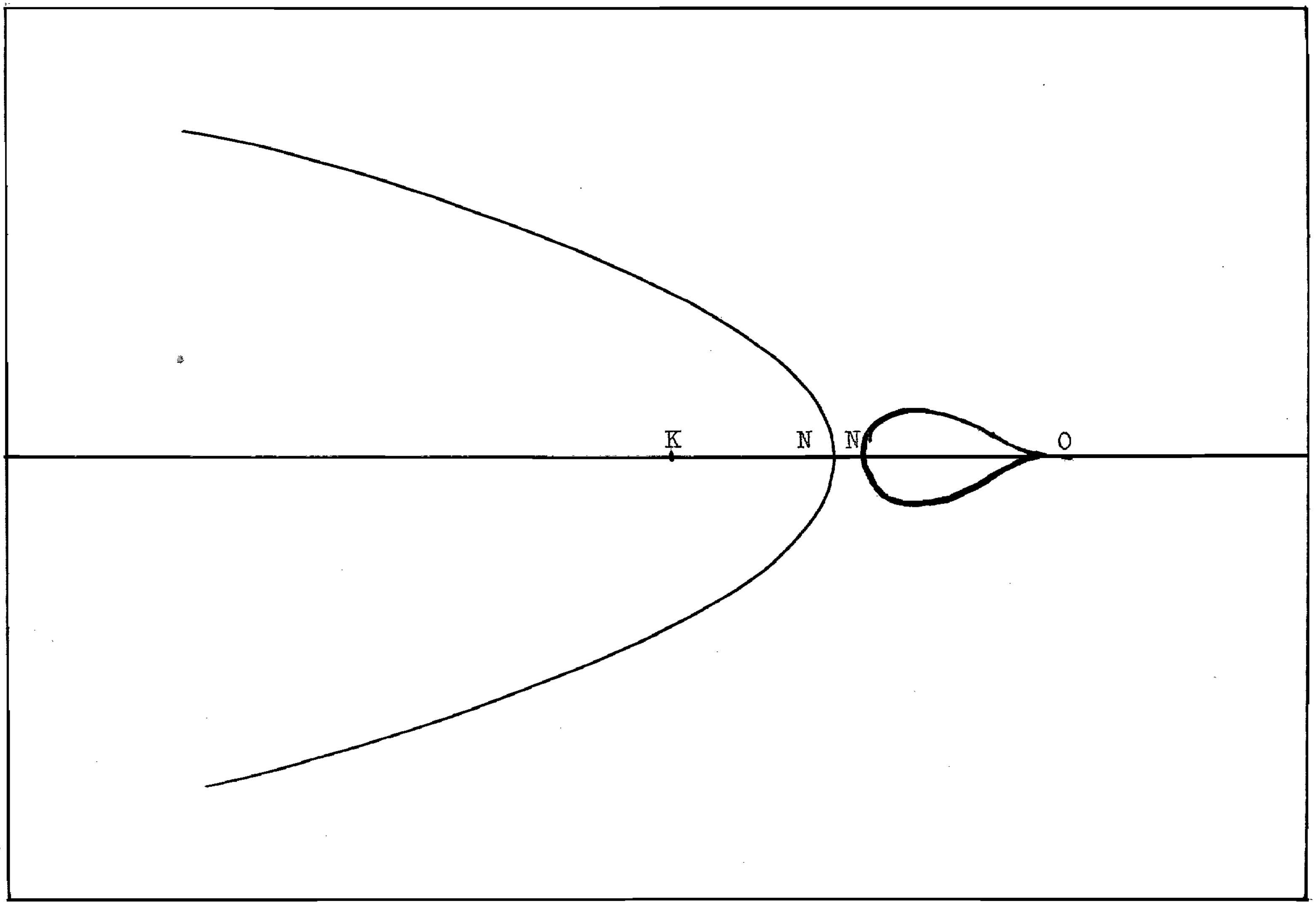

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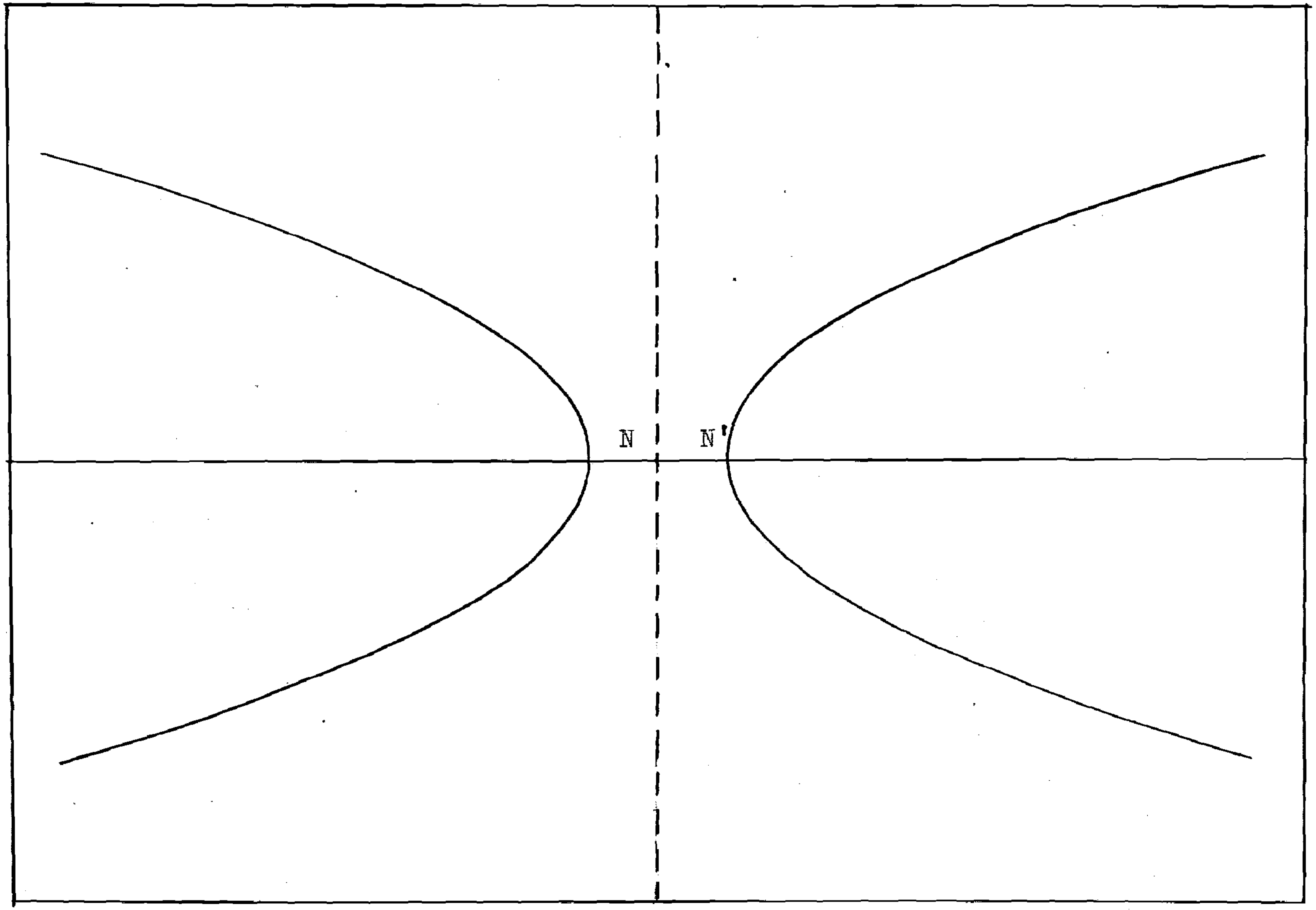

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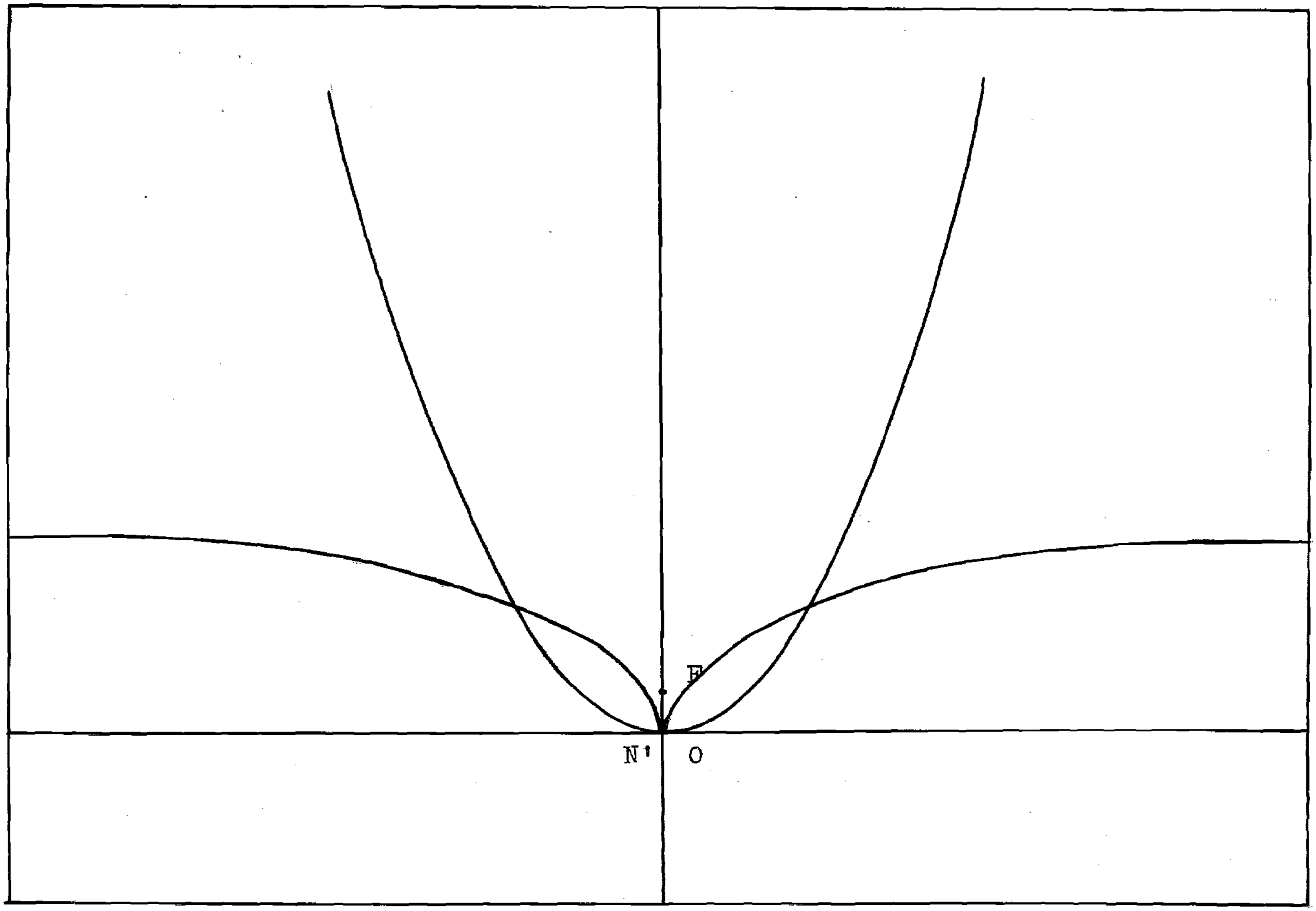

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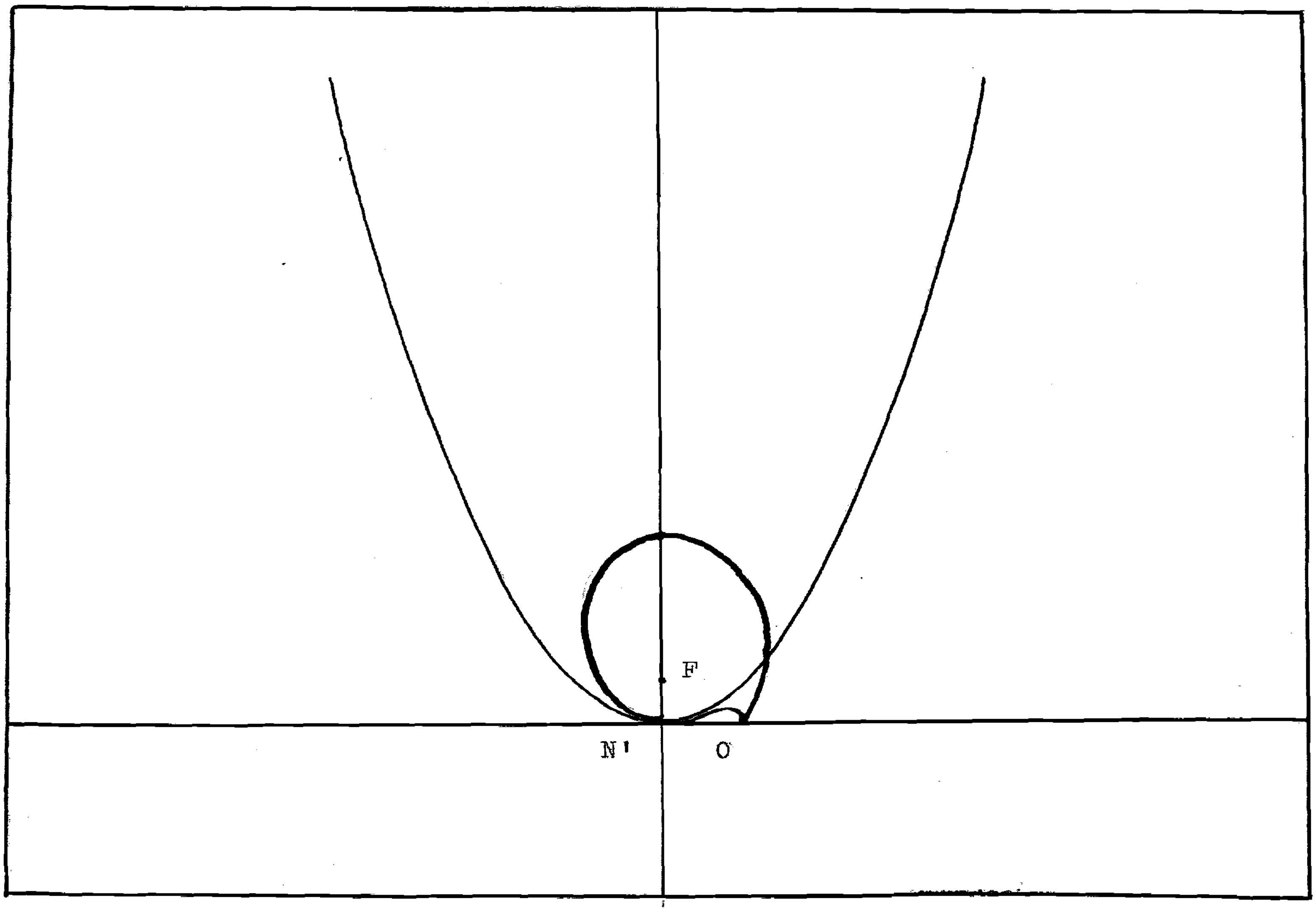

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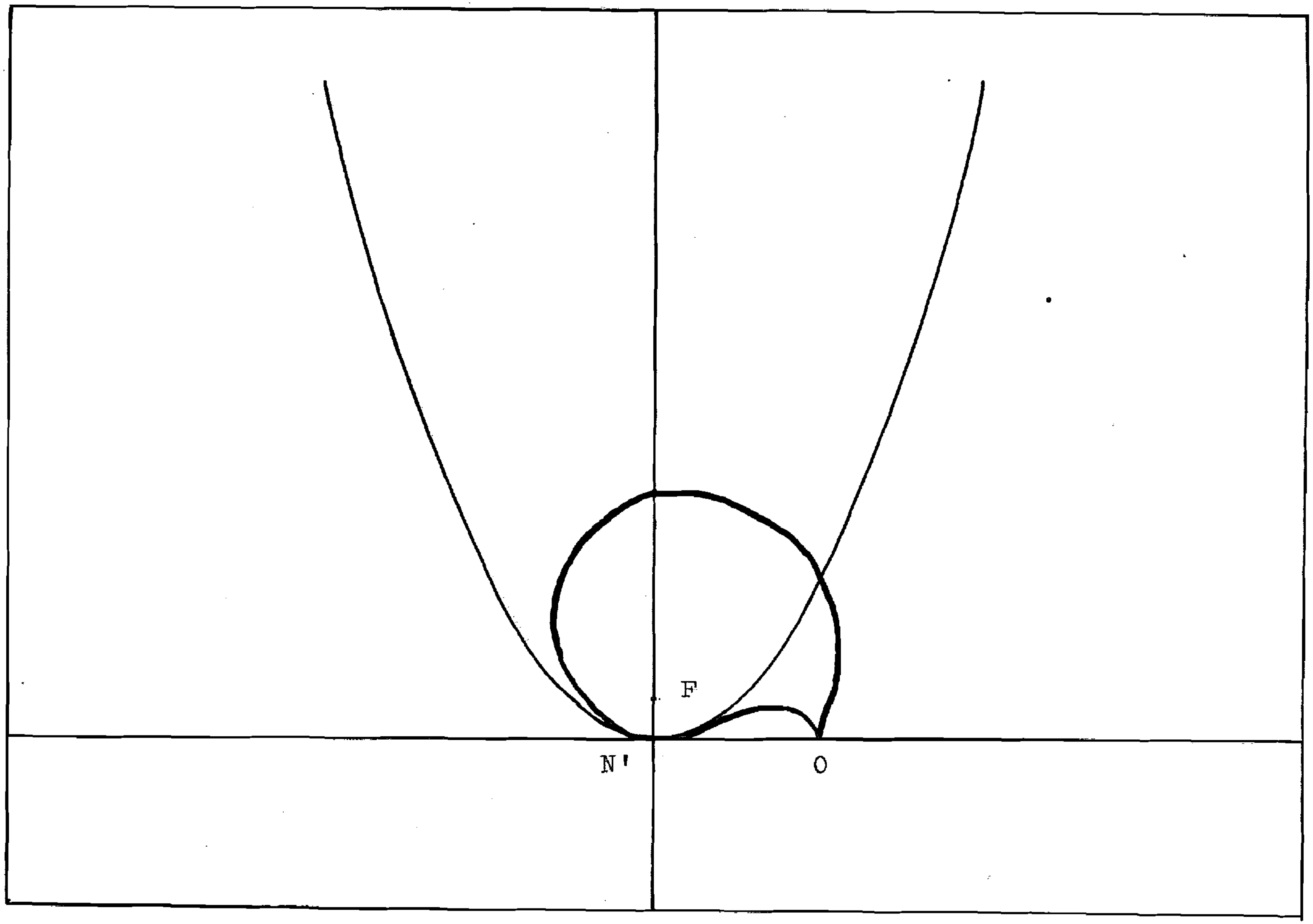

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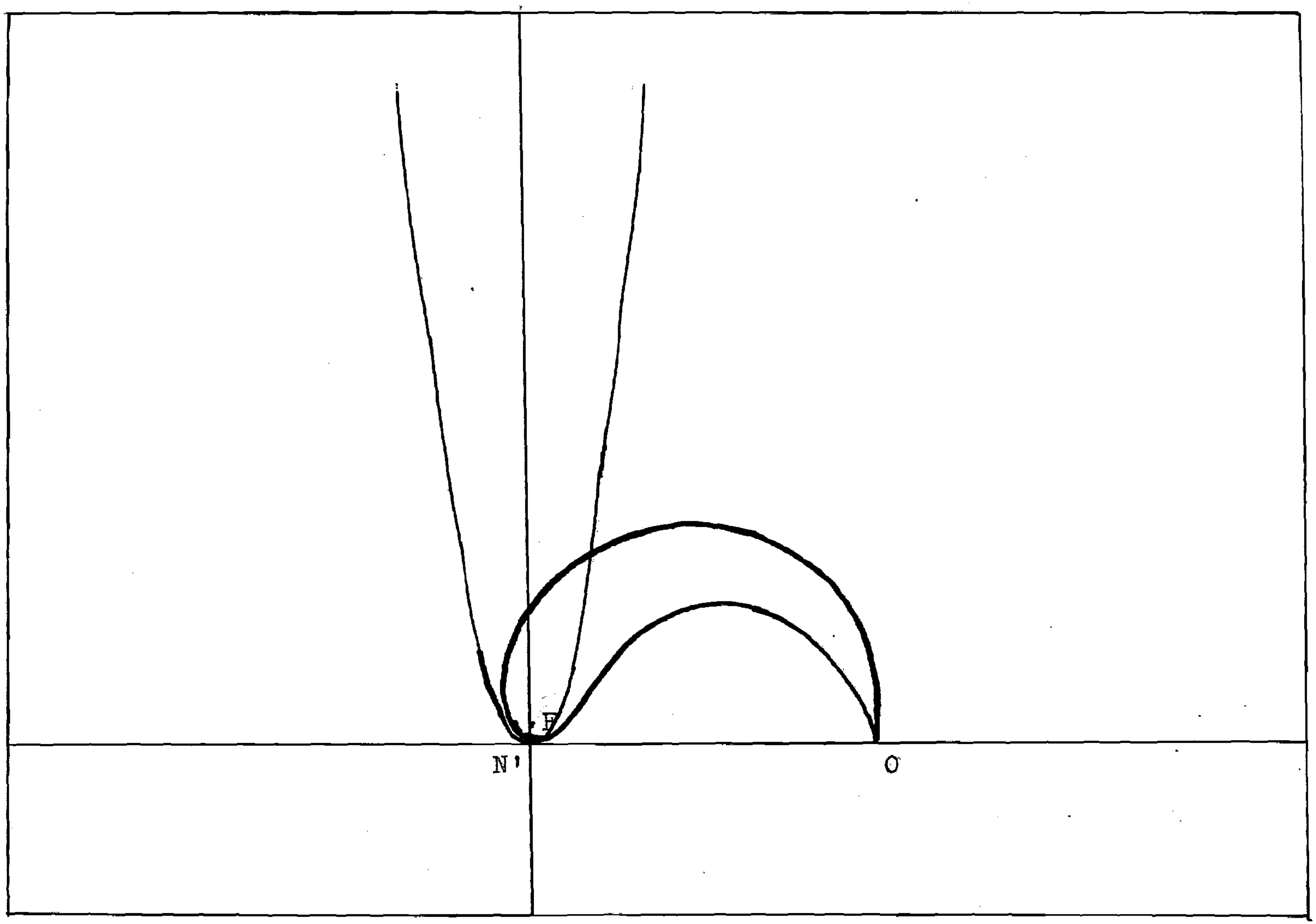

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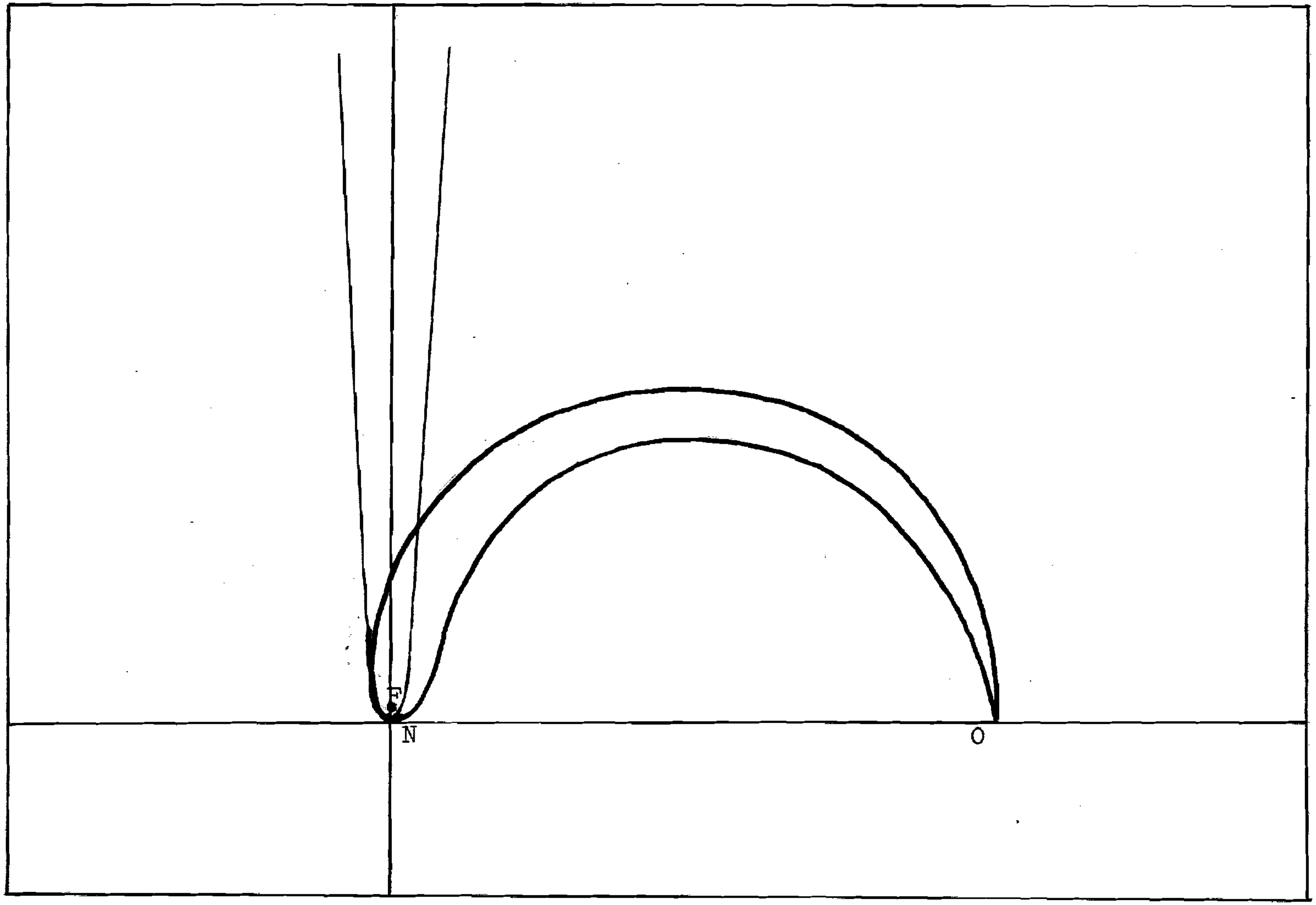

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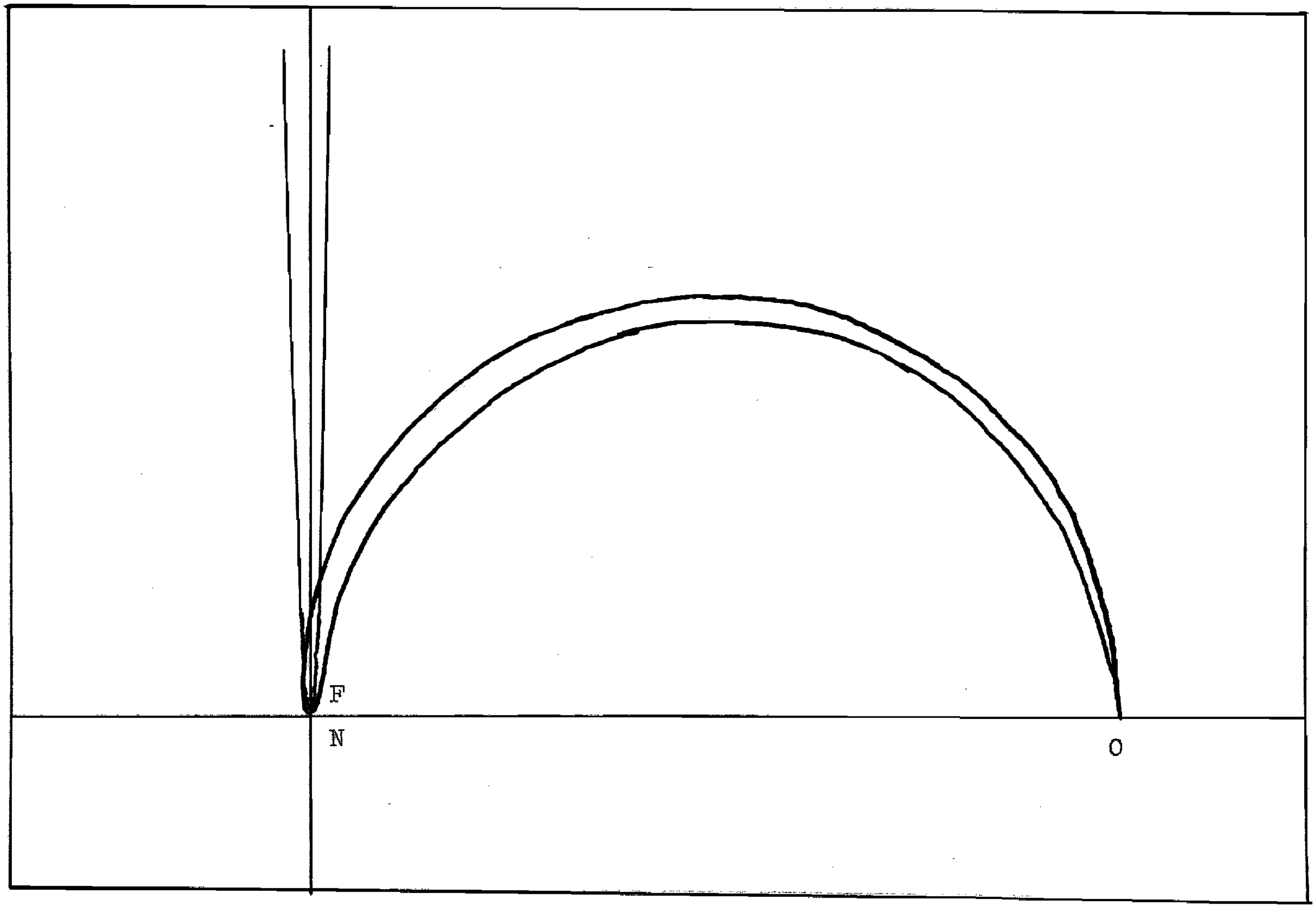

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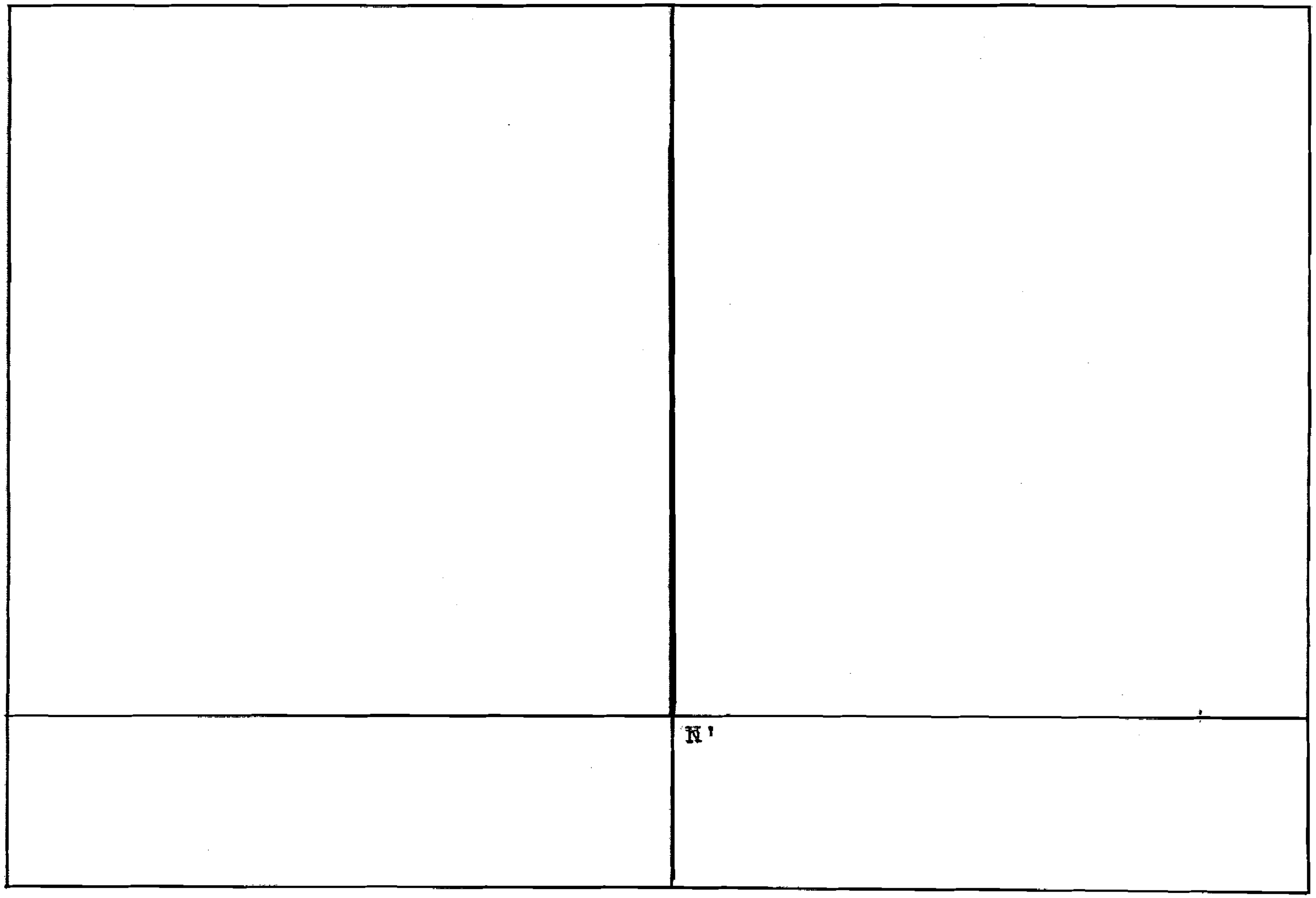

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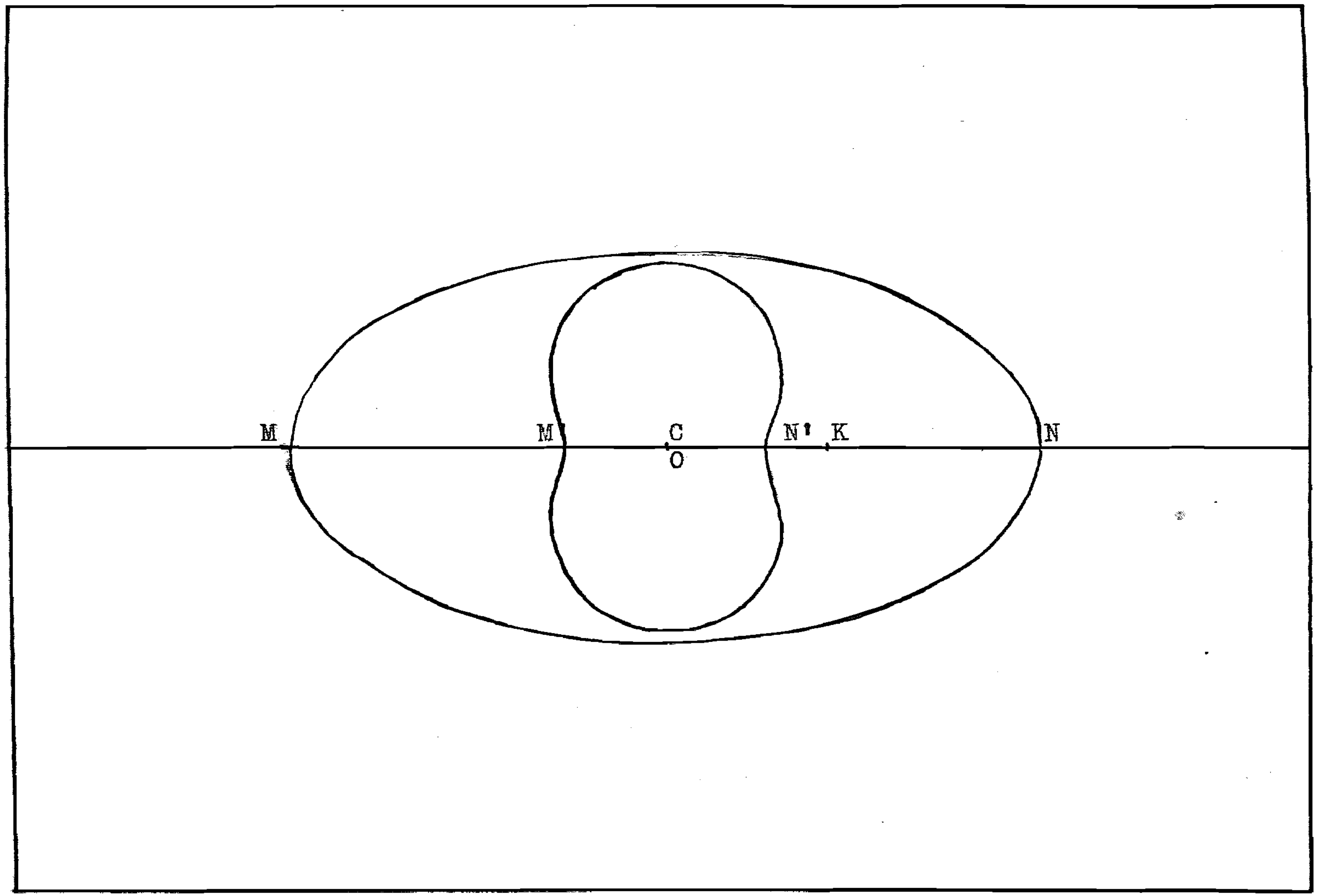

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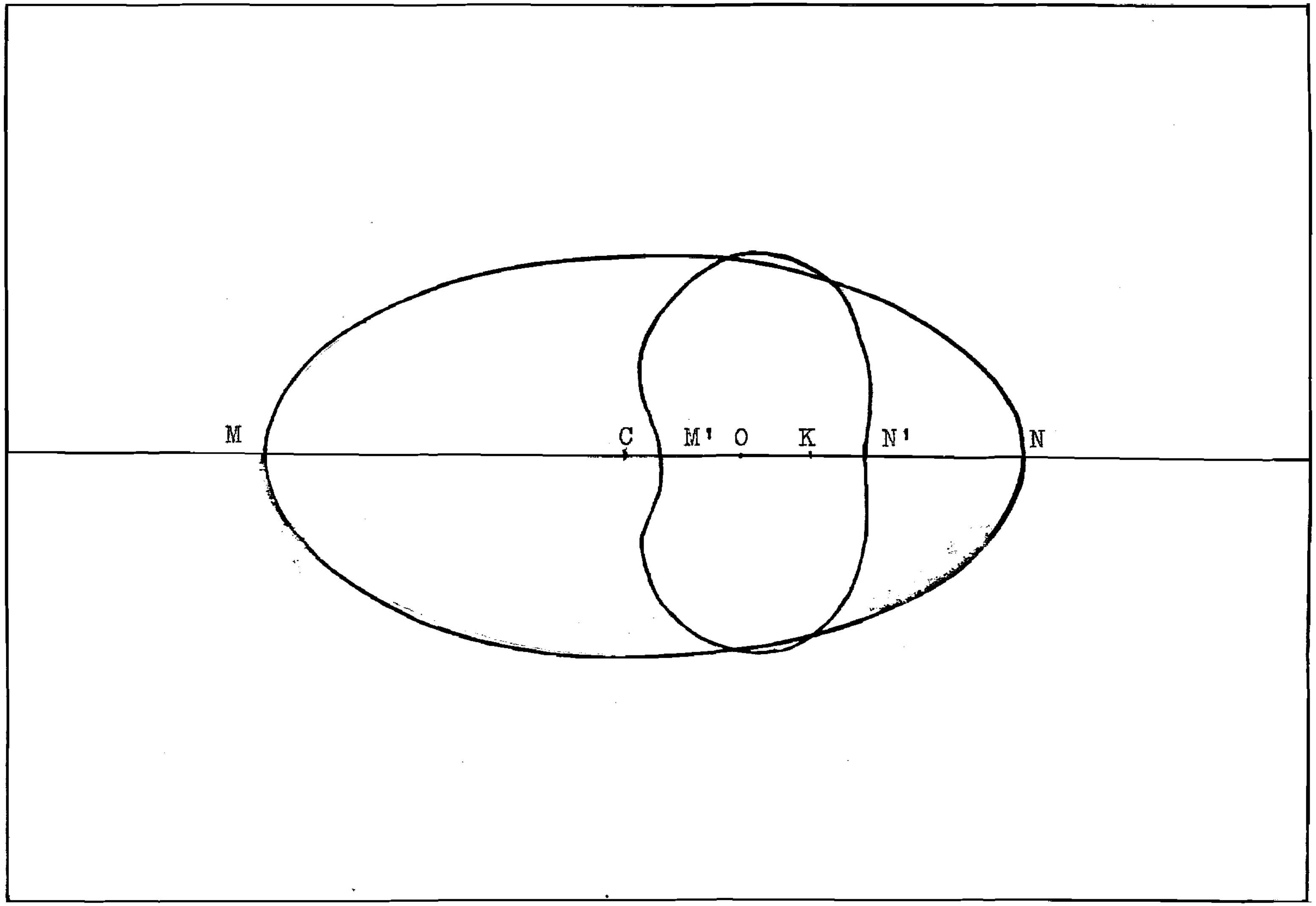

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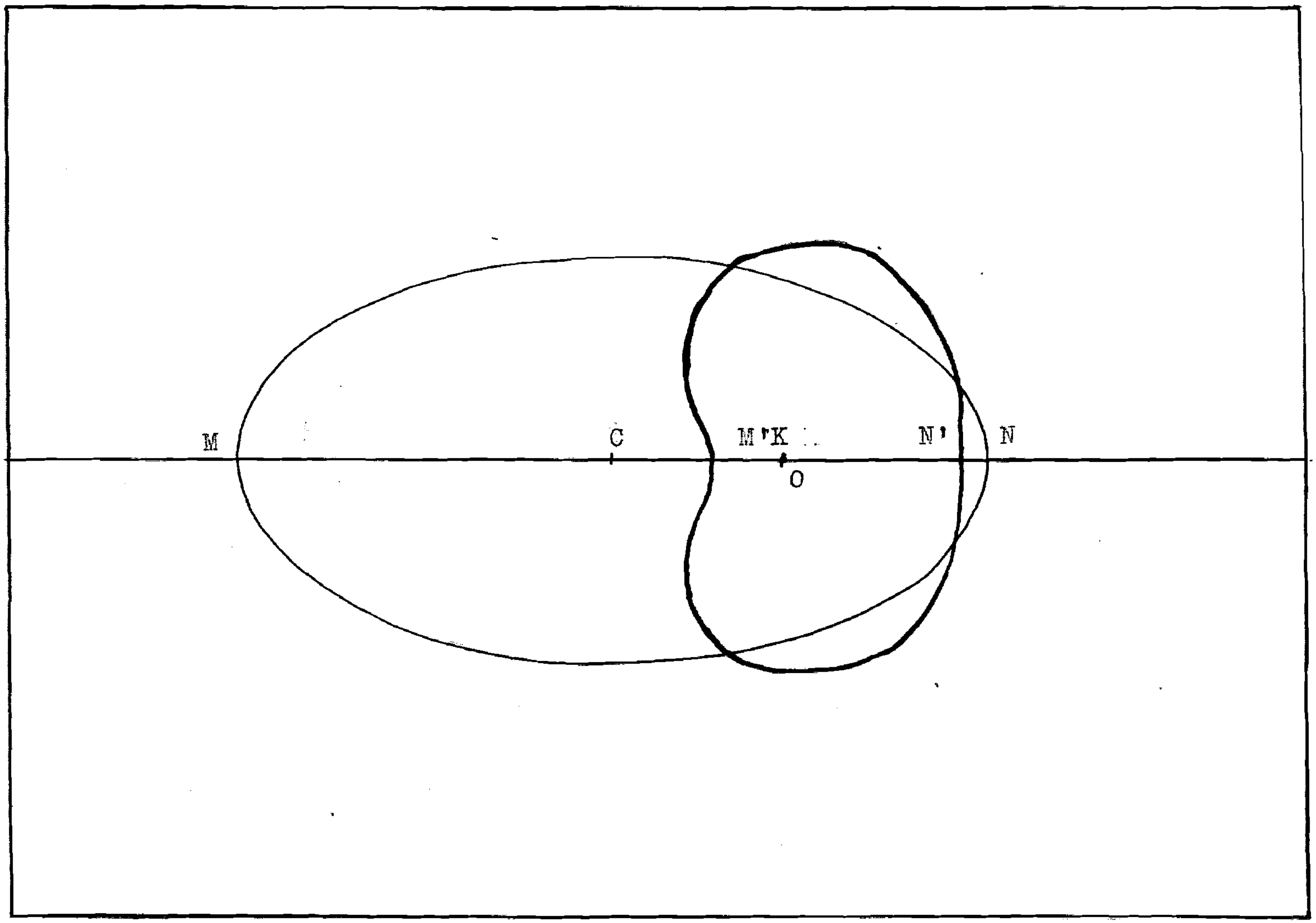

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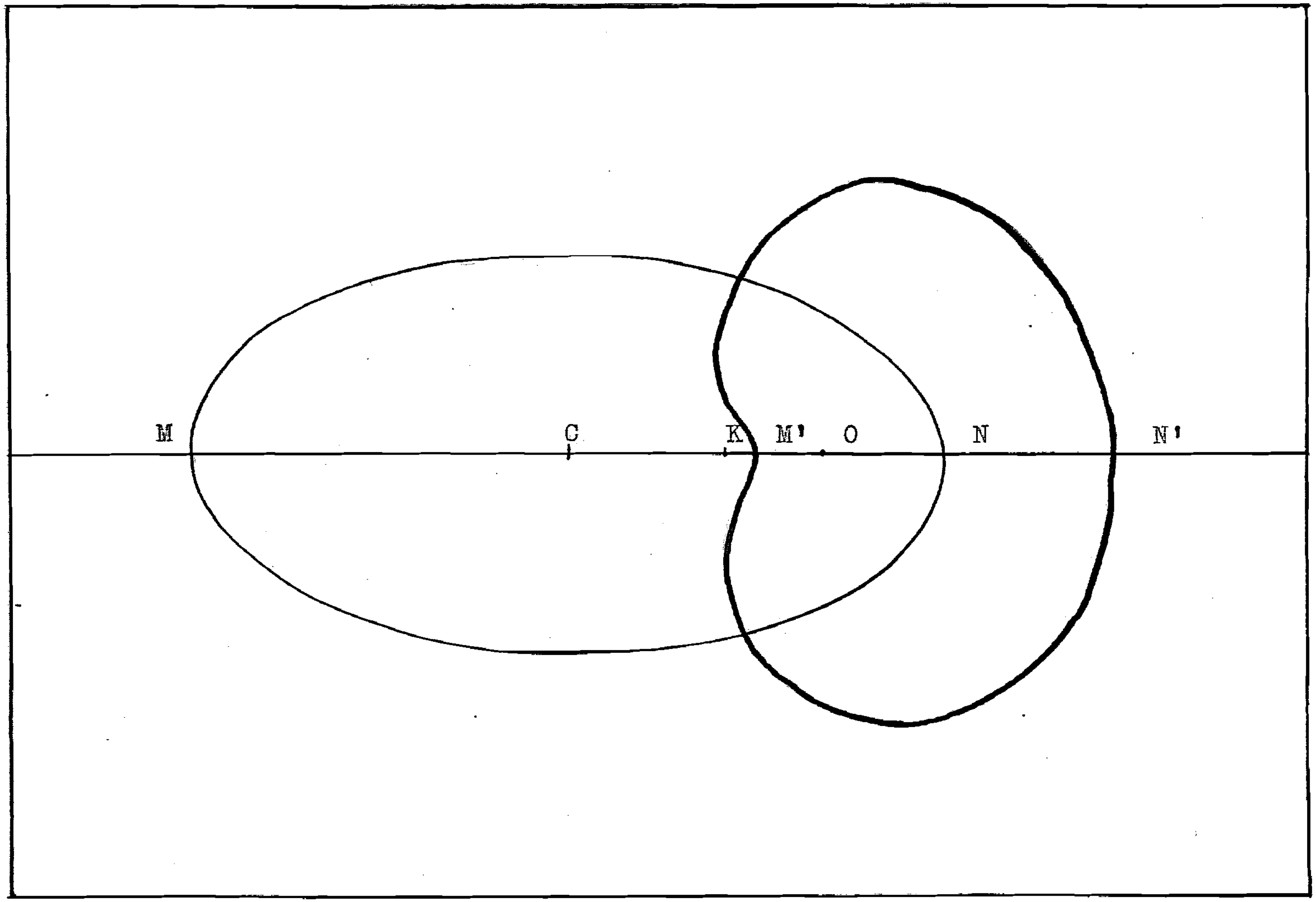

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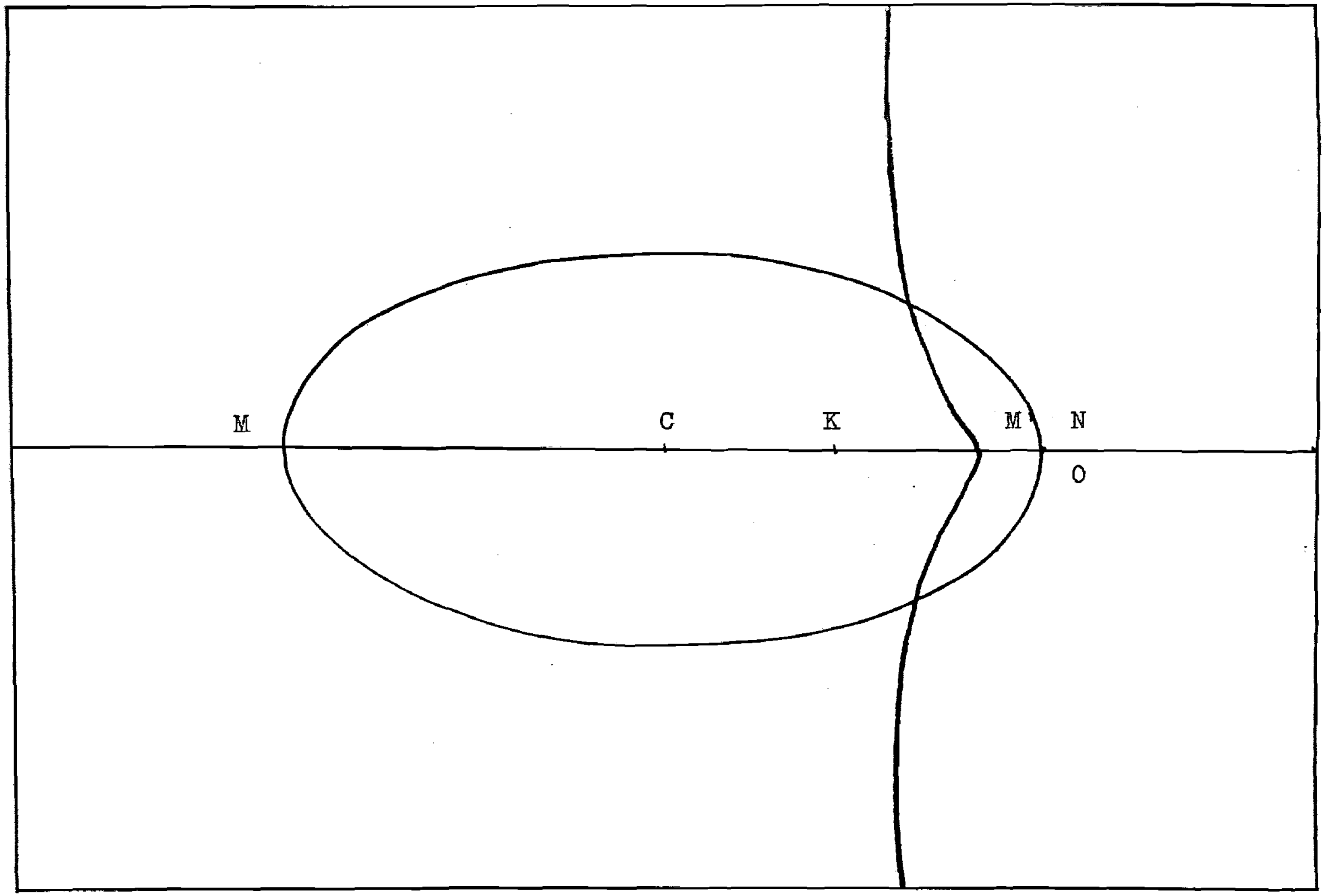

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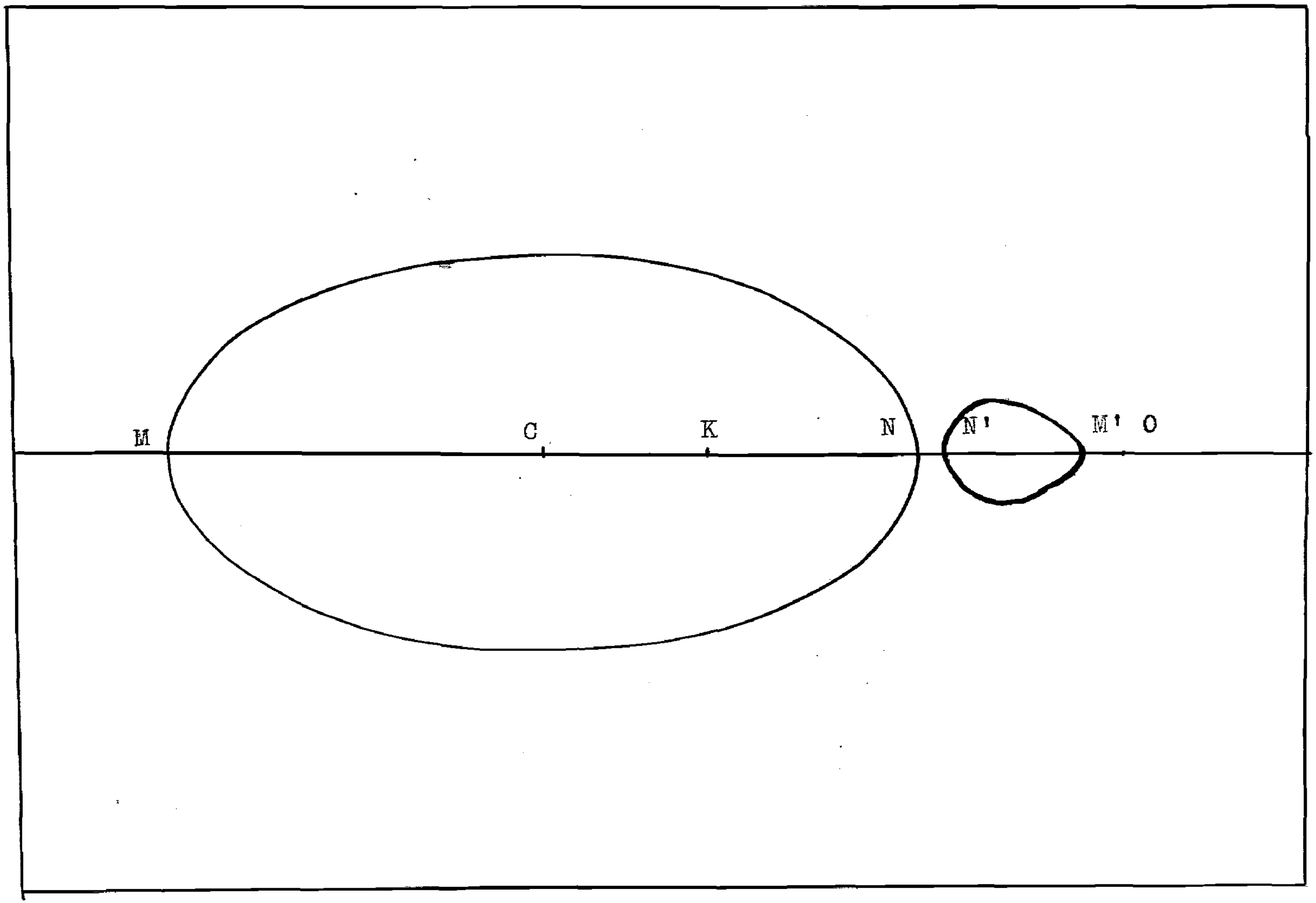

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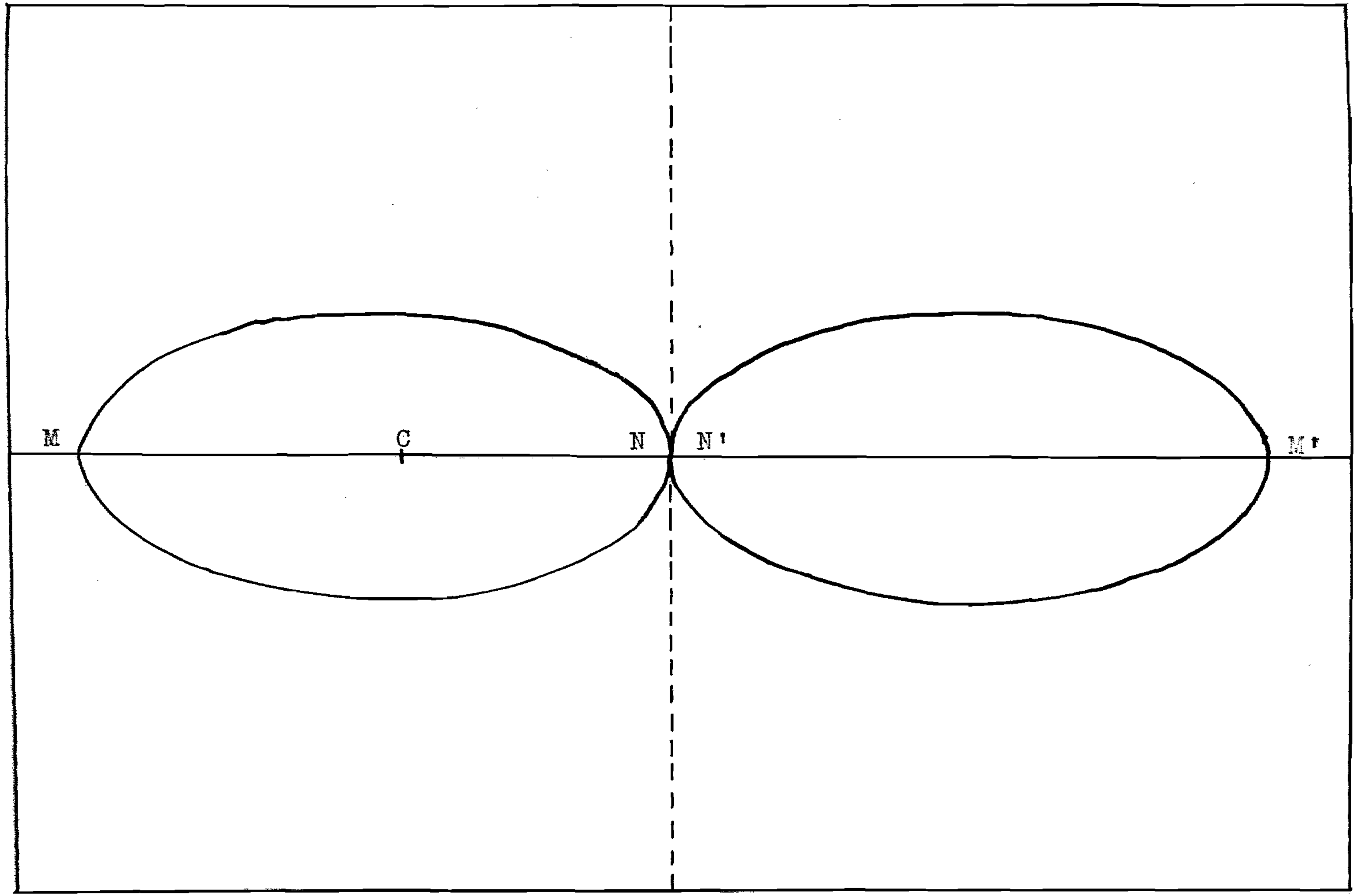

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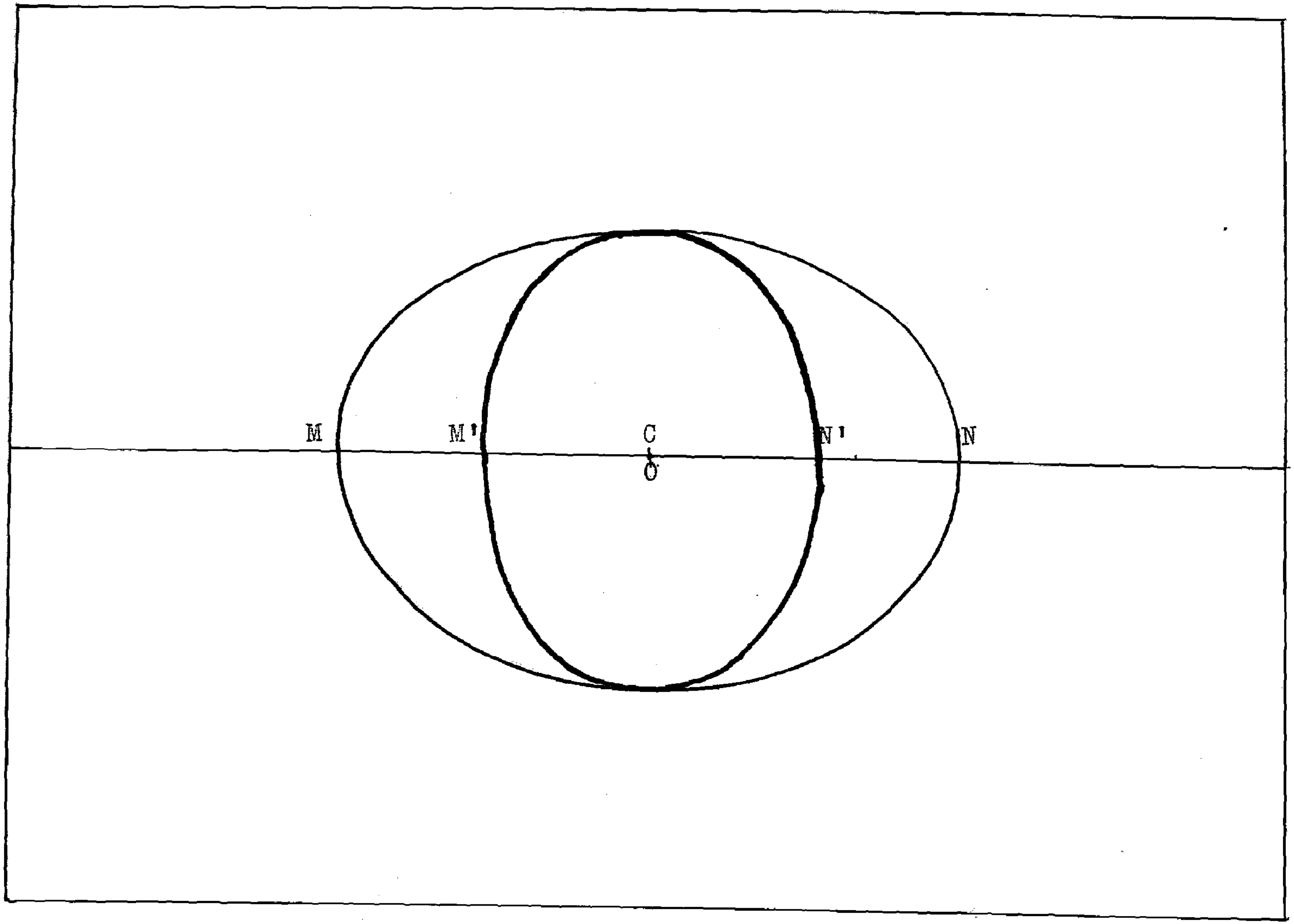

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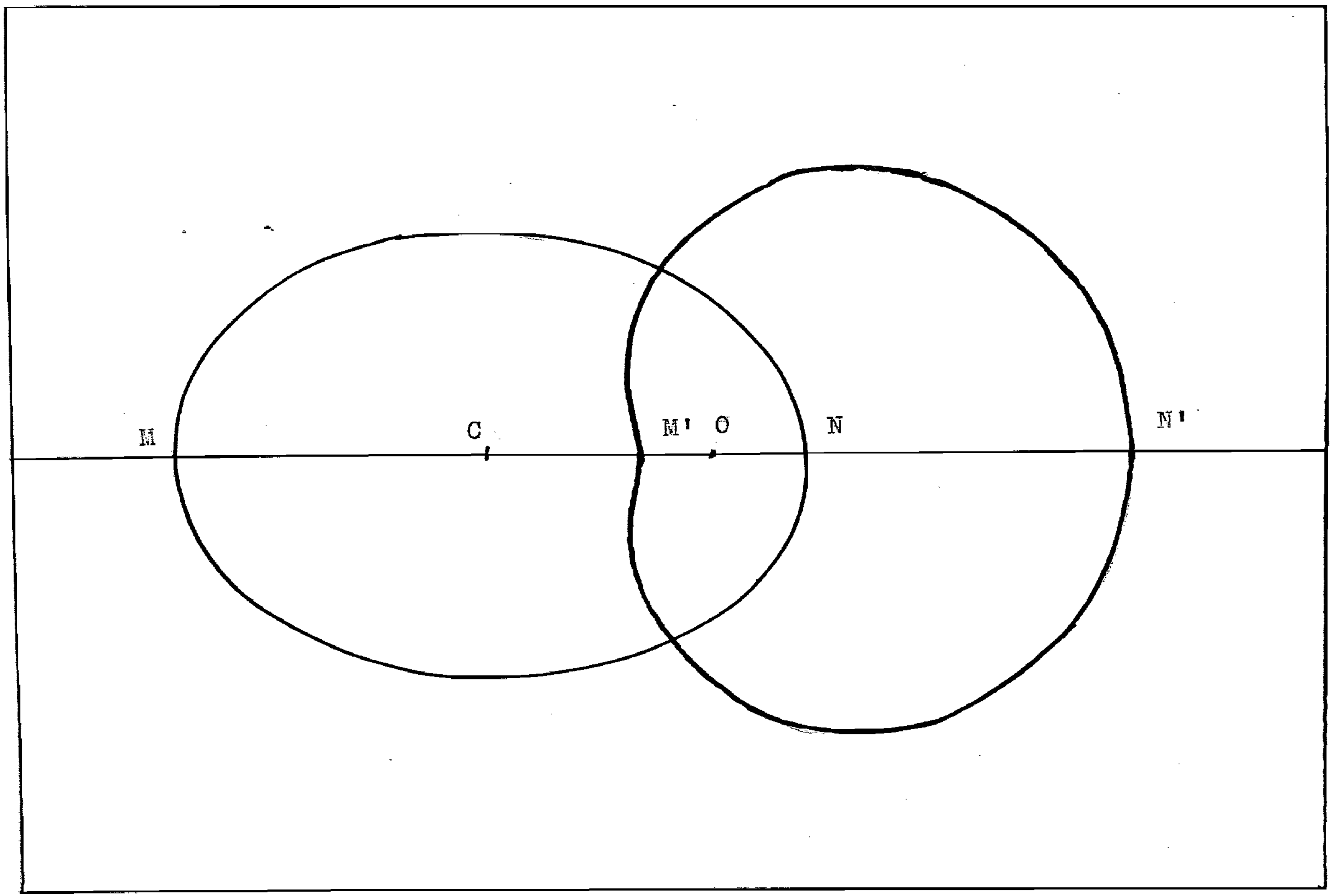

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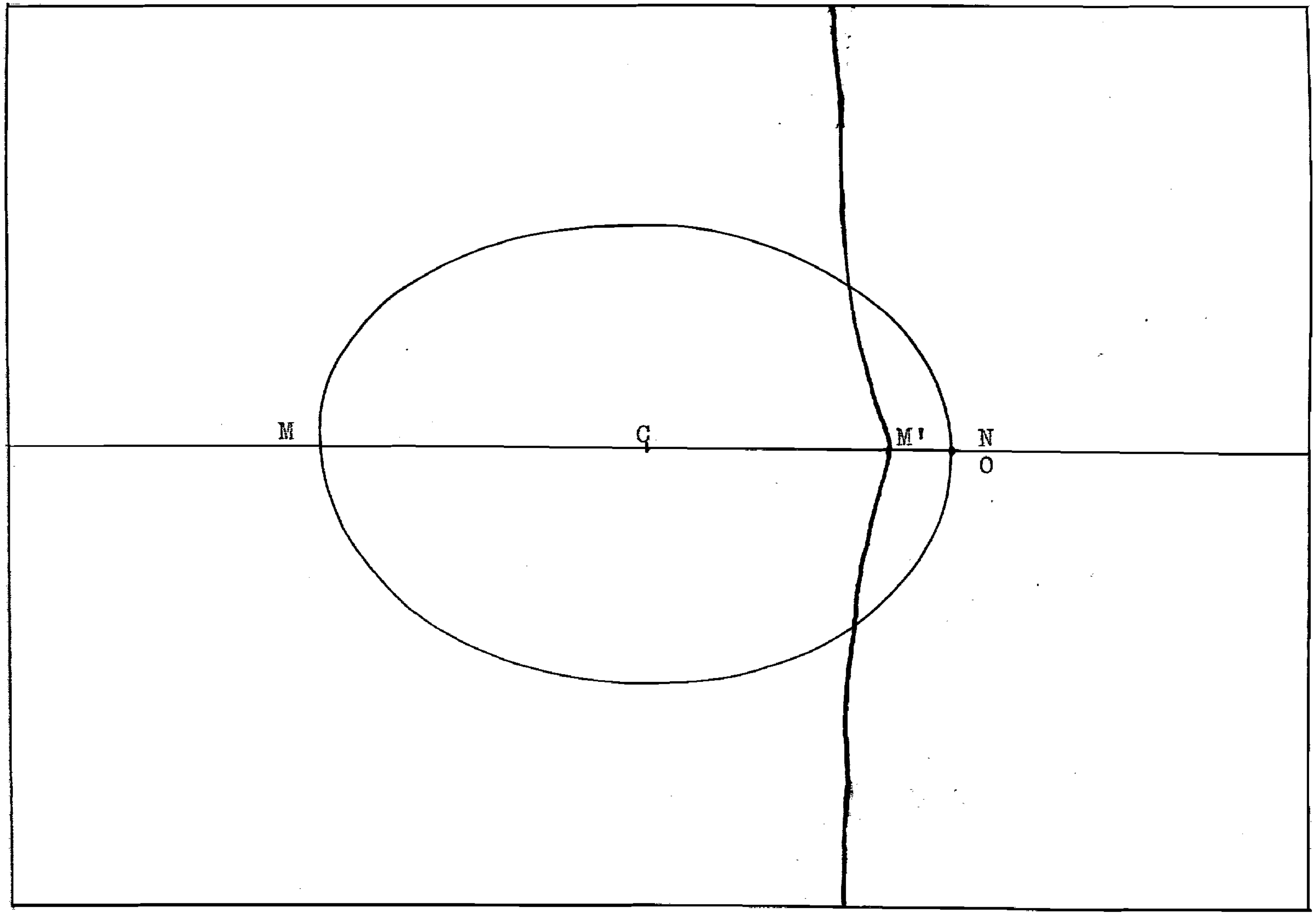

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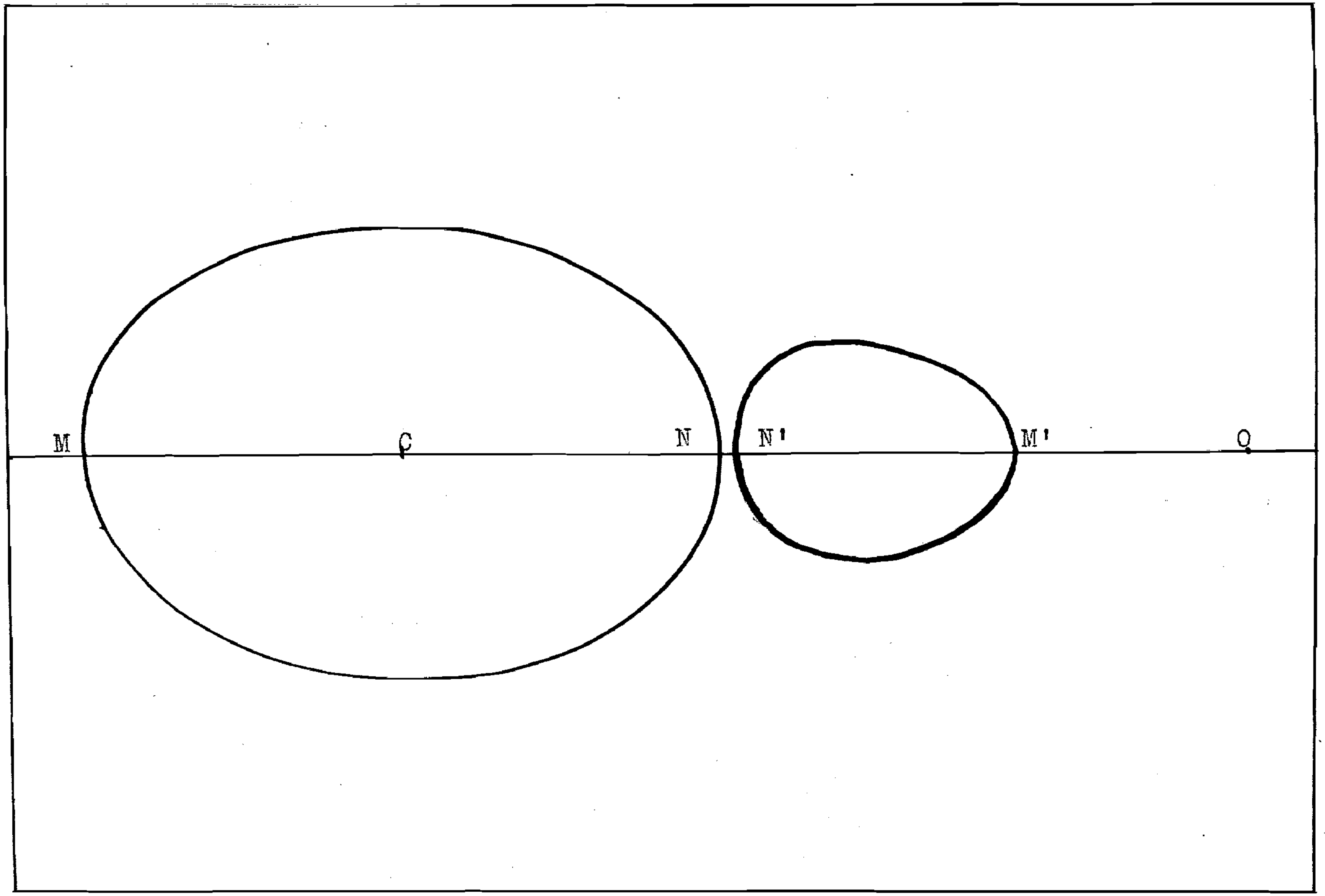

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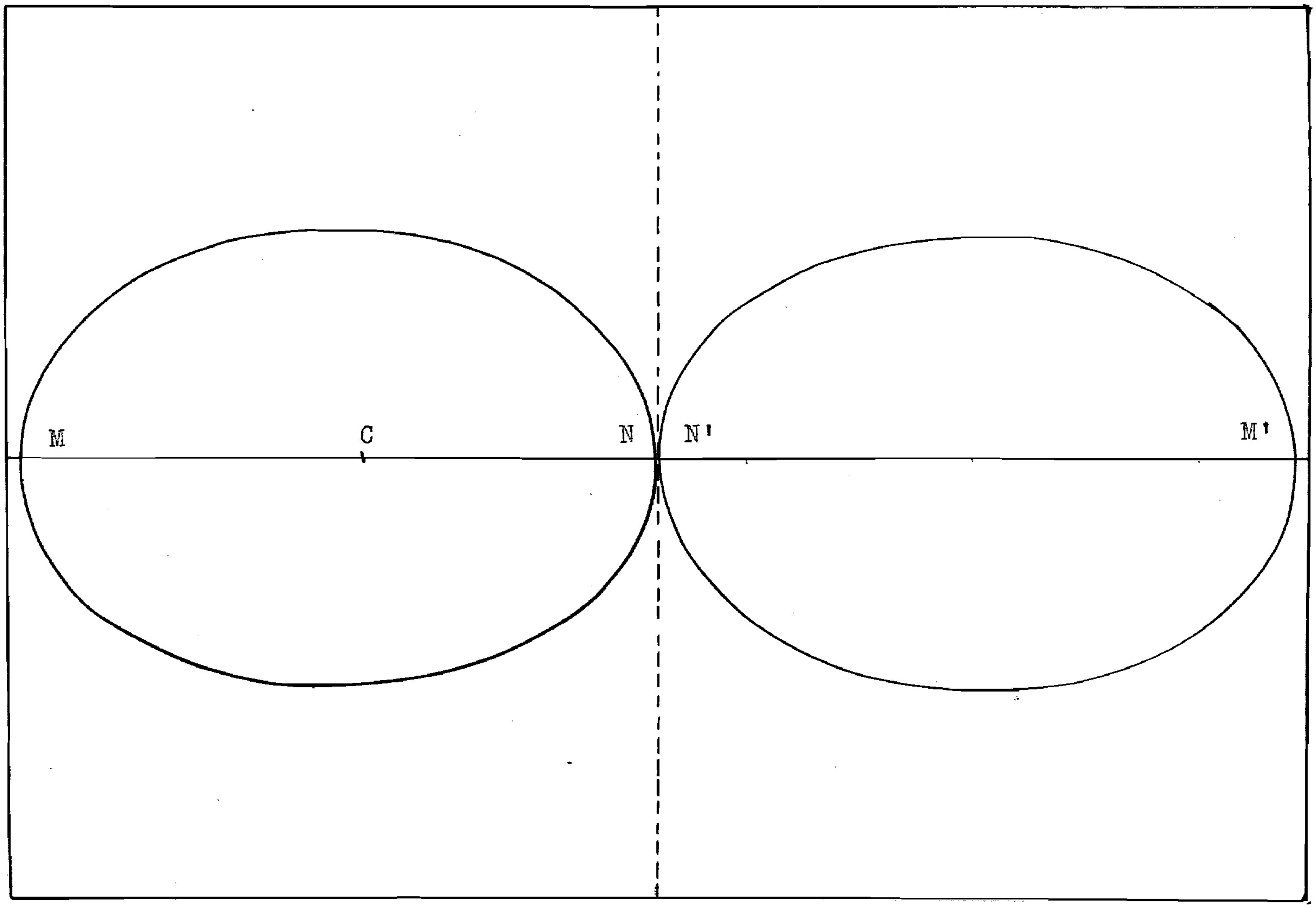

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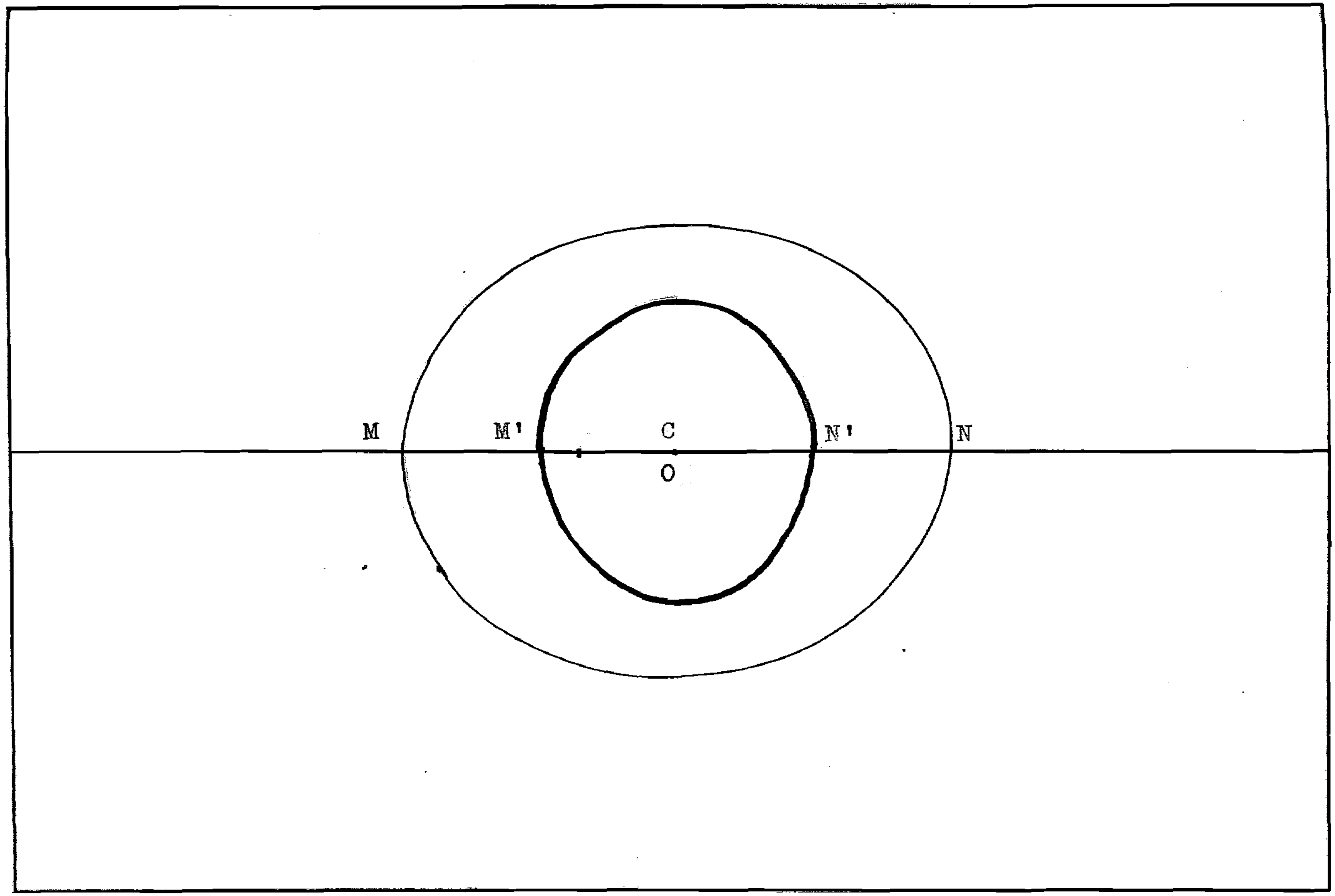

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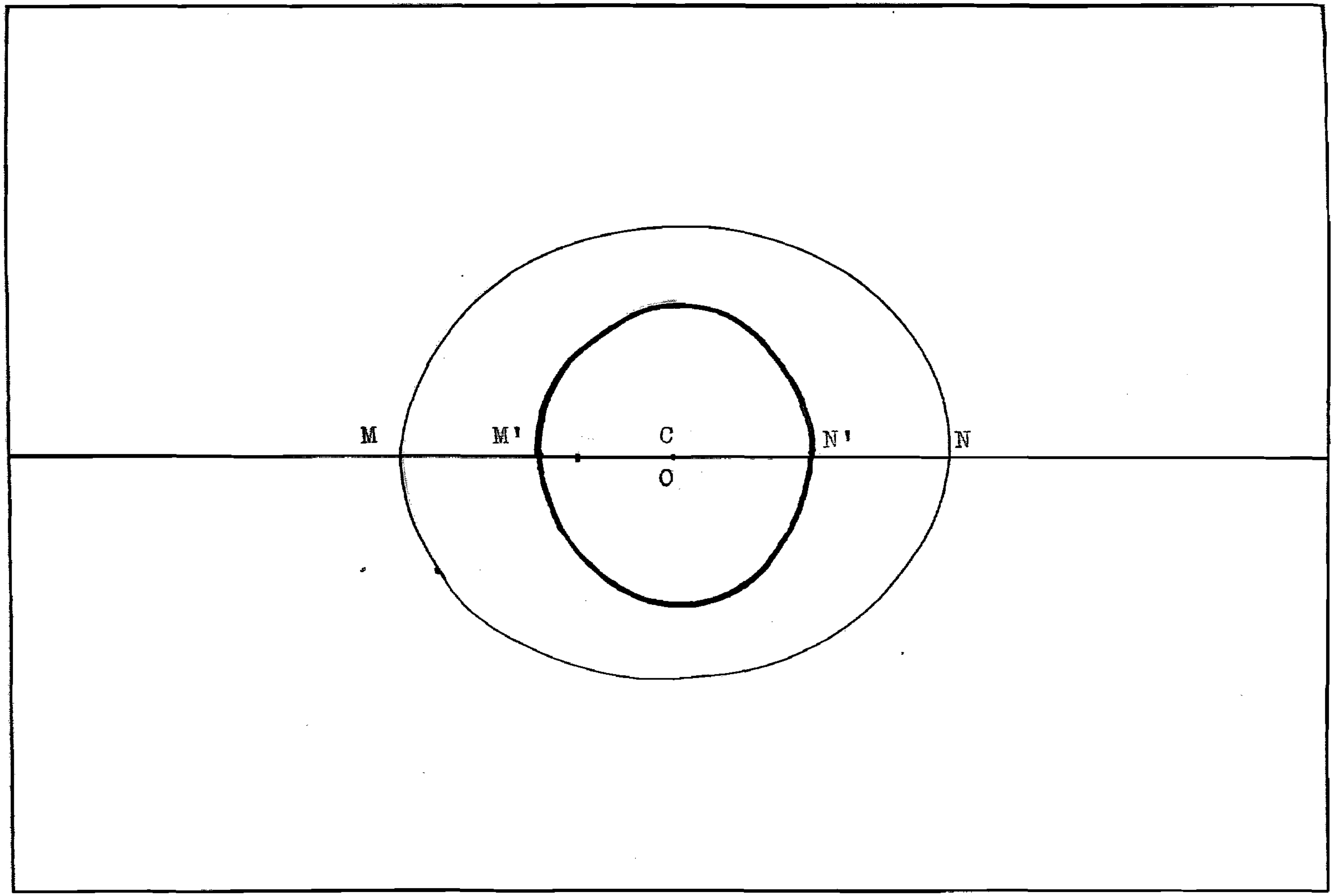

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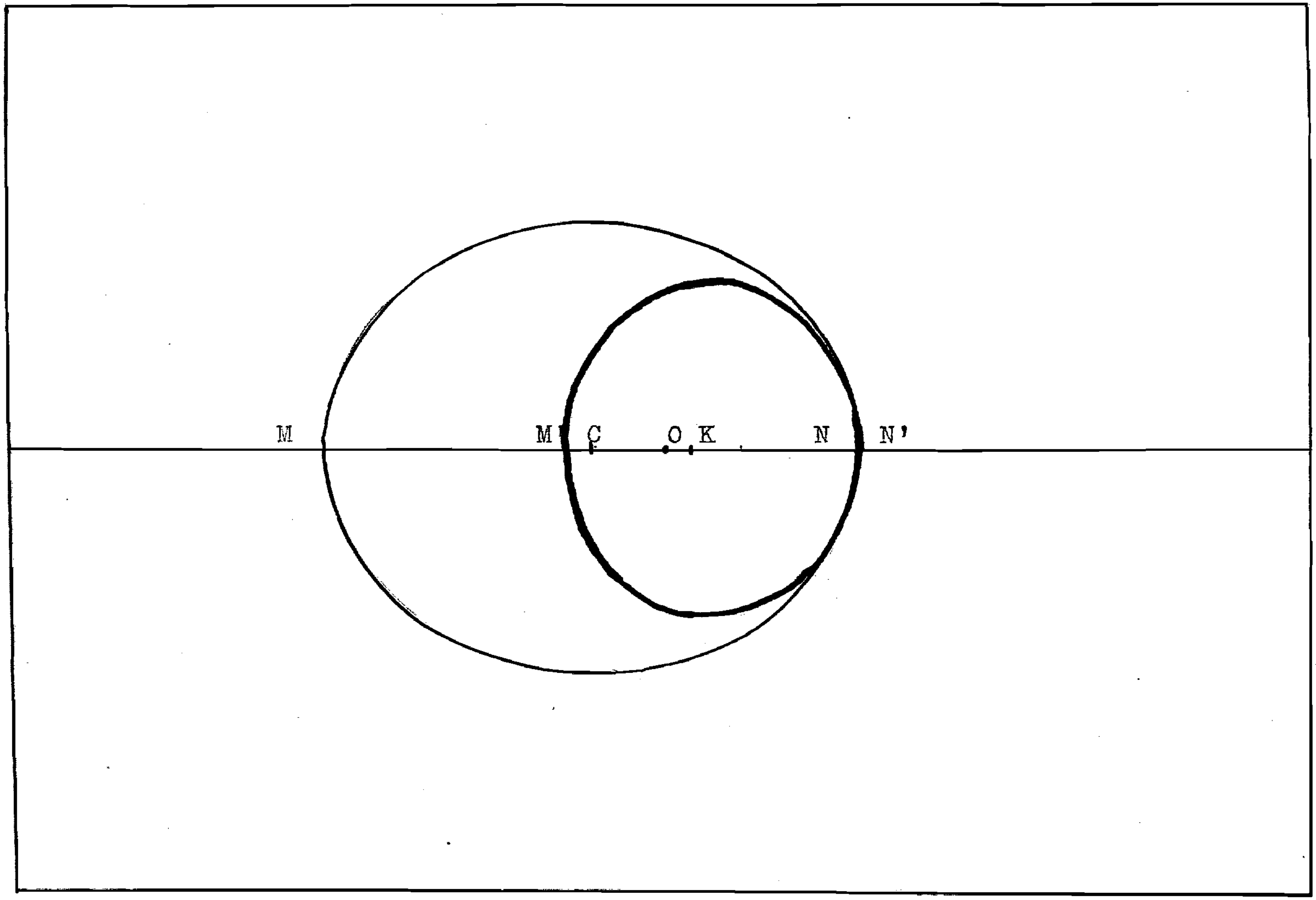

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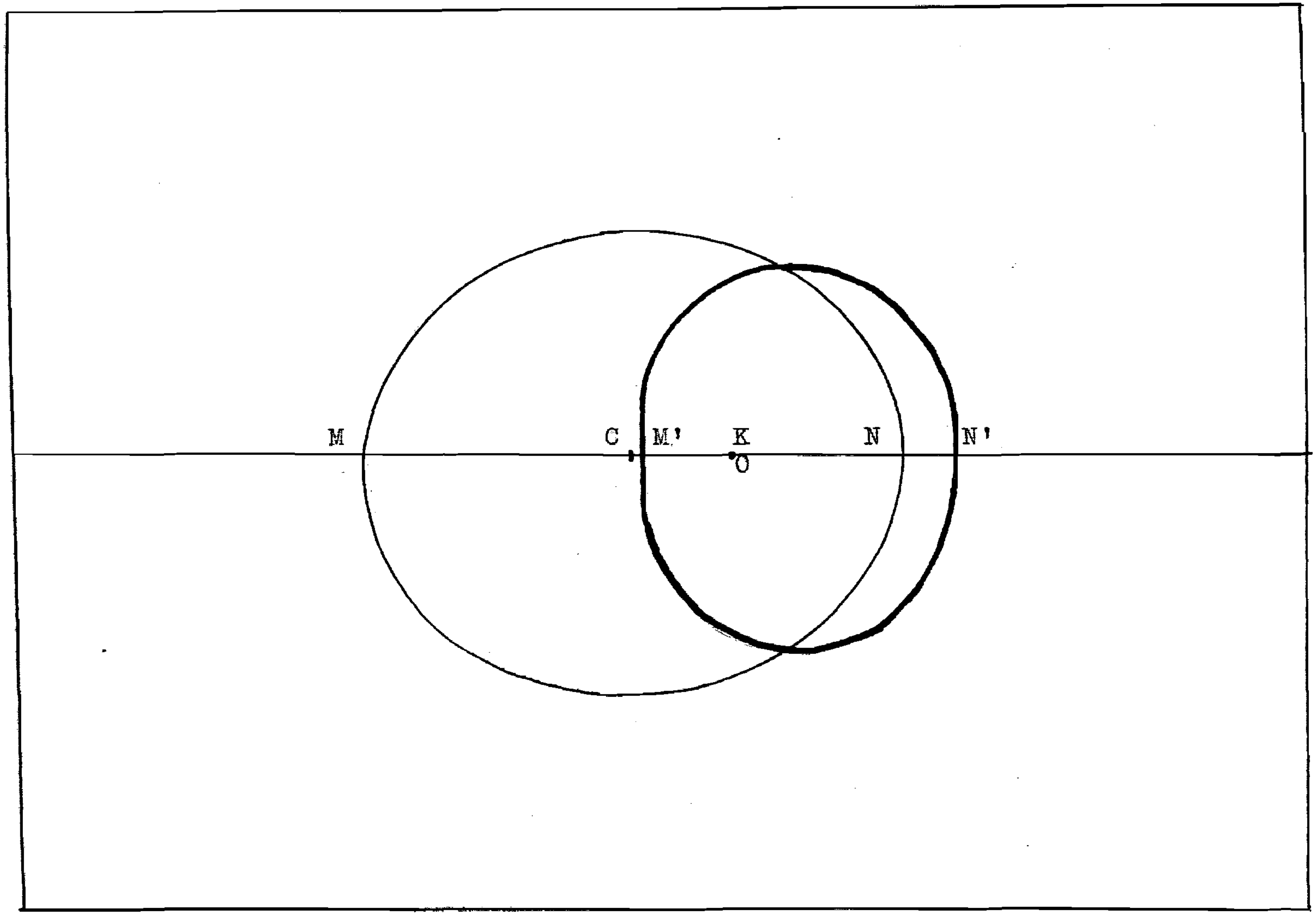

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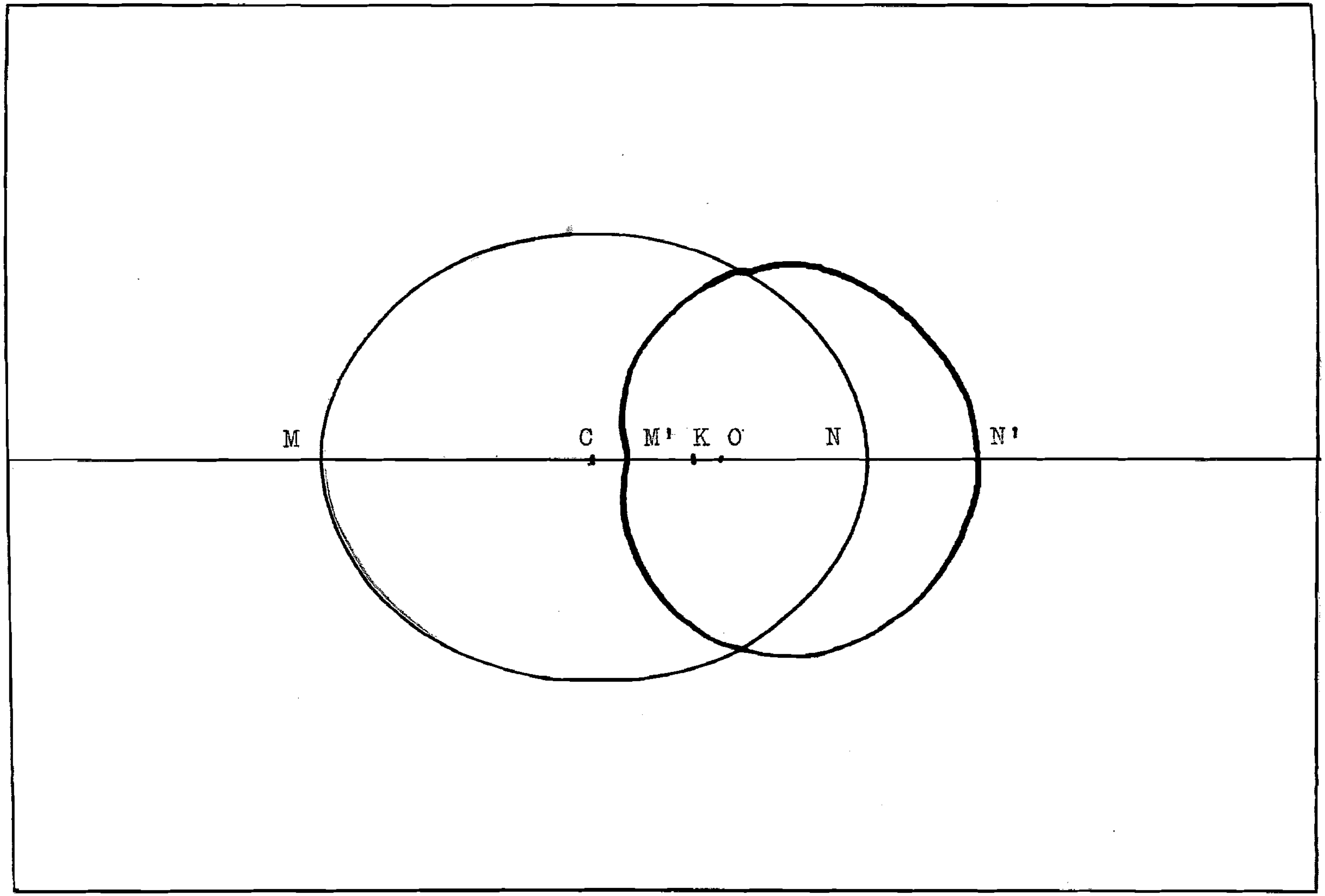

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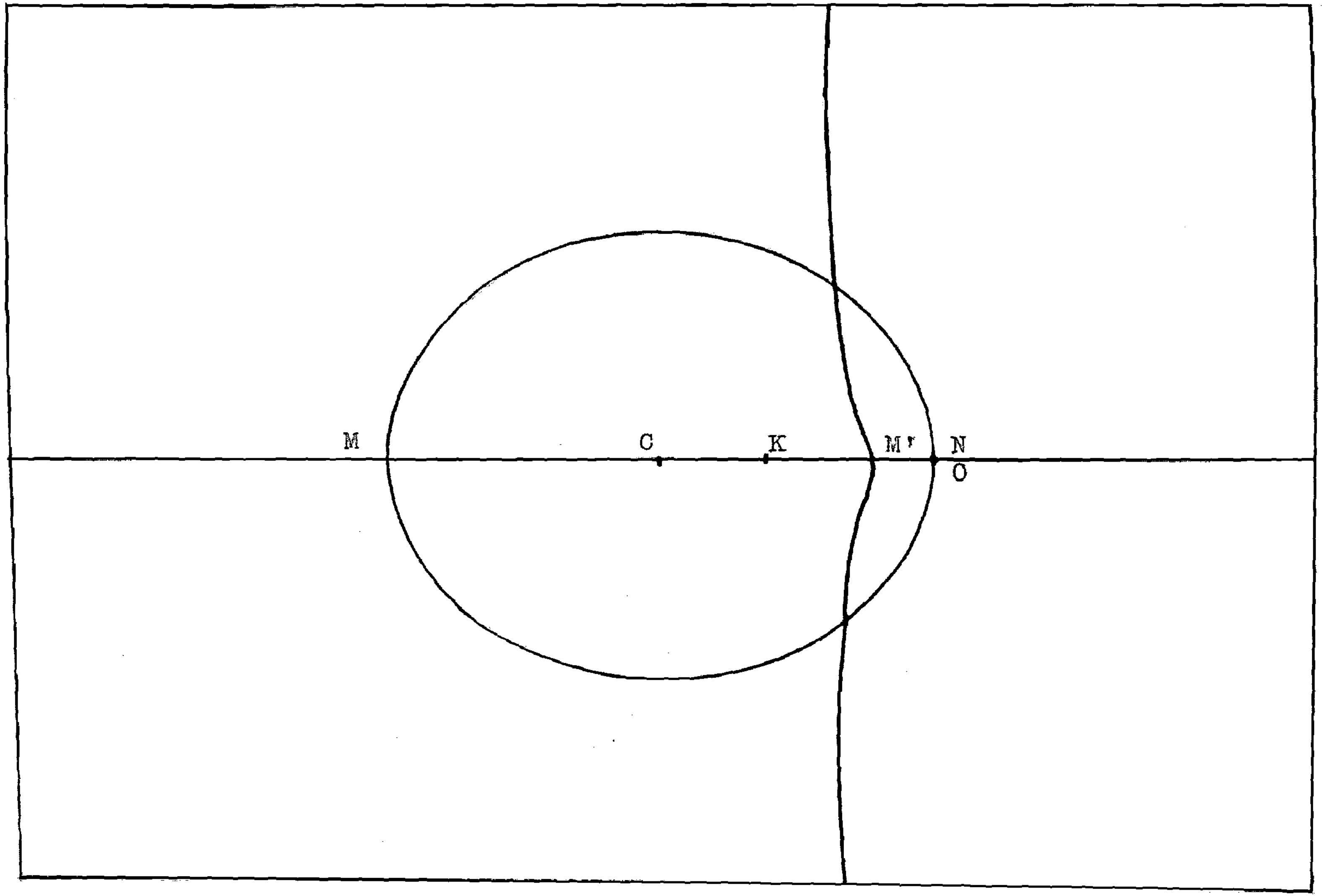

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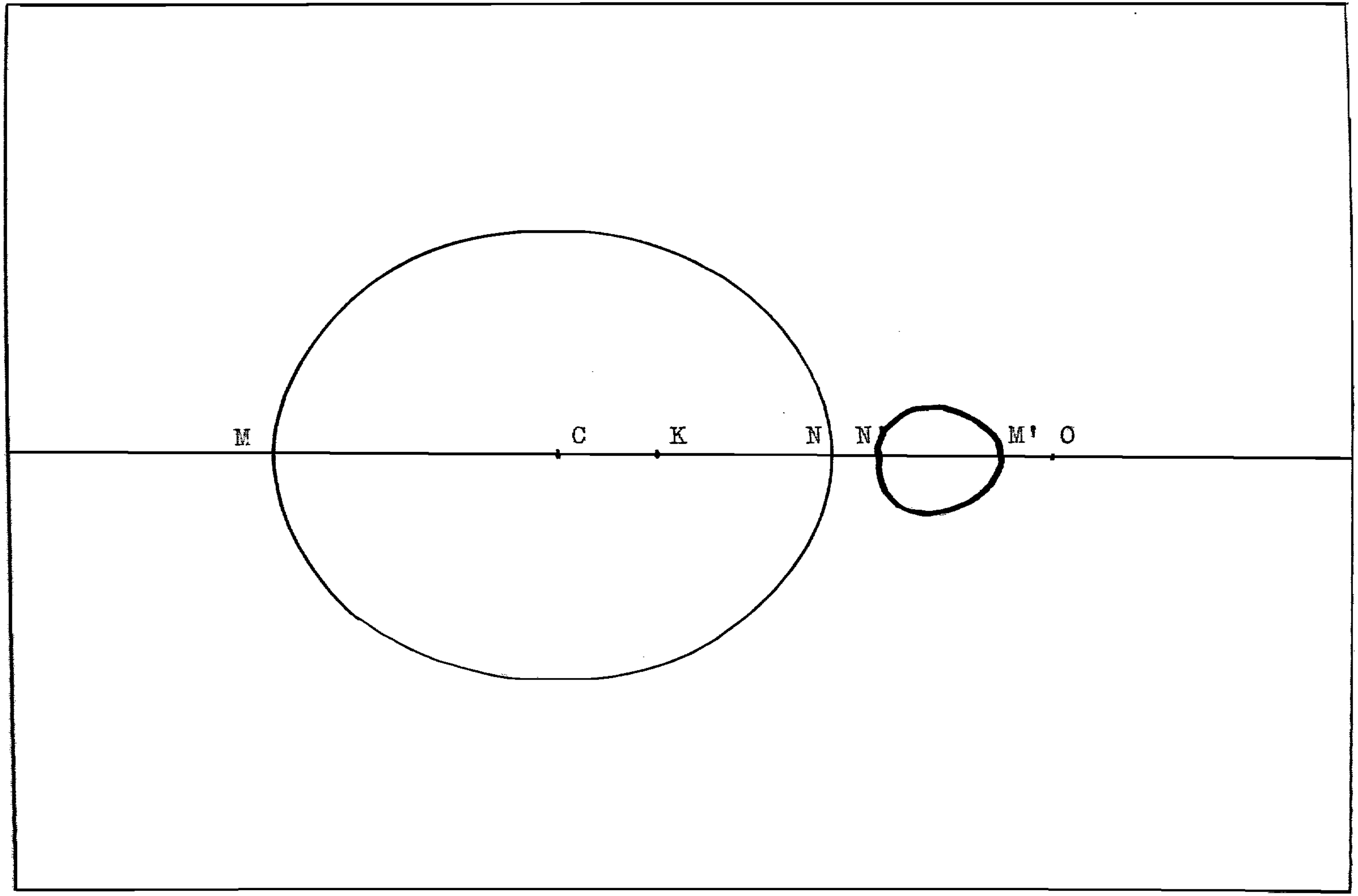

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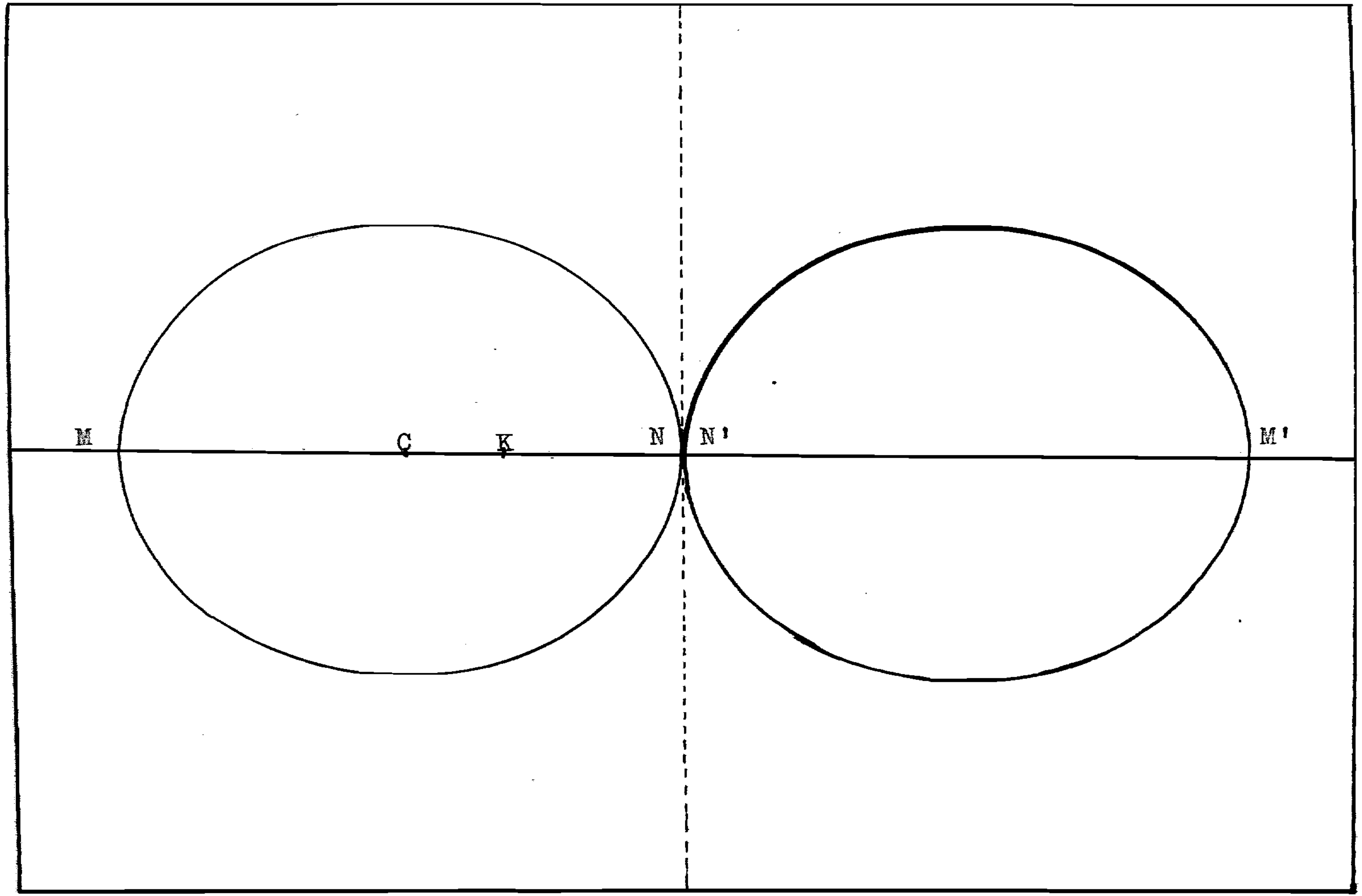

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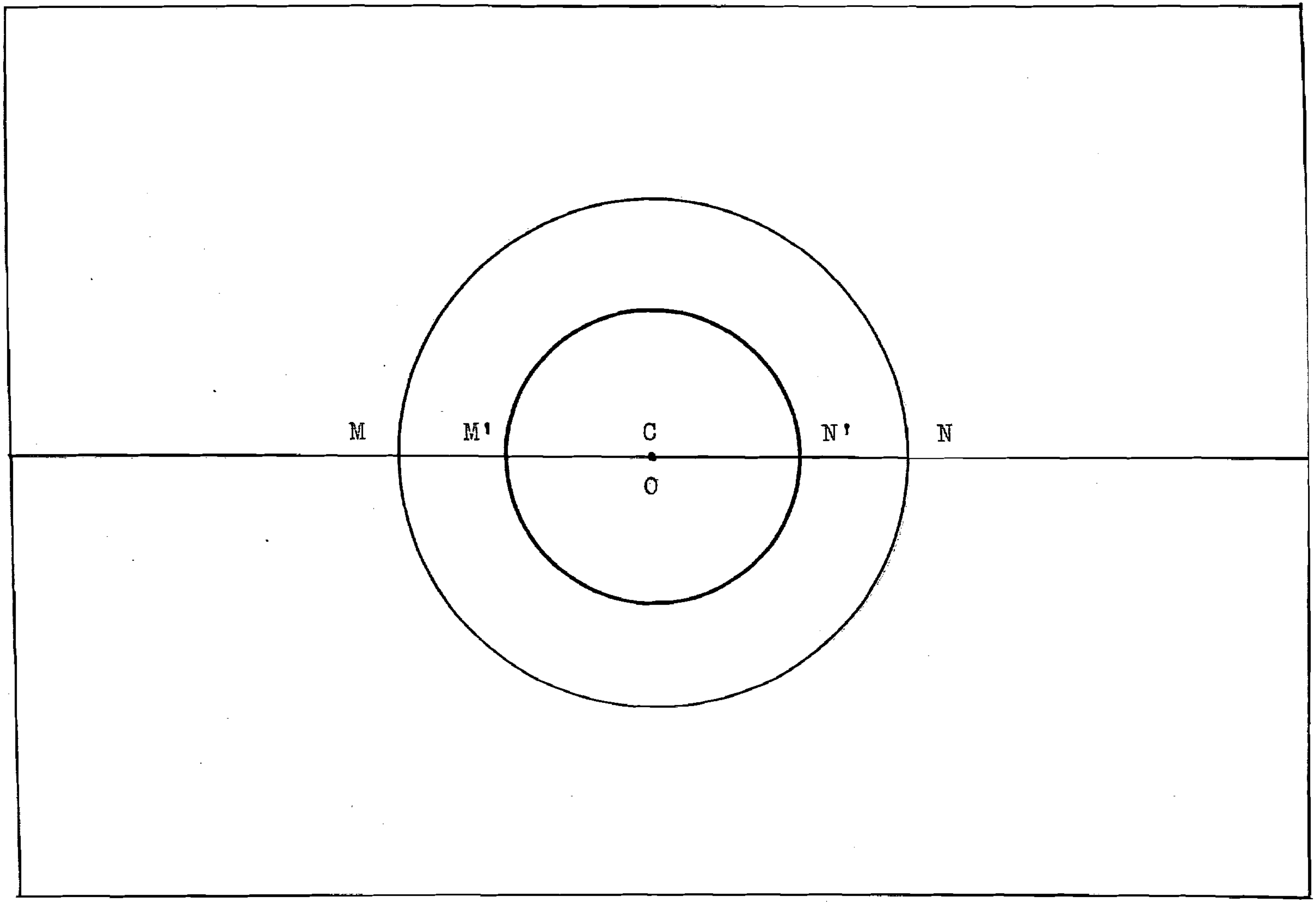

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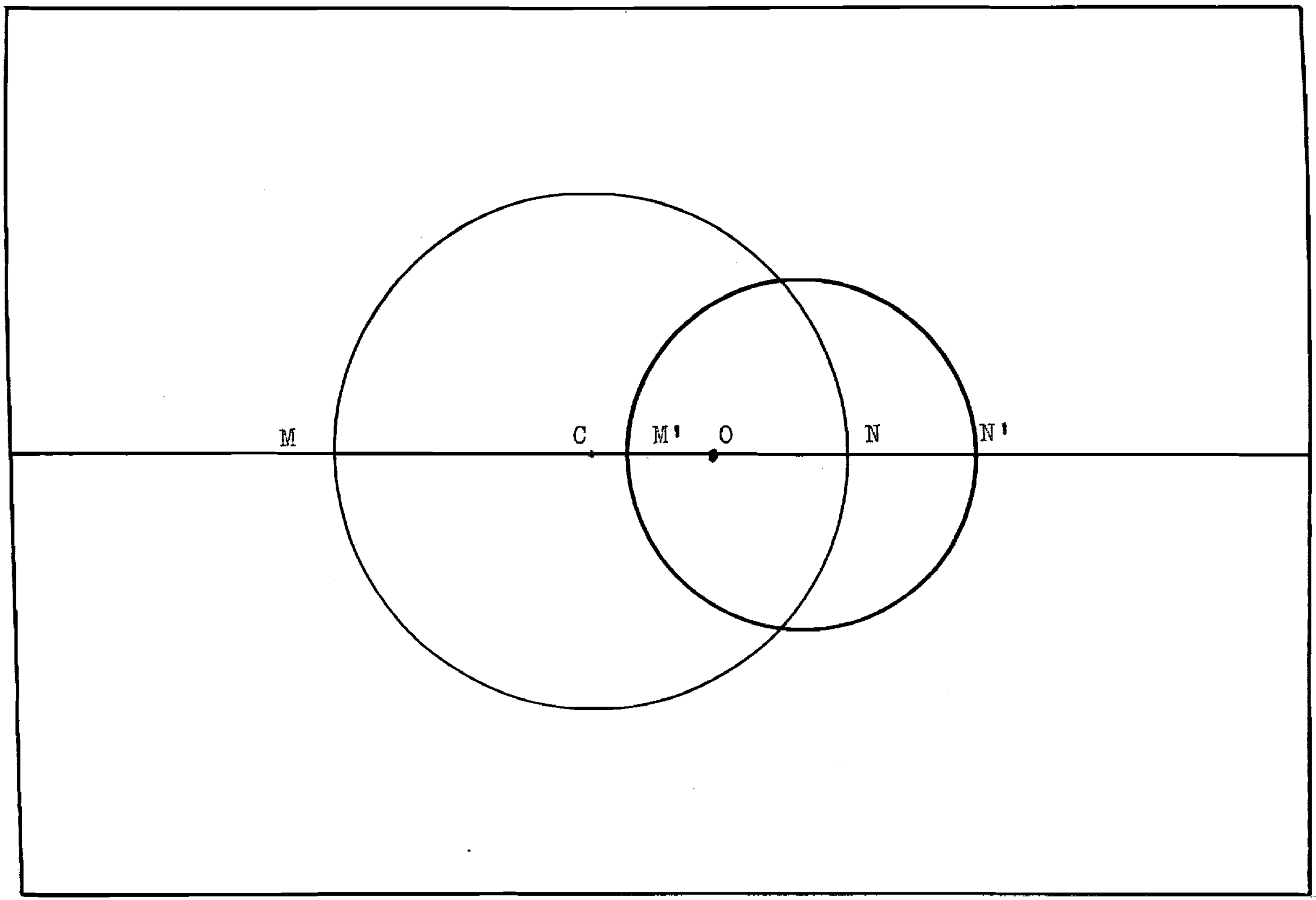

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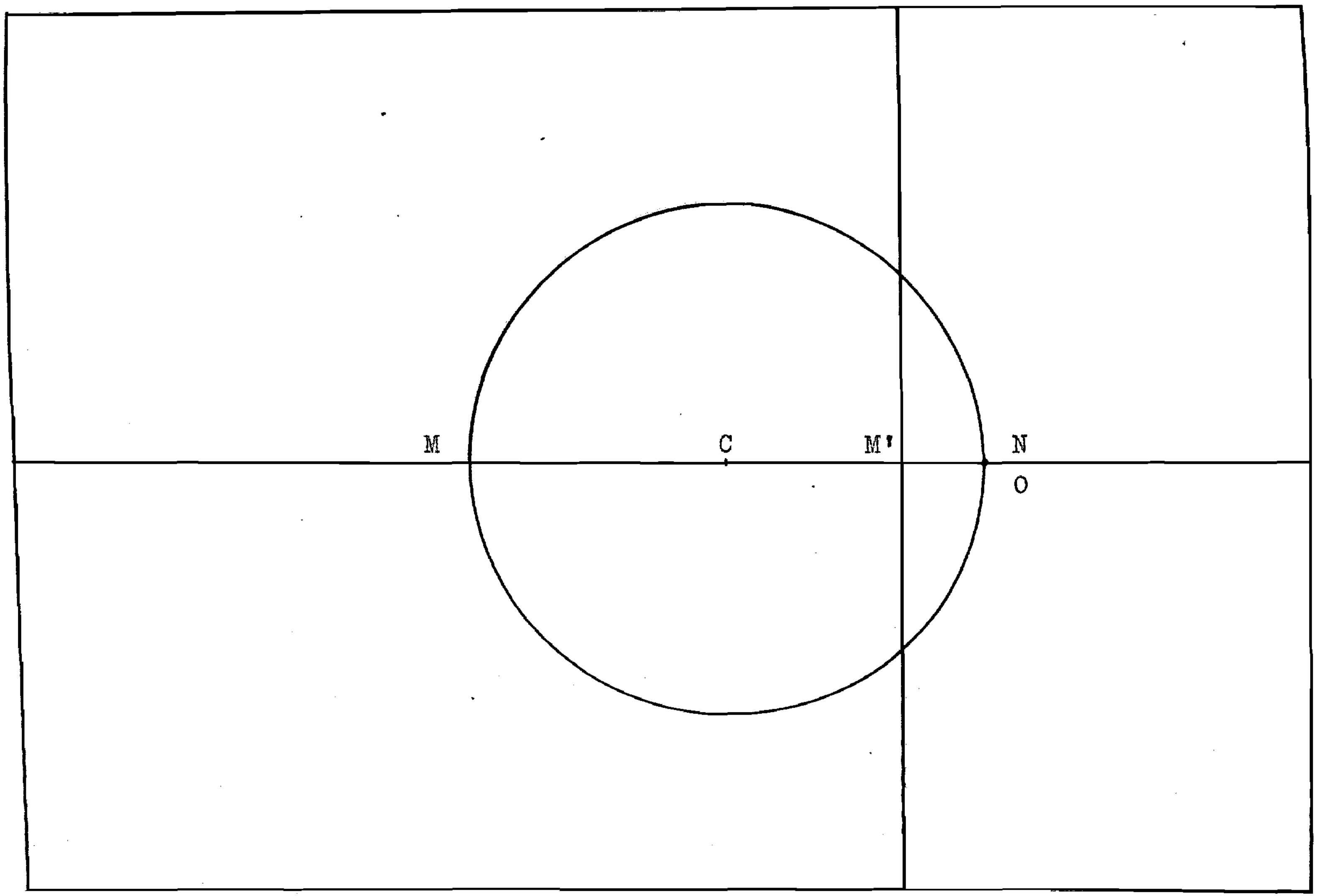

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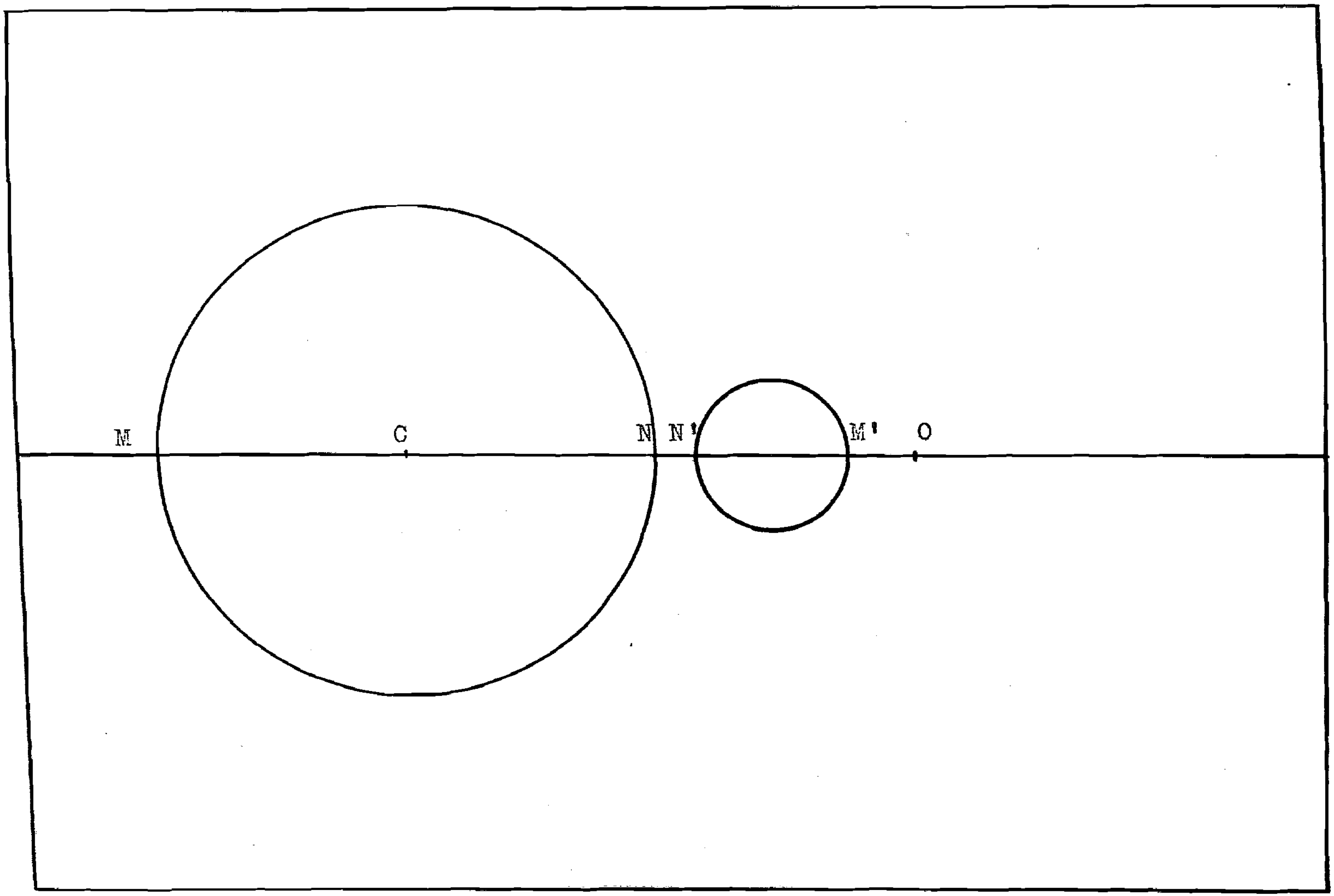

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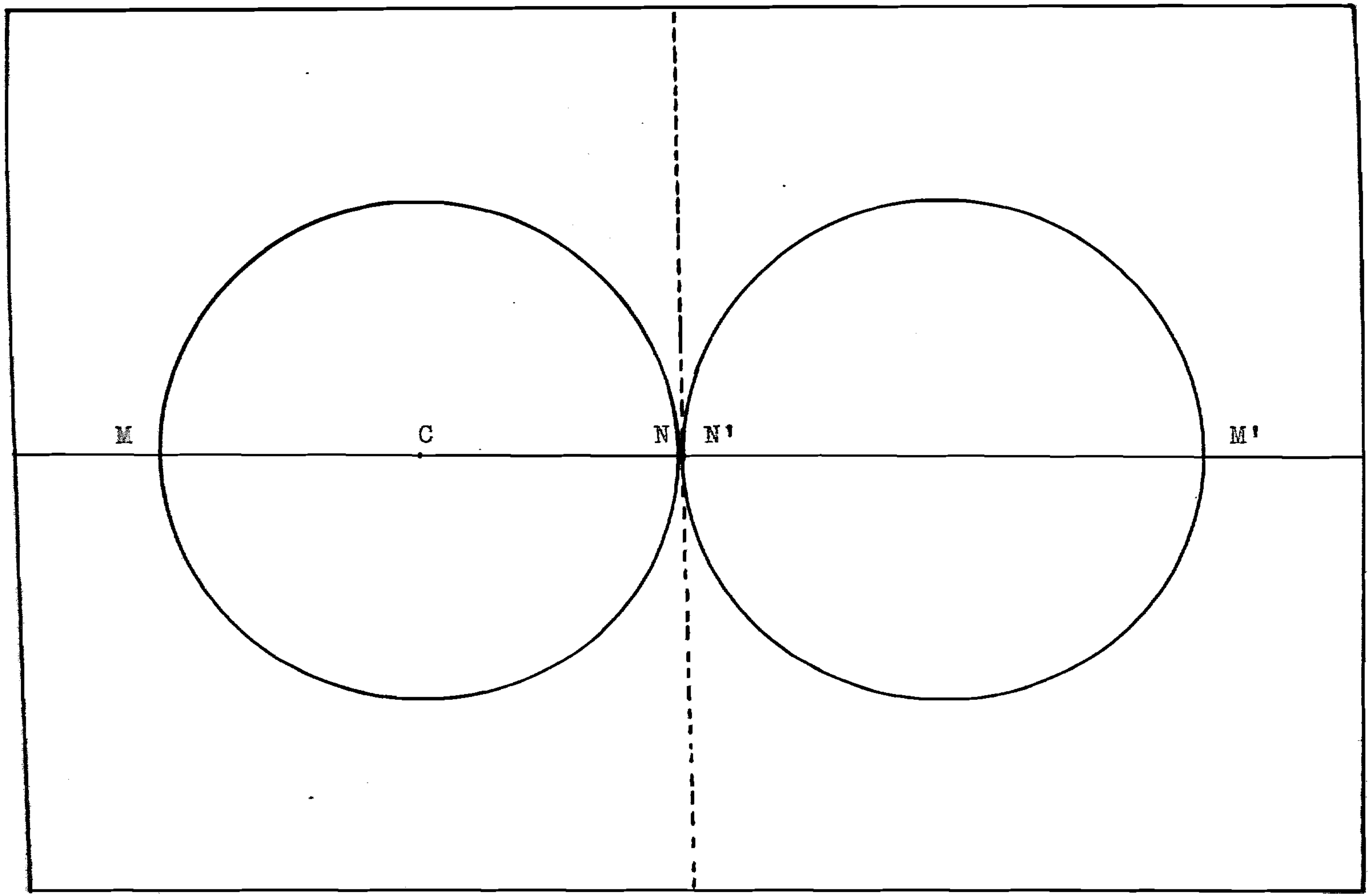

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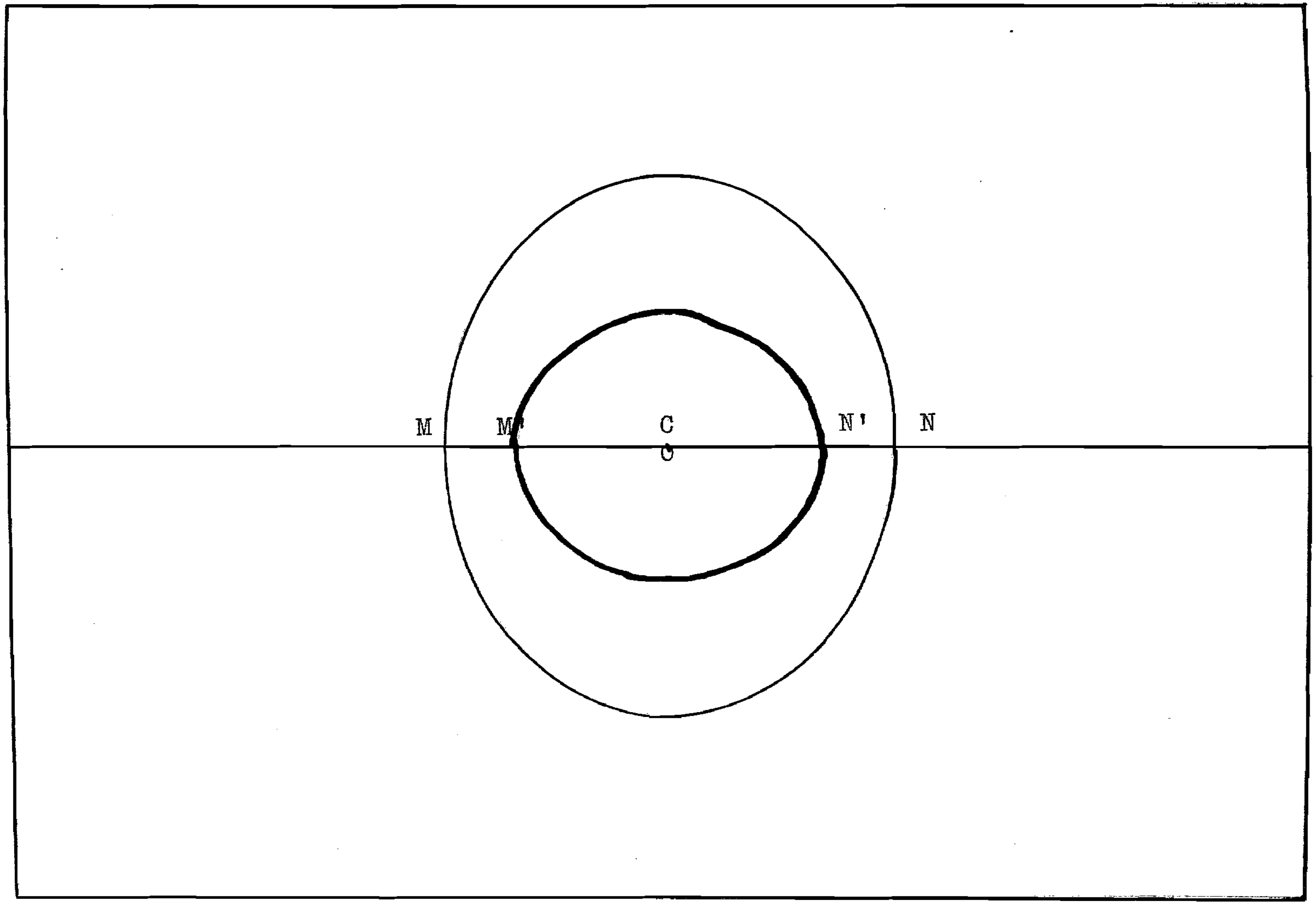

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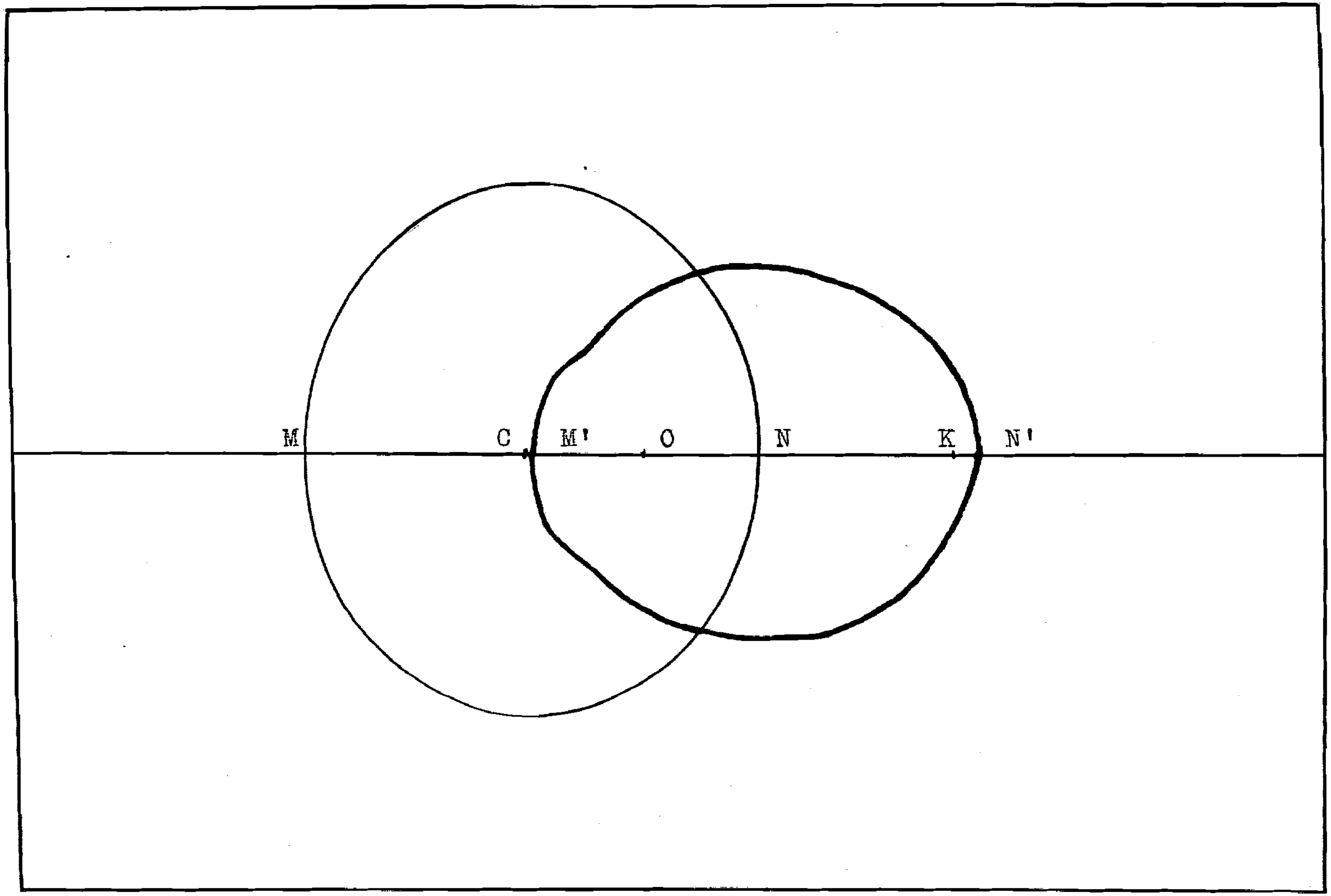

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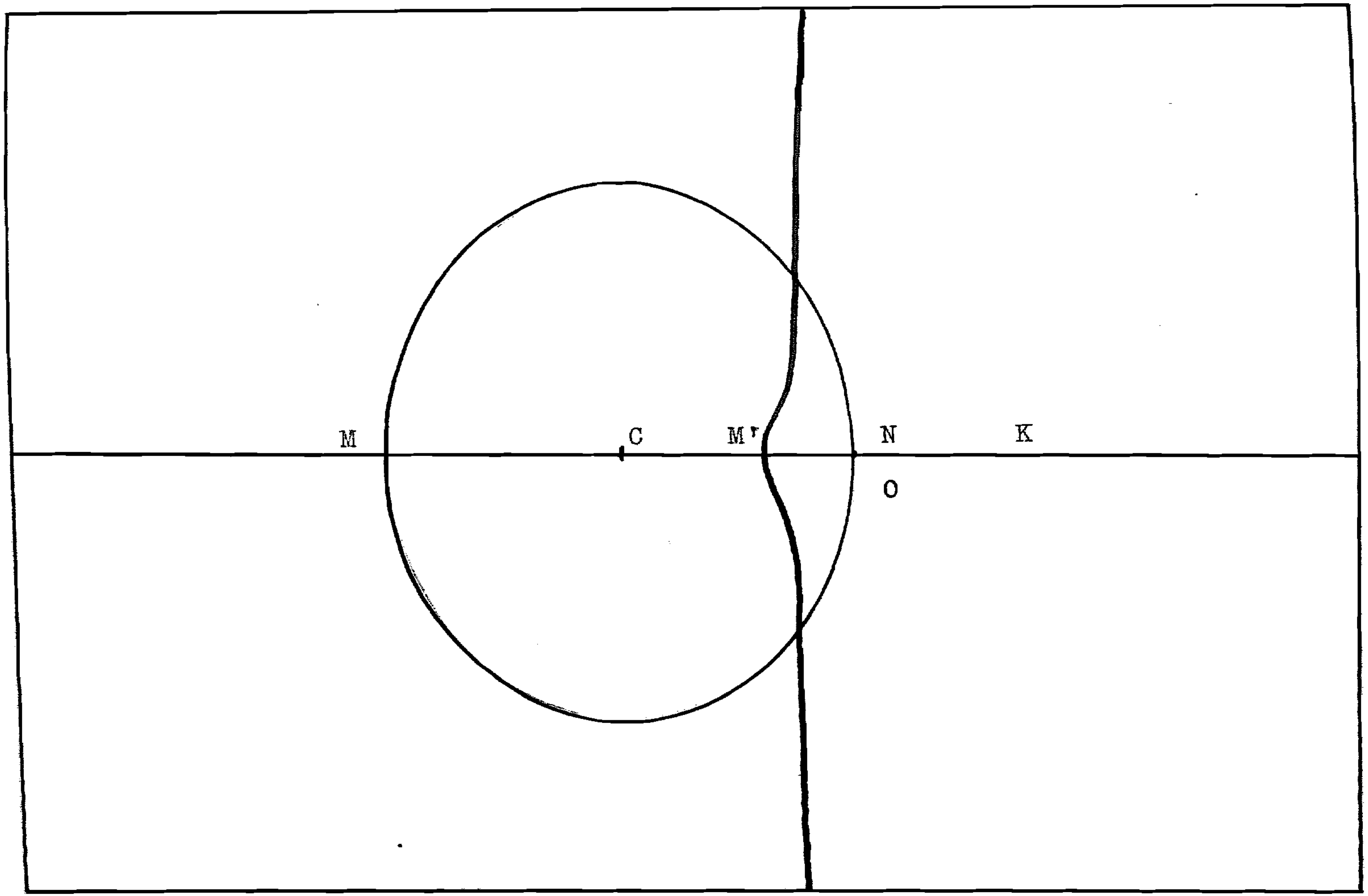

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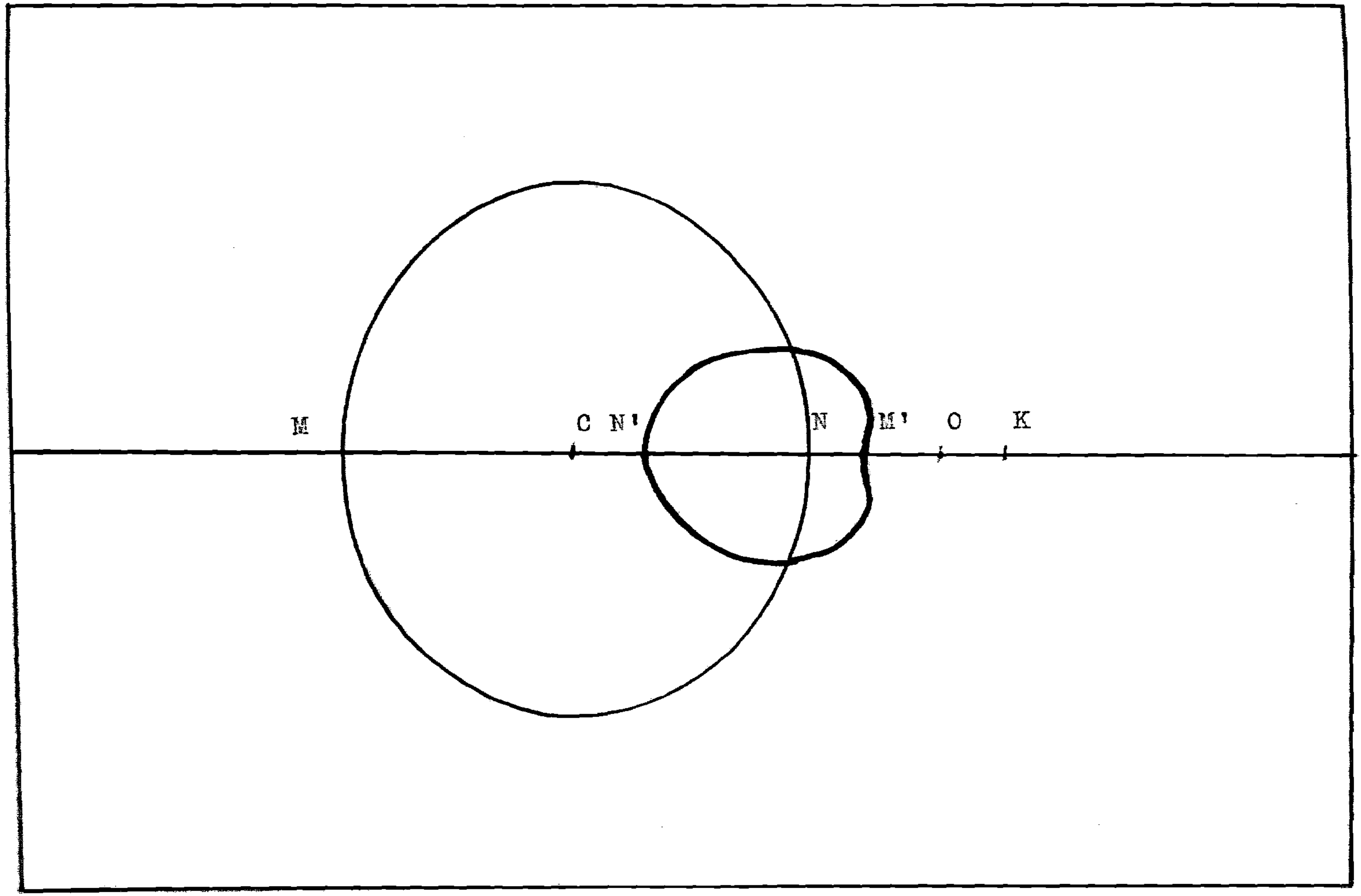

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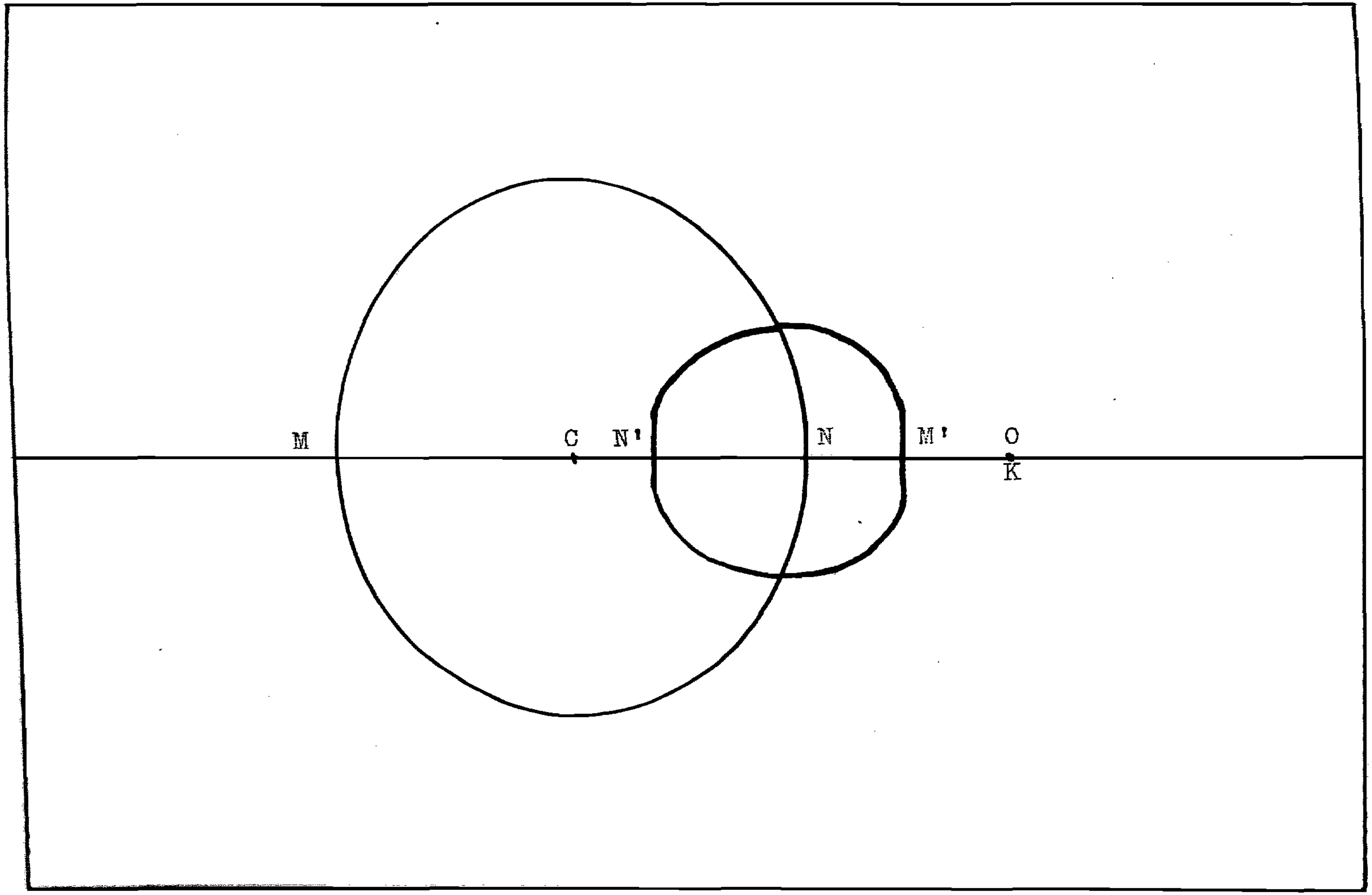

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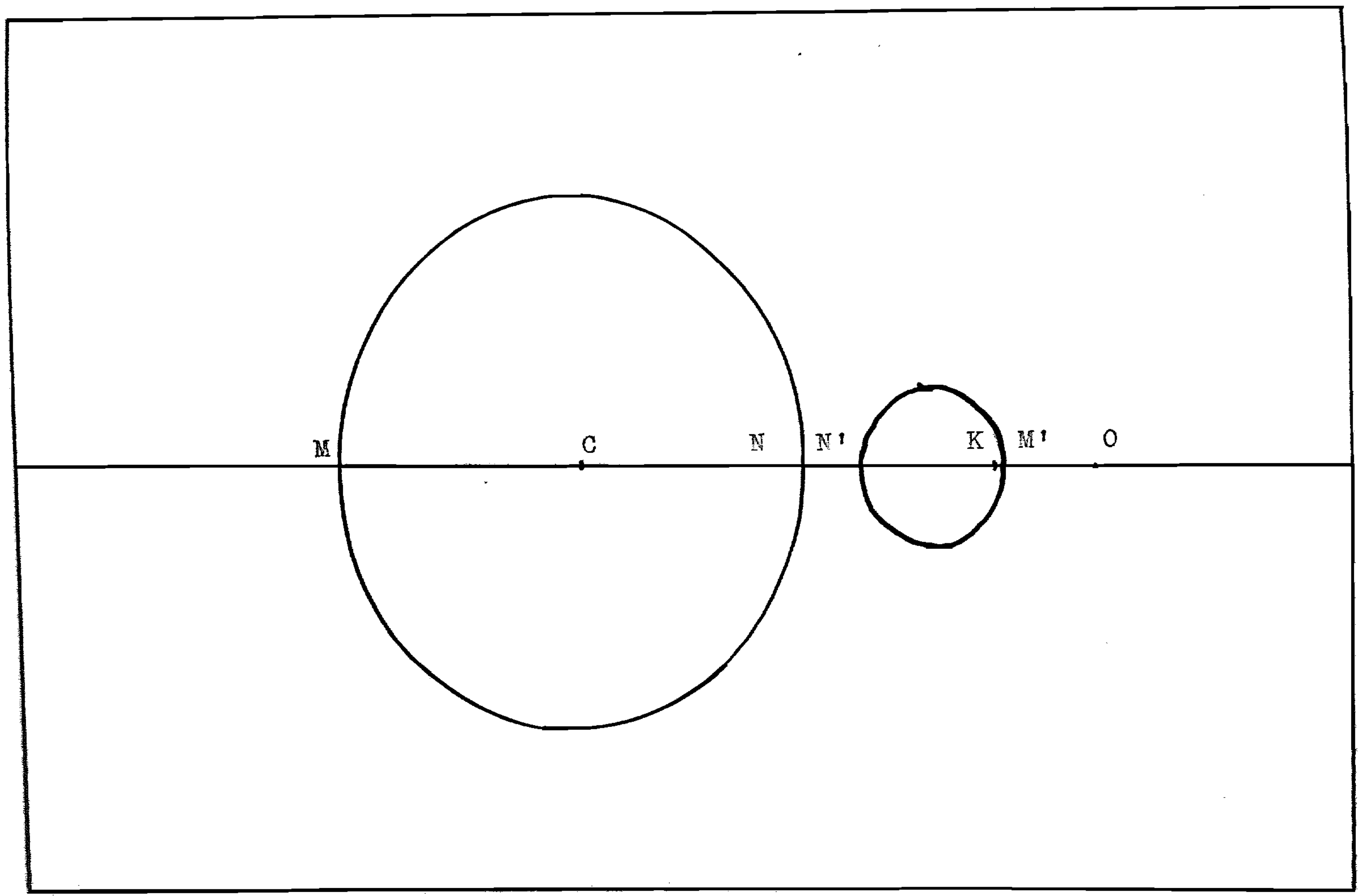

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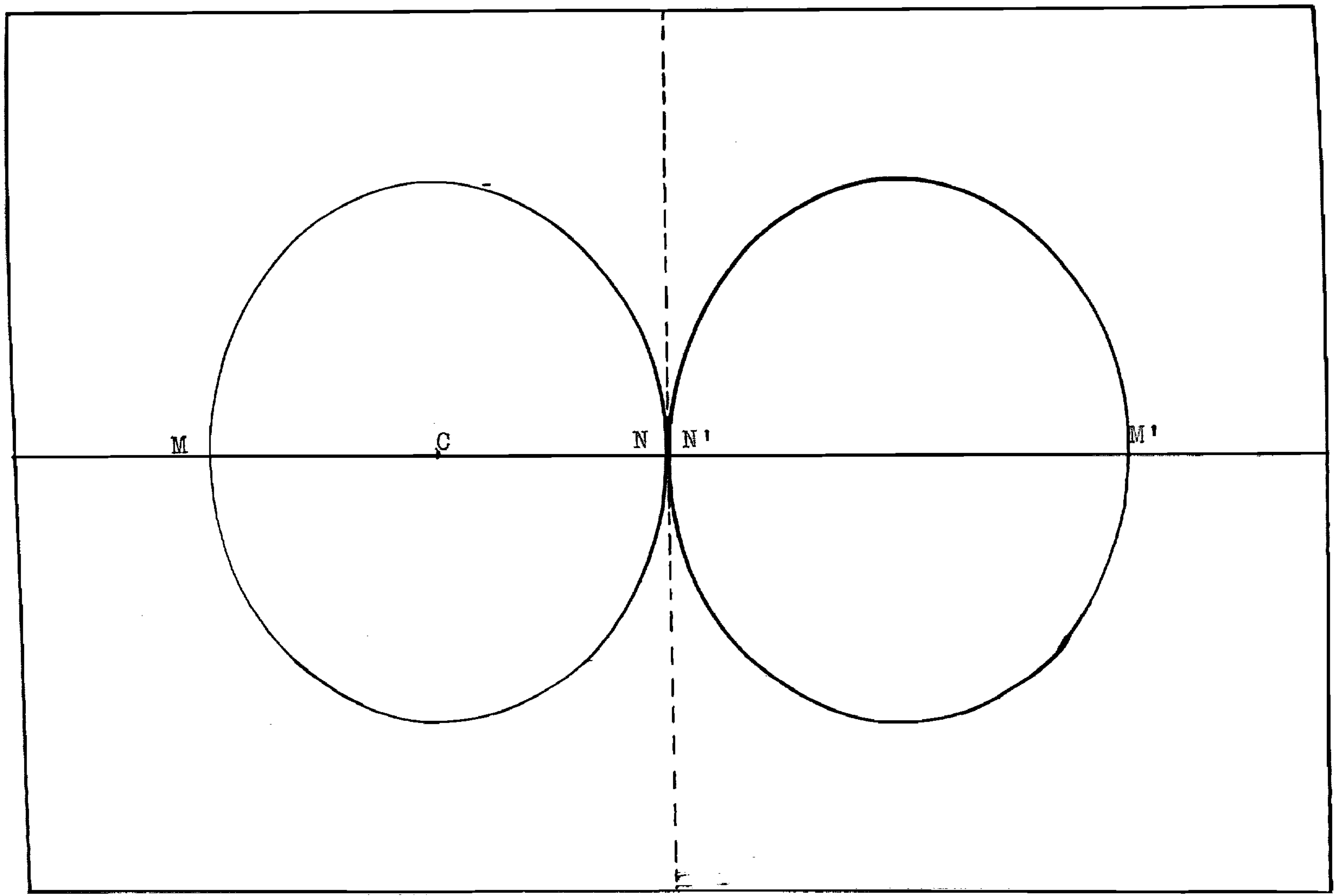

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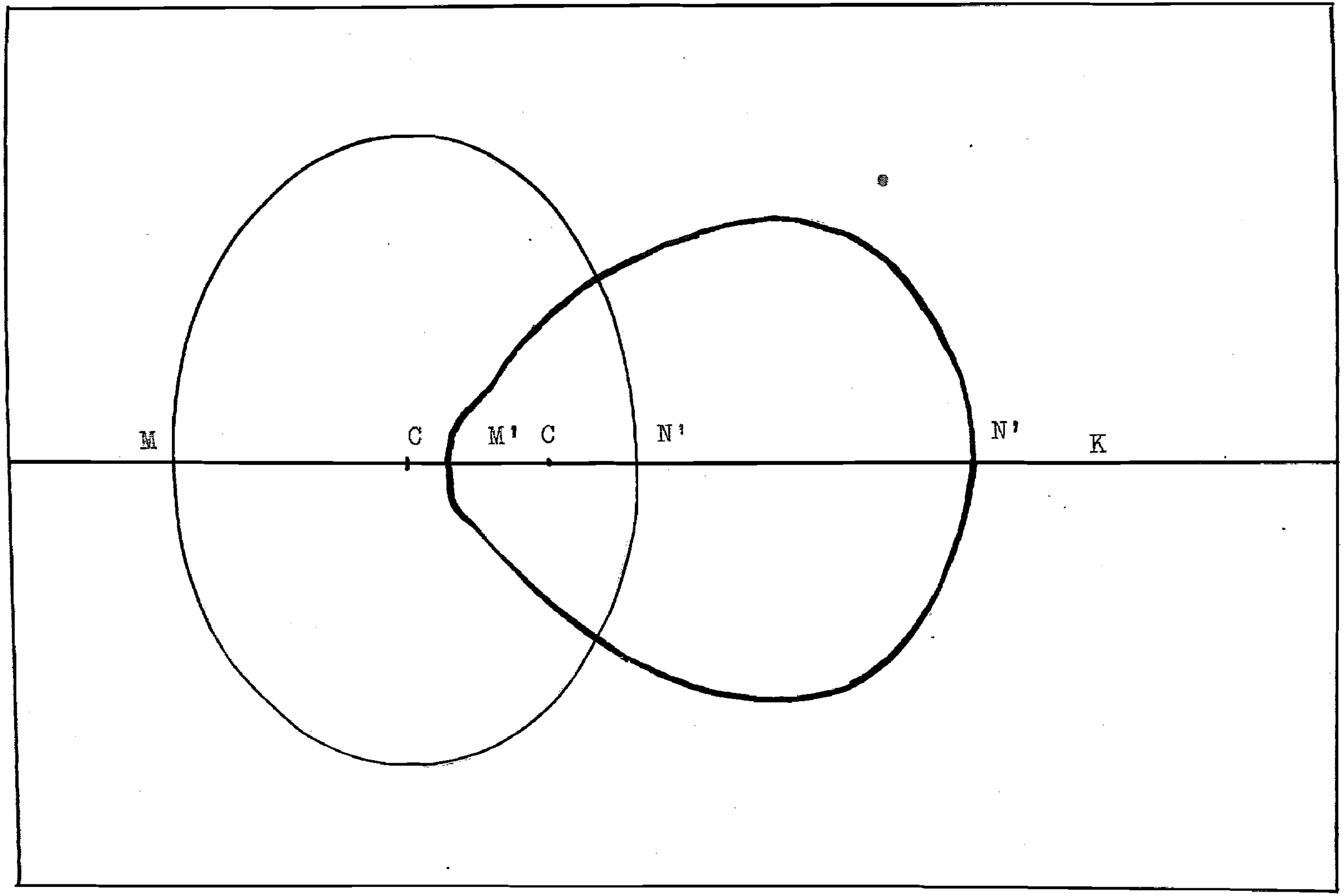

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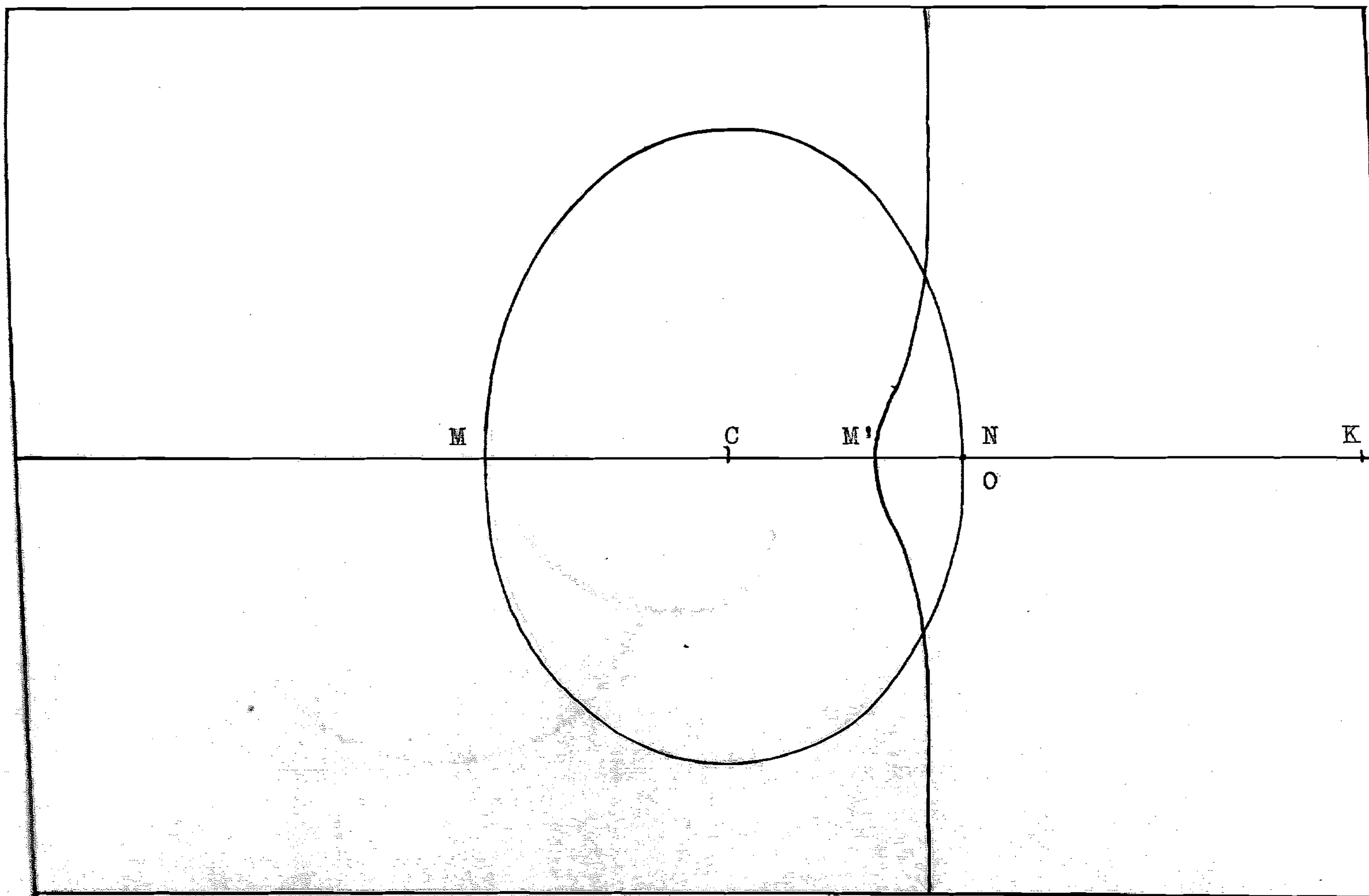

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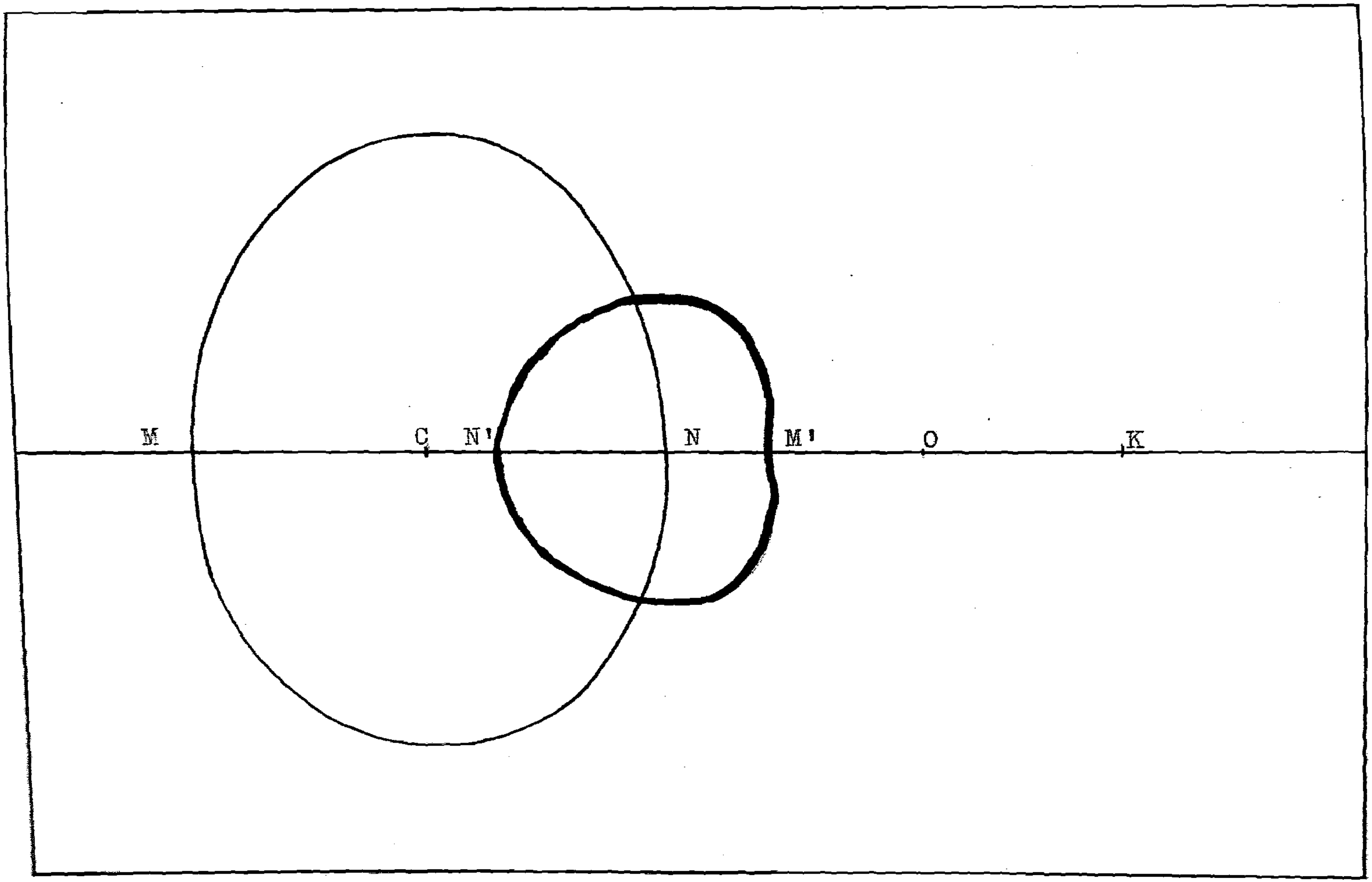

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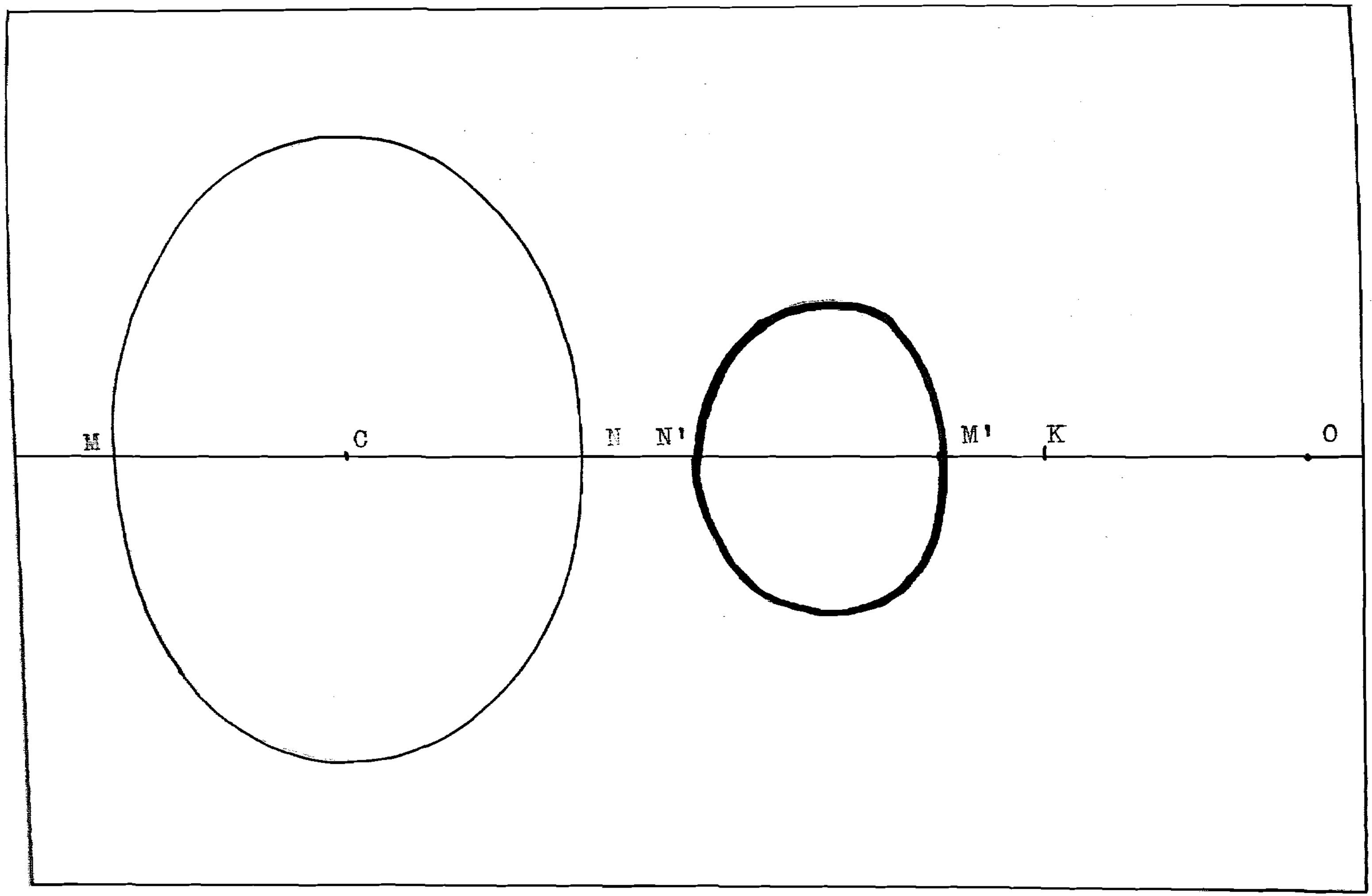

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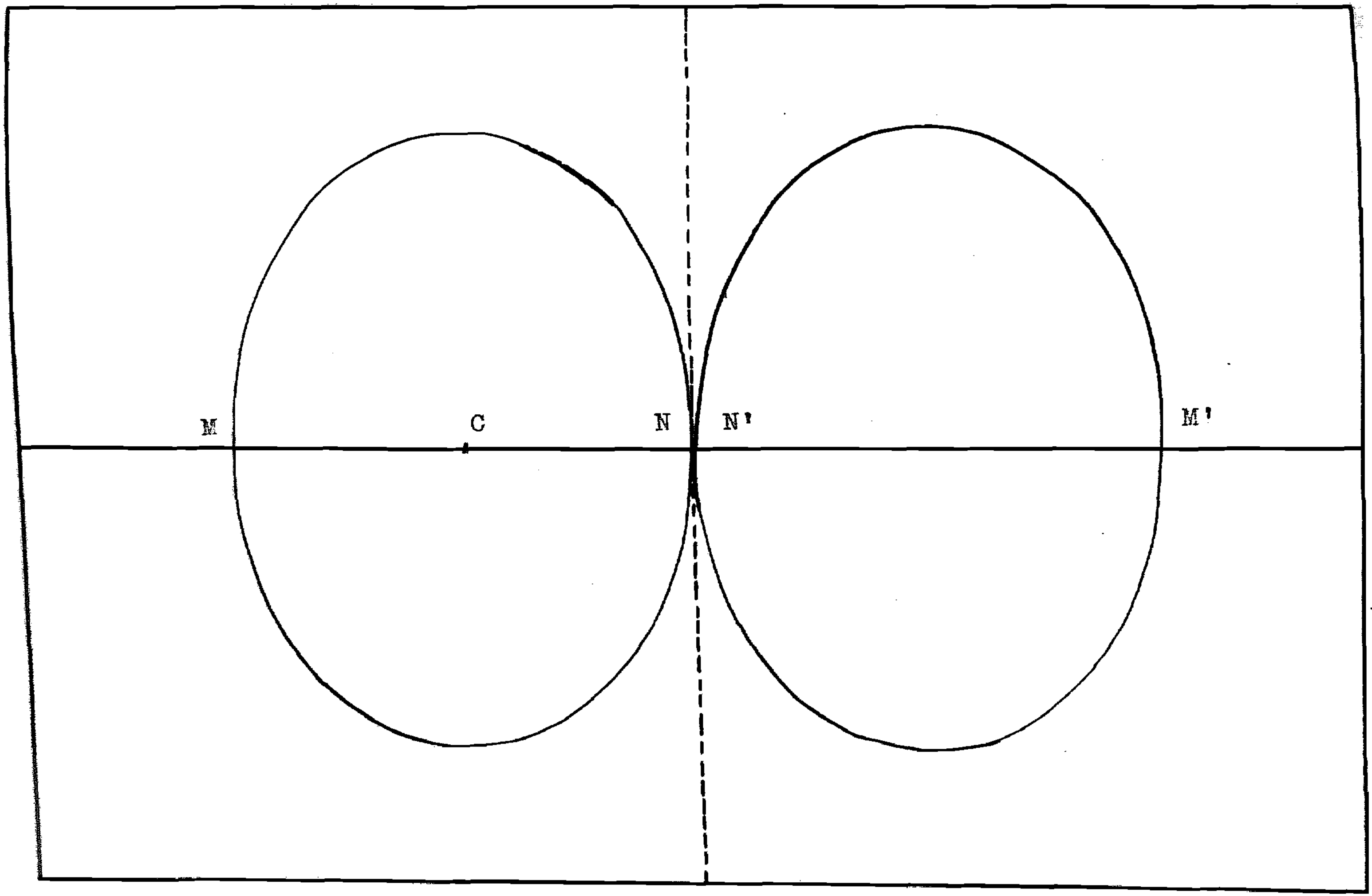

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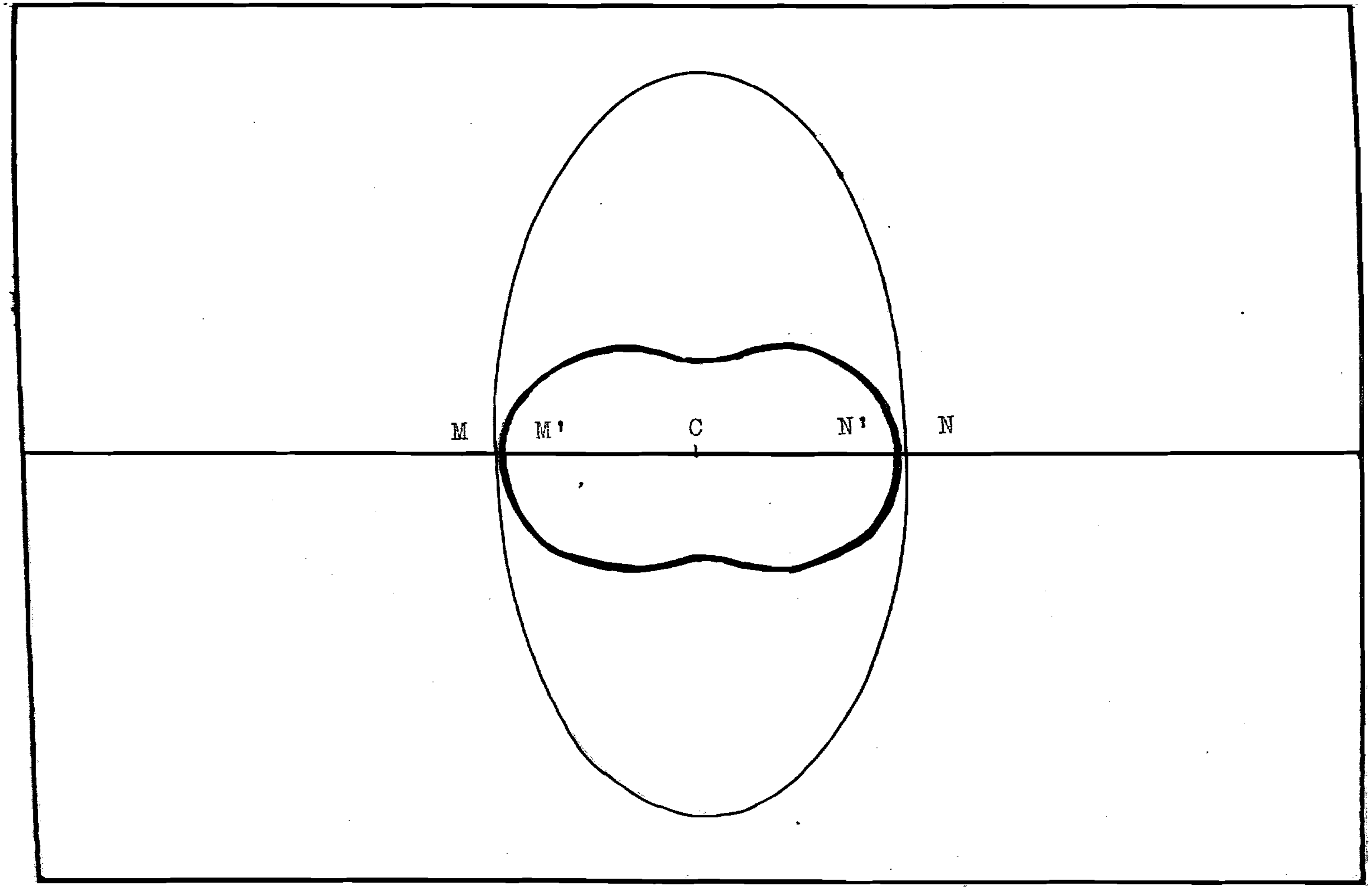

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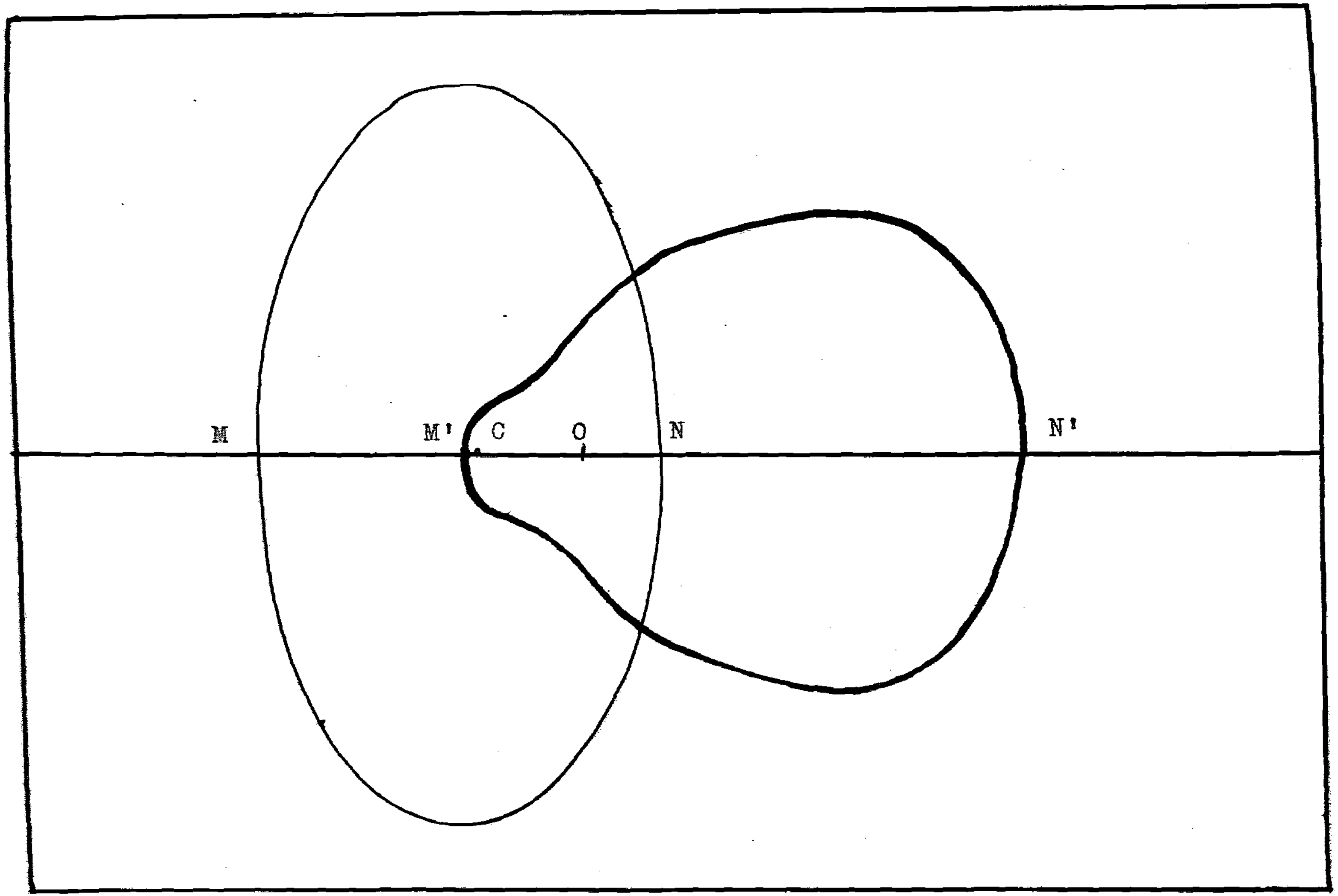

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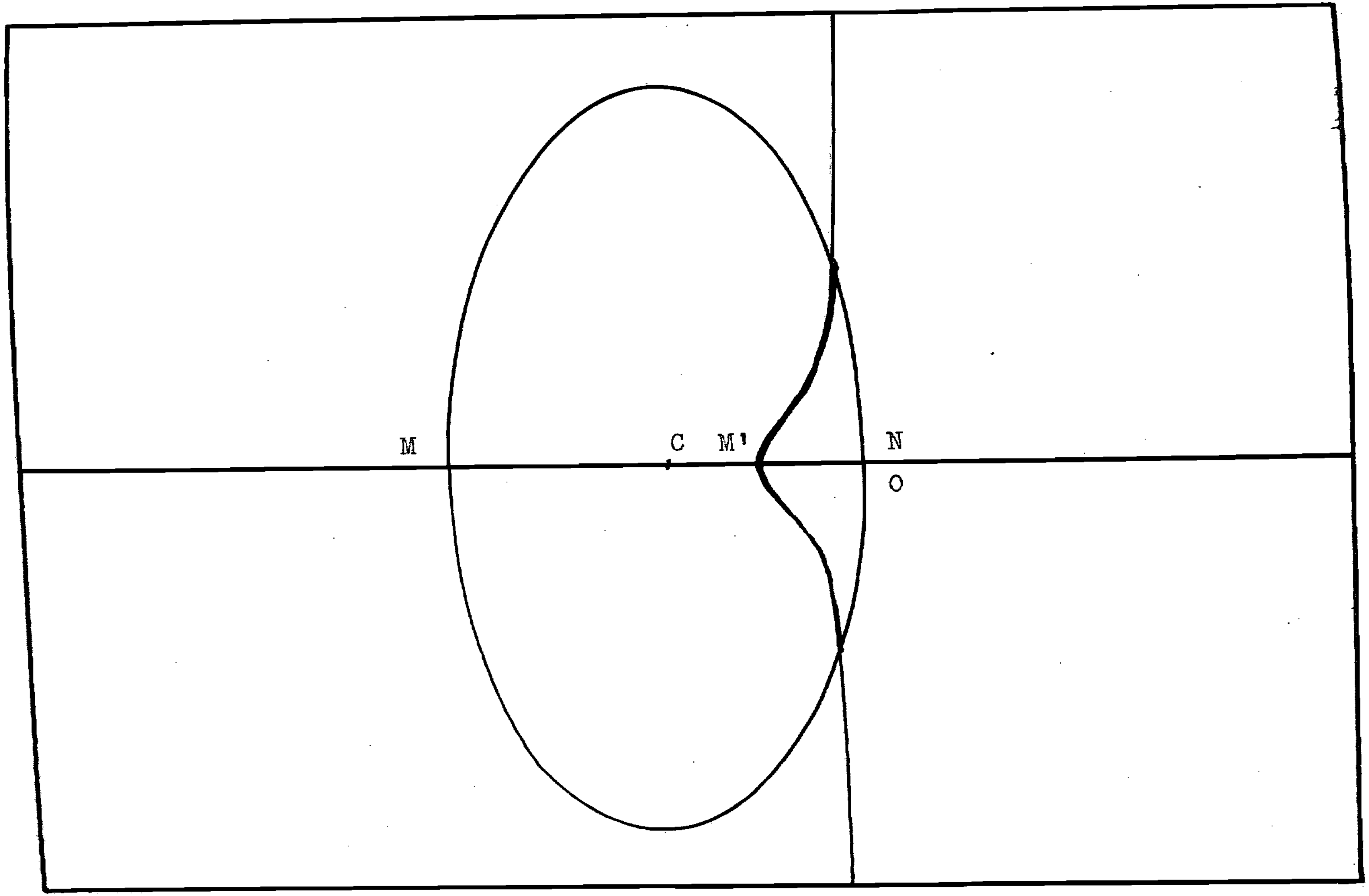

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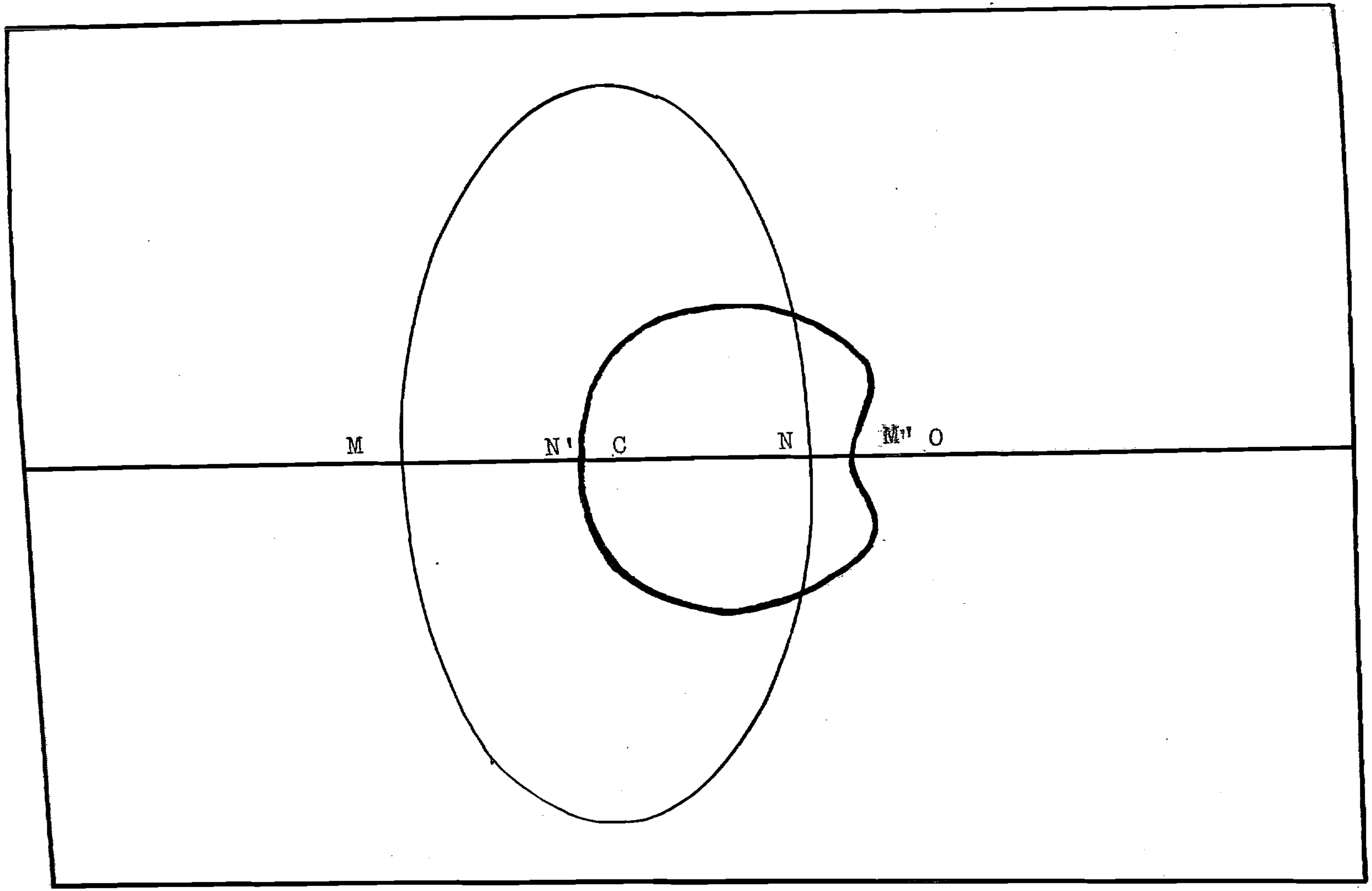

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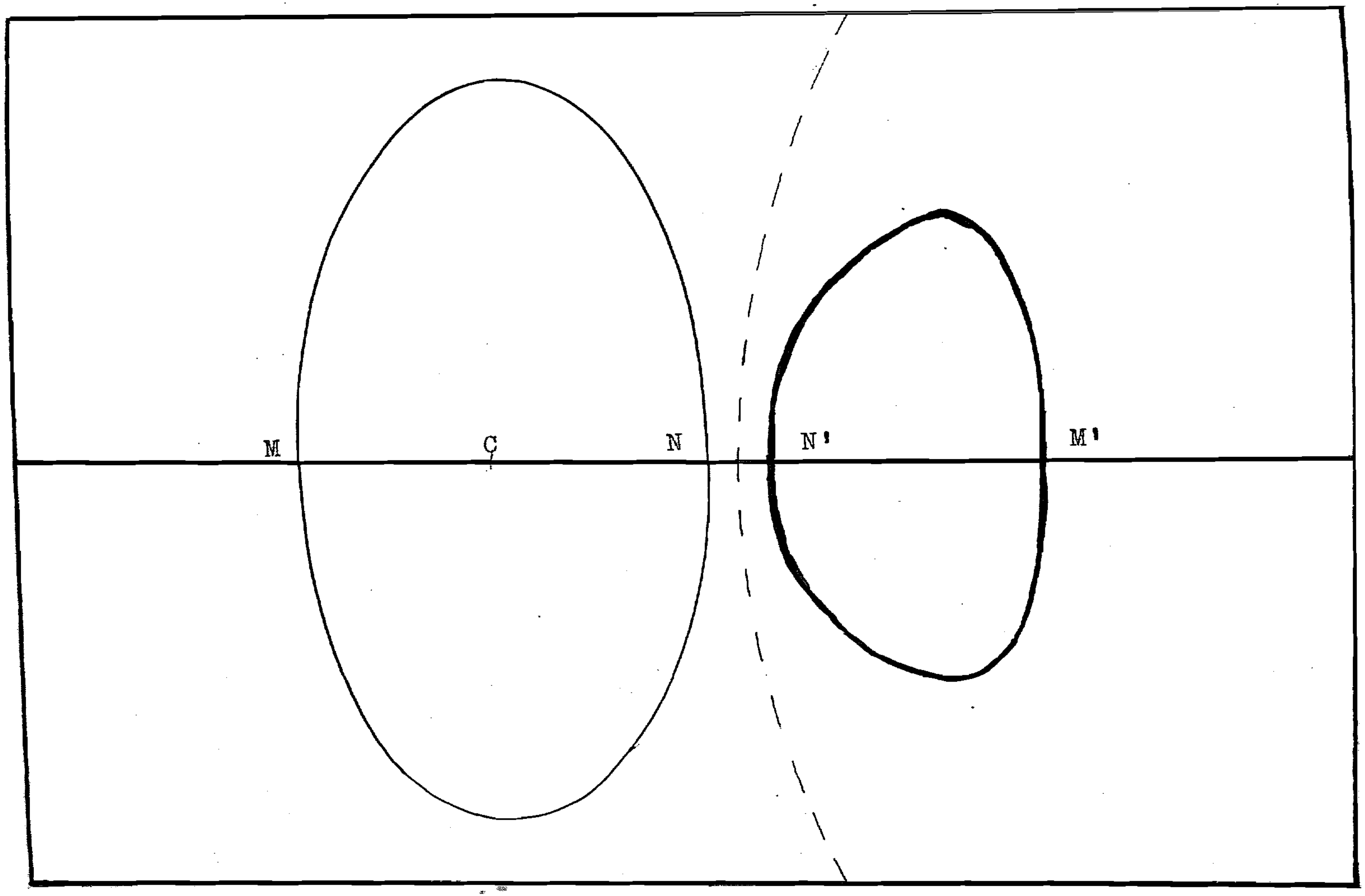

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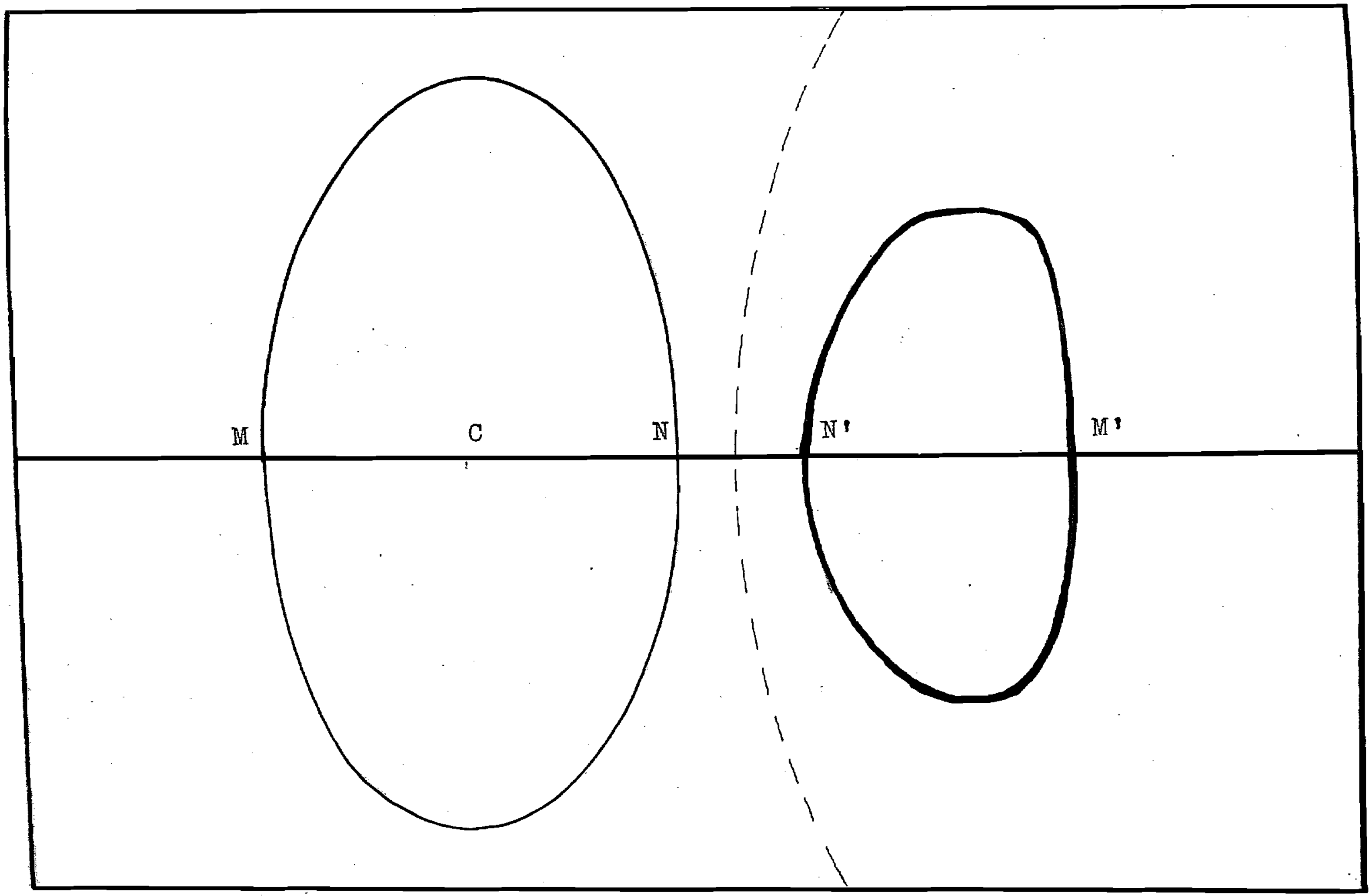

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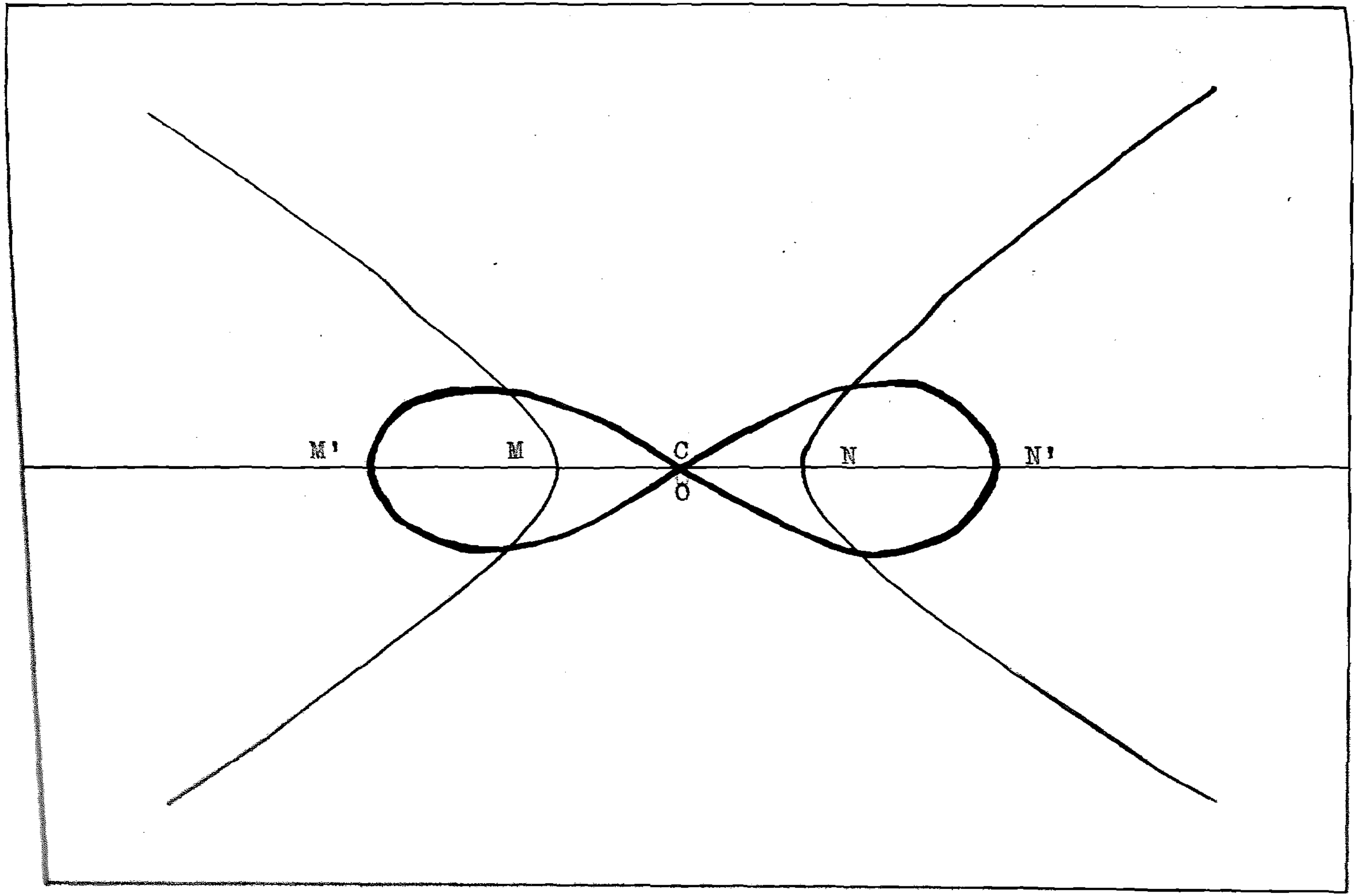

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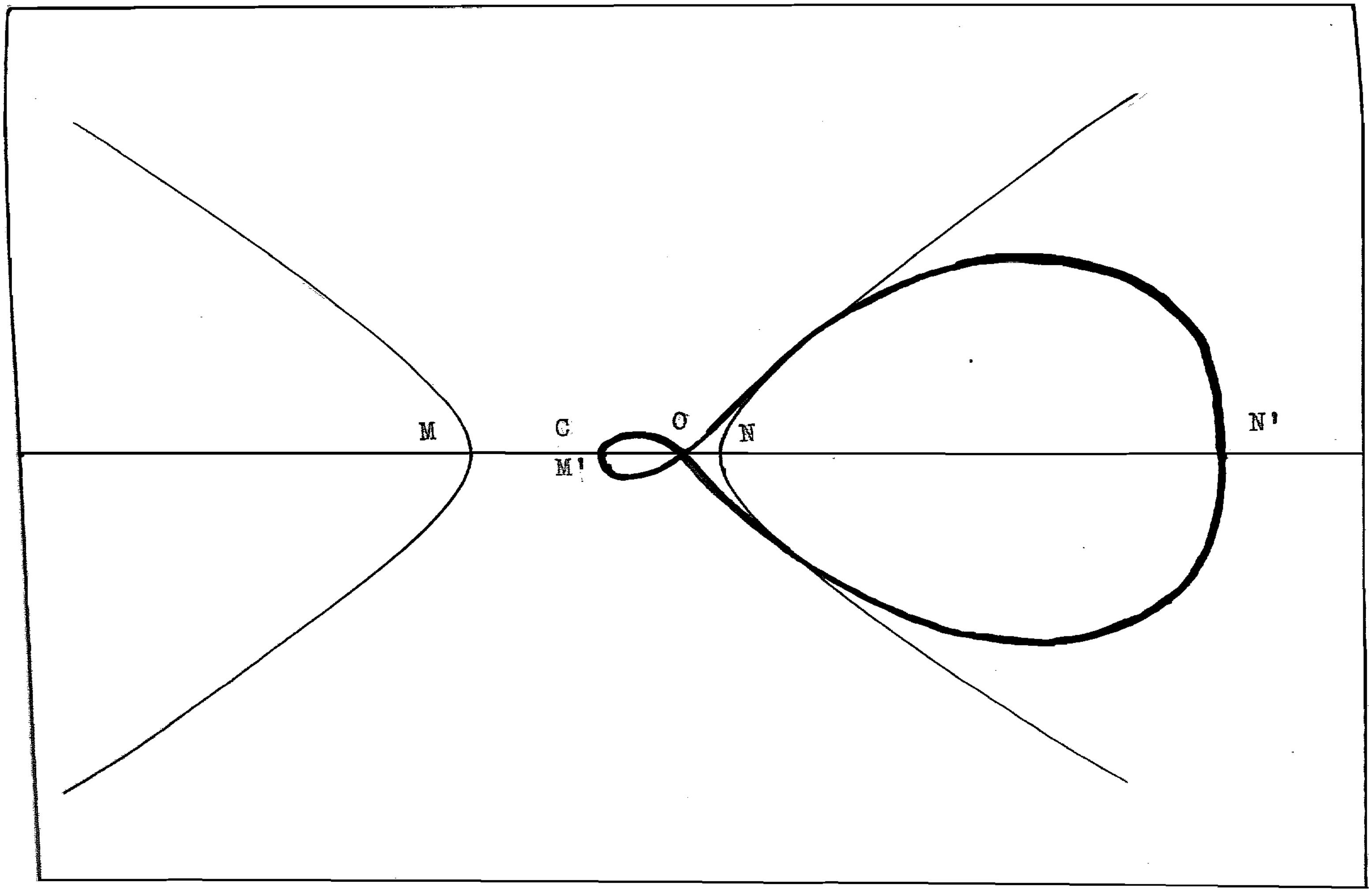

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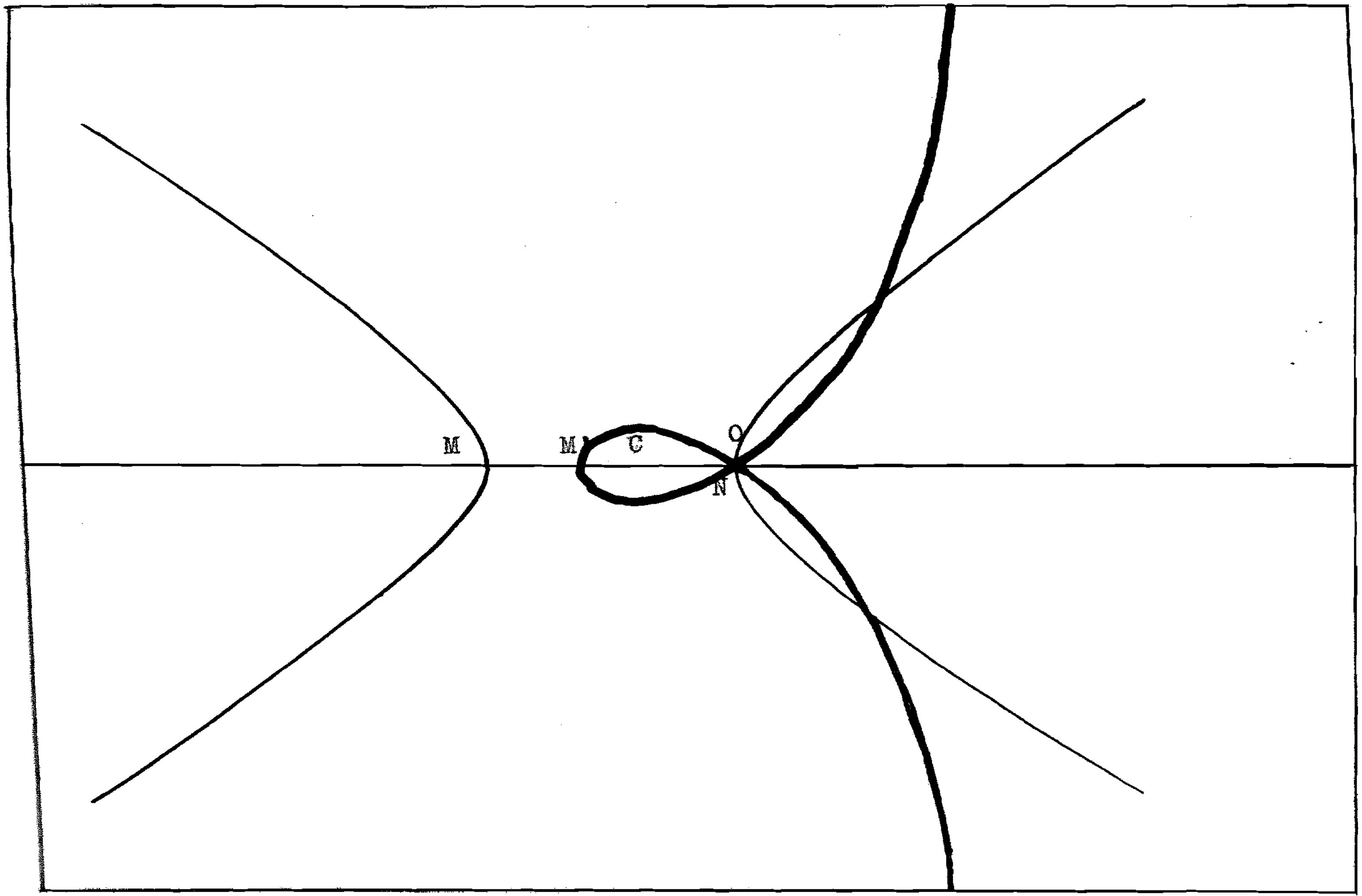

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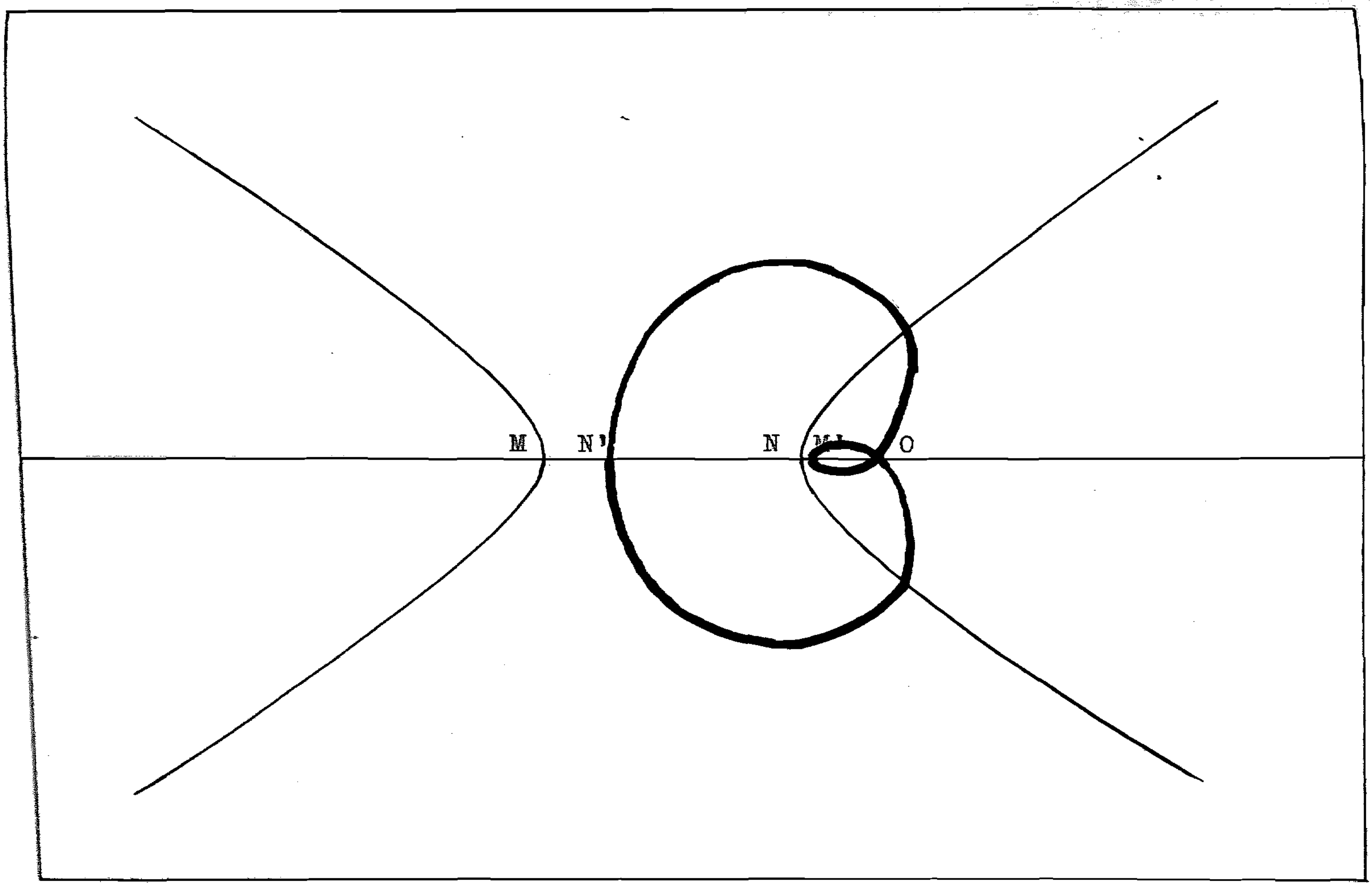

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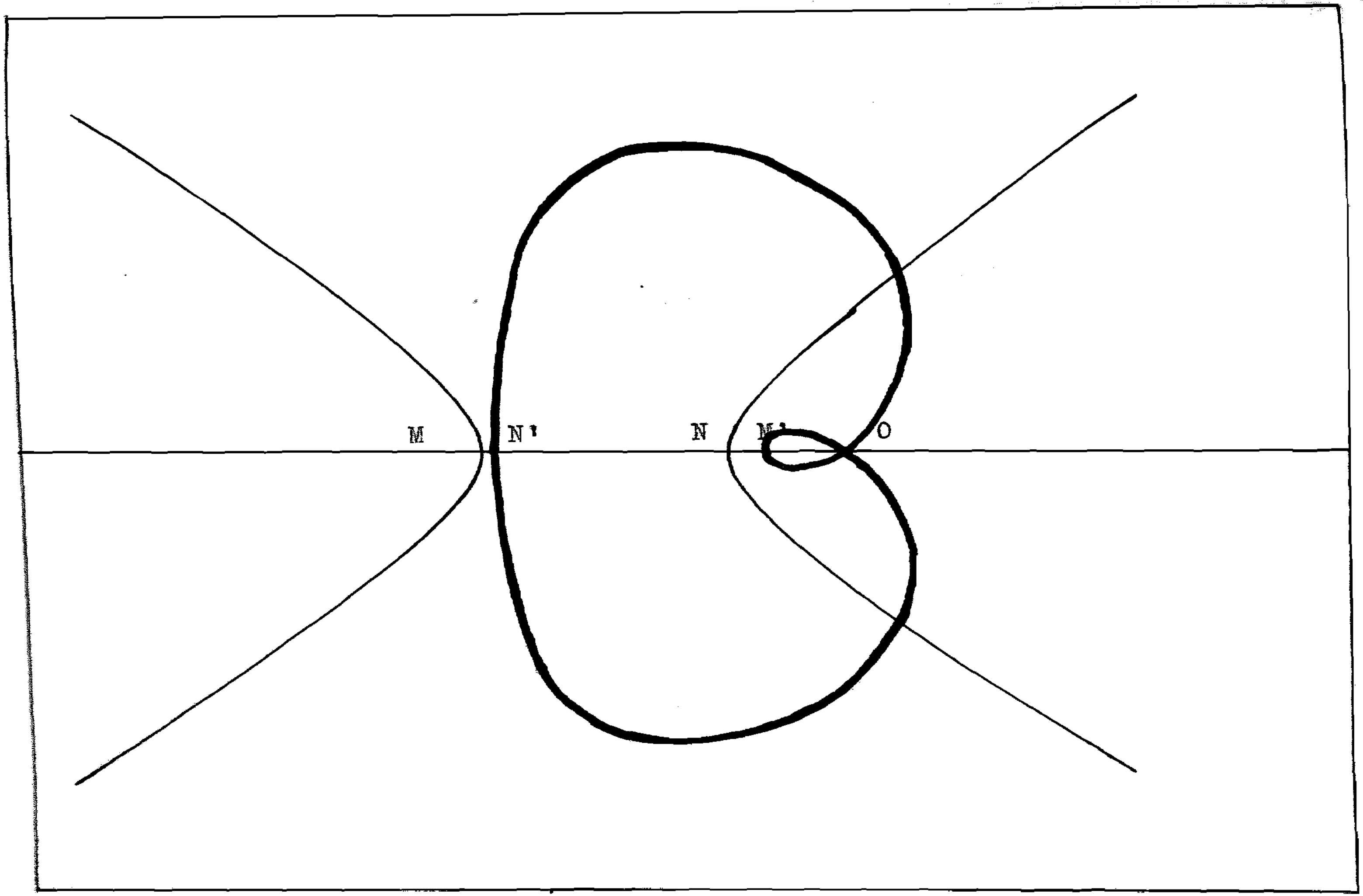

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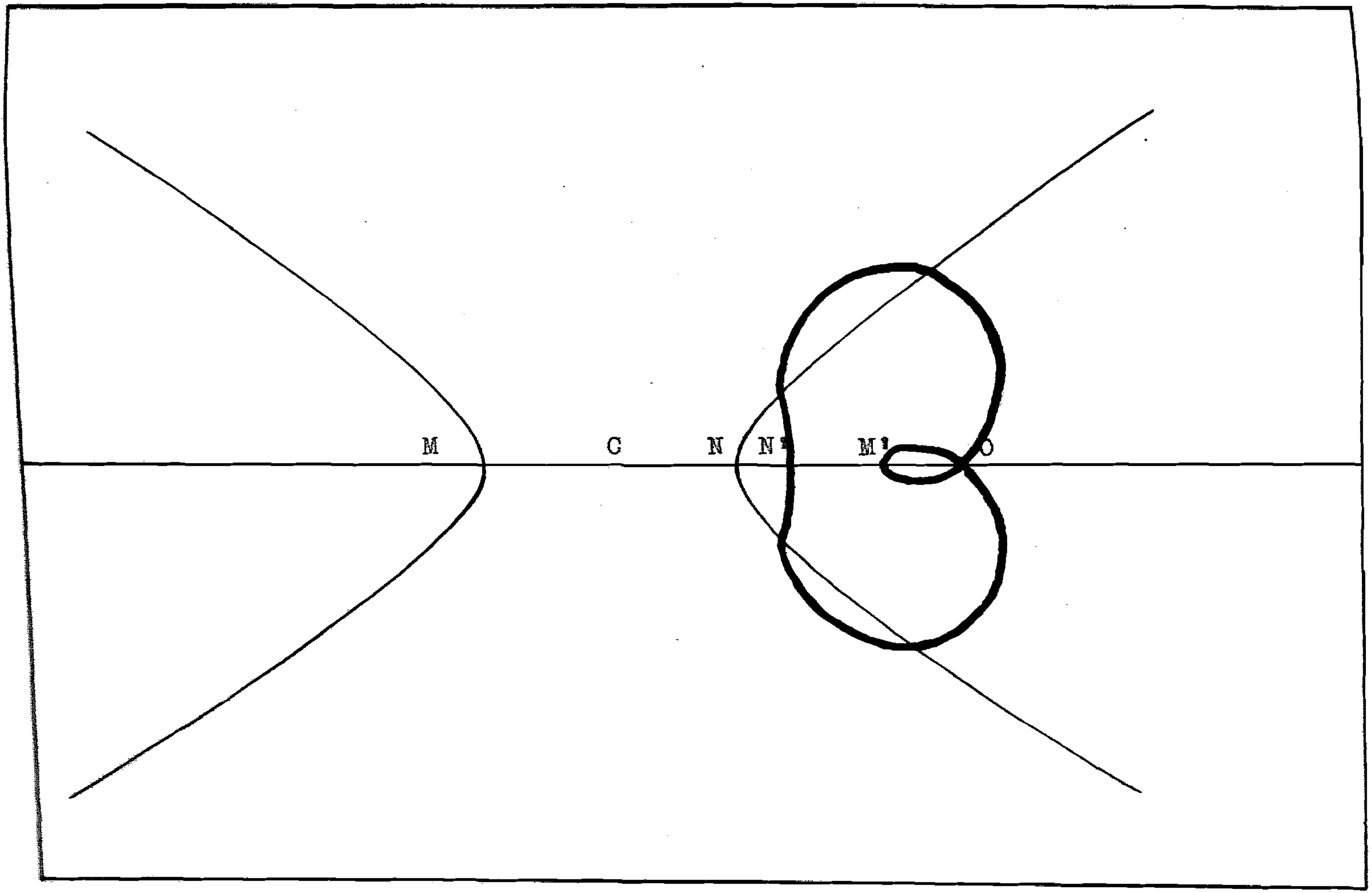

PLATE 67.