AN ABSTRACT OF THE THESIS OF

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Title: Explorations in Continued Fractions

Abstract approved: ______________________

In order to make any type of exploration, some tools are necessary. The fraction 5/3 is chosen as a tool to make explorations in continued fractions.

This thesis mainly consists of the following things. Discussion about the numbers 5/3, 5 and 3 and their significant contributions to the world of mathematics and to the theory of continued fractions in particular. It is to be noticed that 5/3 is the only fraction, when expanded in the form of a continued fraction, that has terms in its expansion whose sum is equal to the average of 5 and 3. Forty theorems in relation to the theory of continued fractions. Also some interesting observations. Famous numbers such as Fibonacci numbers, Theon diameters, et cetera, yield some properties in common to each other when they are studied in the light of continued fractions. A method is explained to get a magic square of order 3 from the magic hexagon of order 3.

EXPLORATIONS IN CONTINUED FRACTIONS

A Thesis

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by

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PREFACE

This thesis consists of 40 theorems which are related to the theory of continued fractions. Special attention has been given to the numbers 5/3, 5 and 3 throughout this thesis,and their contribution to mathematics is explained.

The thesis is divided into 5 chapters. The first chapter deals with the history of continued fractions. The second chapter consists of a discussion about the fraction 5/3 and some related theorems. The theorems are 15 in number,and they are easy to understand. The third chapter consists of 25 interesting theorems and the fourth chapter, some observations and suggestions for further study. The fifth chapter is the summary.

Knowledge of geometry and calculus is not needed to understand this thesis. Any person who has three years introductory algebra can comprehend this material.

In order to assist the reader in understanding certain difficult theorems, necessary material to understand those theorems is given before each such theorem under the title, "Introduction to theorem."

ACKNOWLEDGMENT

I am very grateful to Professor Laird for his encouragement and assistance in writing this thesis. I wish to express my gratitude to Mr. Scott Garten, M.A. for clearing my doubts in many mathematical courses I took at this university. Finally I want to thank all the staff members who taught me mathematics.

"I shall set forth the method of forming fractions which is most pleasing to me today........."

[19, p. 346]

R. Bombelli (1572)

Chapter I: Introductory Material

1.1 Introduction

It is very interesting to know how methods and tools to solve mathematical problems are found and devised.

In the world of mathematics some techniques are found by chance and some by constant struggle. There are some others which are formed by polishing the already existing ones. An example for this type is, "Continued Fractions." This polished key might look like an unpolished one to some people. It may be used again and again in the consideration of different mathematical problems to open many unopened mathematical results so far overlooked or neglected by the simple and the great who have preceded us.

In starting this thesis work, this brightened key of continued fractions was used to see whether the fraction $\frac{5}{3}$ (which is made up of the first two odd primes) might give any interesting results. It was found that it does. Later the reference of some books revealed that the mathematician, Hero, of the school of Alexandria, used $\frac{5}{3}$ as his first approximation to $\sqrt{3}$. In fact the four Heronian approximations to $\sqrt{3}$ are $\frac{5}{3}$, $\frac{26}{15}$, $\frac{265}{153}$, and $\frac{1351}{780}$ where the last two approximations are convergents of a continued fraction for $\sqrt{3}$ [5, p. 152-157]. It is to be noted that the continued fractions may be used to obtain approximations to irrational numbers. [9,p.459]

It is interesting to note that this thesis work starts with exploration of the uses of $\frac{5}{3}$ which is a fraction of antiquity.

1.2 Definition, types and convergents of continued fractions

An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots}}}$$

is called a continued fraction. In general, the a_i and b_i may be unrestricted in character, and the number of terms may be finite or infinite.

A simple continued fraction is one in which each $b_i = 1$ and the a_i are positive integrers, except that a_1 may be positive, negative, or zero. A more convenient way of writing a simple continued fraction is $a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots$ or, yet more simply, $(a_1,a_2,a_3,a_4,\ldots)$. [21, p. 15]

Another way of writing the expression at the beginning of the section 1.2 is $a_1 + \frac{b_1}{a_2+} \frac{b_2}{a_3+} \frac{b_3}{a_4+} \cdots$

The a_i are called terms or the partial quotients of the continued fraction. [13,p.47] There are some types in the continued fractions; for example:

1.) Simple continued fractions:

Definition: The continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}$$

Where each b_i is equal to 1 and each a_i is an integer such that $a_i > 0$ for $i > 1$, is a simple continued fraction. [21,p.15]

Example: $-3 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{5 + \cfrac{1}{2}}}}$ [11,p.97]

2.) Periodic continued fraction:

Definition: A periodic continued fraction is one of the form

$$(b_0, b_1, b_2, \ldots, b_{n-1}, \overline{c_0, c_1, \ldots, c_{m-1}})$$

for some nonnegative integer n and for some positive integer m. It shall be assumed that n and m are chosen as small as possible. It might then be said that $c_0, c_1, c_2, \ldots c_{m-1}$ is the period (of length m) and that the period begins after n terms. If n = o, one has a purely periodic continued fraction.

Example: $\sqrt{3} = (1,1,2,1,2,1,2,1,2,\ldots)$

The terms 1,2 repeat indefinitely. This is usually written as $\sqrt{3} = (1,\overline{1,2})$ and this continued fraction has the period 1,2 and that the period begins after one term. [13,p.125]

3.) Symmetric continued fractions:

Definition: A symmetric continued fraction is a finite simple continued fraction in which the partial quotients read the same both ways.

Example 1: $\frac{247}{77} = (3,4,1,4,3)$

Example 2: $\frac{425}{132} = (3,4,1,1,4,3)$

[21,p.21]

4.) Finite continued fraction:

Definition: If the number of terms is finite, then the continued fraction is called a finite continued fraction.

Example: $\frac{5}{3} = (1,1,2)$

[13,p.97]

5.) Infinite continued fraction:

Definition: If the number of terms of a continued fraction is infinite then it is called an infinite continued fraction.

Example: $\sqrt{3} = (1,\overline{1,2})$

[13,p.110-116]

Convergents of Continued Fractions

Let $z_0, z_1, z_2, \ldots, z_k$ be real numbers, all of which, except possibly the first, are positive, and consider the continued fraction

$$x = z_0 + \cfrac{1}{z_1 + \cfrac{1}{z_2 + \ddots + \cfrac{1}{z_{k-1} + \cfrac{1}{z_k}}}} \qquad \text{(A)}$$

Now clearly x is determined completely by the z's, so one can abbreviate the cumbersome equation (A) by writing $x = (z_0; z_1, z_2, \ldots, z_k)$. The reason for the semicolon in this notation is to emphasize the distinction between (A) and the continued fraction

$$\cfrac{1}{z_0 + \cfrac{1}{z_1 + \cfrac{1}{z_2 + \dots + \cfrac{1}{z_k}}}} = (0;z_0,z_1,z_2,\dots,z_k)$$

Moreover, the number preceding the semicolon plays a rather difficult role from the other z's in that it can be zero or negative. By placing parentheses around the fraction $z_{k-1} + \frac{1}{z_k}$ at the bottom of (A), one gets

$$(z_0;z_1,z_2,z_3,\dots,z_{k-1},z_k) =$$
$$(z_0;z_1,z_2,z_3,\dots,z_{k-2},z_{k-1} + \frac{1}{z_k})$$

The continued fractions

$$(z_0;),(z_0;z_1),(z_0;z_1,z_2),\dots,\quad (z_0;z_1,z_2,z_3,\dots,z_k)$$

are called the convergents of the expansion (A). If one simplifies the first few to ordinary fractions, the following equations are obtained.

$$(z_0;) = \frac{z_0}{1}$$

$$(z_0;z_1) = \frac{z_0 z_1 + 1}{z_1}$$

$$(z_0;z_1,z_2) = \frac{z_0z_1z_2 + z_0 + z_2}{z_1z_2 + 1}$$

. . . .

Now define the numbers P_n and q_n, for $n = 1,\dots,k$, as being the numerators and denominators of the fractions just written, so that

$$P_0 = z_0, \quad q_0 = 1$$

$$P_1 = z_0 z_1 + 1, \quad q_1 = z_1$$

$$P_2 = z_0 z_1 z_2 + z_0 + z_2, \quad q_2 = z_1 z_2 + 1$$

.
.
.

and refer to P_n and q_n as the numerator and denominator of the n^{th} convergent of (A). [11,p.77-78]

1.3 Scholars and their contributions to the history of continued fractions

1.) Euclid (300 B.C.): Euclid found the greatest common measure of two lines and used the same principle to find the greatest common divisor of two numbers. [16,p.418]

The algorithm for expanding $\frac{p}{q}$ into a continued fraction is identical with the Euclidean Algorithm for finding the greatest common divisor of p and q. This algorithm gives the equations

$$p = a_1 q + r_1 \qquad (a_i,\ r_i \text{ are integers and } o \leq r_i < q)$$

$$q = a_2 r_1 + r_2$$

$$r_1 = a_3 r_2 + r_3$$

. . . .

. . . .

which may be written

$$\frac{p}{q} = a_1 + \frac{r_1}{q}$$

$$\frac{q}{r_1} = a_2 + \frac{r_2}{r_1}$$

$$\frac{r_1}{r_2} = a_3 + \frac{r_3}{r_2}$$

. . . .

. . . .

and when these are combined the continued fraction is secured. [21,p.16-17]

Example: Let p = 11 and q = 7

$$11 = 1 \cdot 7 + 4$$

$$7 = 1 \cdot 4 + 3$$

$$4 = 1 \cdot 3 + 1$$

which may be written

$$\frac{11}{7} = 1 + \frac{4}{7}$$

$$\frac{7}{4} = 1 + \frac{3}{4}$$

$$\frac{4}{3} = 1 + \frac{1}{3}$$

when these are combined one gets the following expression

$$\frac{11}{7} = 1 + \frac{4}{7}$$

$$= 1 + \cfrac{1}{\frac{7}{4}}$$

$$= 1 + \cfrac{1}{1 + \frac{3}{4}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{\frac{4}{3}}}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{3}}}$$

This is the earliest important step in the theory of continued fractions. Further traces of the general idea are found occasionally in the Greek and Arab writings. [16,p.419]

2.) Aryabhata (b.476): Continued fractions both ascending and descending appear to have been known already to the Hindus, though not in our present notation. [3,p.188]

Hindus, Aryabhata in particular had used continued fractions to solve linear indeterminate equations. [9,p.254]

3.) Bombelli (1572): The modern theory of continued fractions may be said to have begun with Bombelli (1572).

He is the first mathematician who tried to use the concept of continued fractions. He used this concept in finding the approximate values of the square roots of numbers that are not perfect squares. [18,p.80]

4.) Cataldi (1613): The next writer who considered these fractions and put them in modern form was Cataldi (1613).

5.) Daniel Schwenter (1625): Daniel Schwenter was the first to make any material contribution towards determining the convergents of continued fractions. He devoted his attention to the reduction of fractions involving large numbers and determined the rules now in use for calculating the successive convergents. [6,p.131]

6.) Lord William Brouncker (1620-1684): Lord Brouncker's beautiful equality is

$$\pi = \cfrac{4}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}}$$

Brouncker's expression increased interest in the theory of continued fractions. [3,p.188]

7.) Wallis (1695): John Wallis, an English mathematician, in his OPERA Mathematica, I (1695), introduced the term, "Continued Fractions" for the first time. [9,p.255]

8.) Euler (1707-1783): The theory also attracted the attention of mathematical giants such as Euler who also bowed down and became a contributor.

The foundations of the theory of continued fractions were stated by Euler in his INTRODUCTIO (Chapter 18). There he showed how to go from a series to a continued fraction representation of the series, and conversely. Some of Euler's interesting results are

(i) $$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{4} + \cdots}}}$$

(ii) Every rational number can be expressed as a finite continued fraction. [9,p.459-460]

9.) Johann Heinrich Lambert (1728-1777):

Lambert proved that if x is rational but not zero, then neither e^x nor $\tan x$ can be a rational number. Lambert's proofs rest on the expression for e as a continued fraction, given by L. Euler. [3,p.246]

10.) Lagrange (1736 - 1813): Lagrange used continued fractions to find approximations to the irrational roots of equations and he also got approximate solutions of differential equations in the form of continued fractions. In a 1768 paper, Lagrange proved the converse of a theorem that Euler had proved in his 1744 paper. The converse states that a real root of a quadratic equation may be written as a periodic continued fraction. [9,p.460]

It is to be noticed that continued fractions can be converted into divergent or convergent series and conversely. It is very important to listen to Morris Kline who said, "The continued fraction is the only intermediary between the series and the integral; that is, given the series one obtains the integral through the continued fraction."

11.) Laguerre (1879): In 1879 Laguerre proved that the integral

$$y = \int_0^\infty \frac{xe^{-t}}{1+xt} \, dt$$

could be expanded into the continued fraction

$$\frac{x}{1+}\ \frac{x}{1+}\ \frac{x}{1+}\ \frac{2x}{1+}\ \frac{2x}{1+}\ \frac{3x}{1+}\ \frac{3x}{1+}\ \ldots$$

12.) Stieltjes (1856-1895): Stieltjes used continued fractions as a tool to find a "sum" for divergent series. He studied continued fraction expansions of divergent series and wrote two celebrated papers during 1894-95 on this subject. This work, which is the beginning of an Analytic theory of continued fractions, considered questions of convergence and the connection with definite integrals and divergent series. [9,p.1114-1116]

Thus the simple and the great are attracted by the theory of continued fractions and thus make its story a longer one.

1.4 Some important uses of continued fractions

1.) Continued fractions may be used to obtain approximations to irrational numbers. For example, to approximate $\sqrt{2}$ one can write

$$\sqrt{2} = 1 + \frac{1}{y} \qquad (1)$$

From this one finds

$$y = 1 + \sqrt{2} \qquad (2)$$

By adding 1 to both sides of (1) and using (2) it follows that

$$y = 2 + \frac{1}{y} \qquad (3)$$

Hence again by (1) and (3)

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{y}}$$

and since y is given by (3)

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{y}}}$$

By repeated substitution of the value of y one obtains

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

This continued fraction is simple because the numerators are all 1. It is periodic because the denominators repeat. [9,p.255]

2.) Continued fractions may be used in a method of approximating the real roots of an equation.

3.) Laplace proved that the solution of an equation in the finite differences of the first degree and the second order may be always obtained in the form of a continued fraction. [2,p.411,419]

4.) Aryabhata used continued fractions to solve linear indeterminate equations. [8,p.254]

5.) Periodic continued fractions may be used to attack some difficult number theoretical questions. [5,p.151]

Chapter II: The Fraction $\frac{5}{3}$ and some Related Theorems

2.1 A note on $\frac{M}{N}$

In this thesis the fraction of the form $\frac{M}{N}$ always satisfies the following conditions, unless mentioned otherwise.

1.) M and N are natural numbers

2.) $M > N$

2.2 The fraction $\frac{5}{3}$ and numbers 5 and 3. Their contribution to Mathematics and to the theory of continued fractions.

It is to be noted that the fraction $\frac{5}{3}$ is made up of the first two odd primes. Also $\frac{5}{3} = (1,1,2)$, has 3 terms in its continued fraction expansion.

The denominator of "$\frac{5}{3}$" is 3

The sum of the terms in the continued fraction of $\frac{5}{3}$ is equal to the number of positive divisors of 5 plus the number of positive divisors of 3. i.e. } 4

The numerator of "$\frac{5}{3}$" is 5

The integers 3, 4, and 5 were special integers to the ancient Greek mathematicians, because these integers were involved in the construction of a right angled triangle. The set of integers 3, 4, and 5 can be called the first primitive Pythagorean Triple because they satisfy the following definition.

Definition of Primitive Pythagorean Triple

A set of integers x,y,z such that $x^2 + y^2 = z^2$ and $(x,y,z) = 1$ is called a Primitive Pythagorean Triple. [1,p.190]

It is also to be noticed that the terms in the continued fraction of $\frac{5}{3}$, namely 1,1,2 can be used in the construction of a right angled triangle, where hypotenuse would be $\sqrt{2}$ and the sides being 1 and 1.

It is quite obvious that $\frac{5}{3} = 1.6666...$ is an approximation to the 'golden proportion,' d, where d = 1.6180...

$$\frac{5}{3} = 1.666 \qquad \text{(Correct to 3 decimal places)}$$

$$d = \frac{1+\sqrt{5}}{2} = 1.618 \qquad \text{(Correct to 3 decimal places)}$$

Because of this striking association of $\frac{5}{3}$ with the golden ratio, a brief history of golden ratio is given below for it helps the reader to understand its significance in mathematics.

Brief history of golden ratio

A great invention of the Greeks was a certain rectangle called the golden rectangle. The proportions of the golden rectangle have made it famous. Euclid (300 B.C.) suggested the problem of constructing the regular pentagon with compass and straight edge. This ultimately involves the ratio

$$d = \frac{1+\sqrt{5}}{2} = 1.6180$$

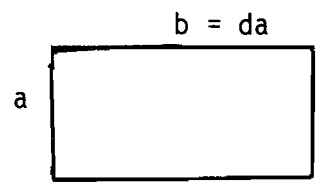
d is called the golden ratio because it is significant both in mathematics and in its applications. One observes that the value d is the positive root of the quadratic equation $x^2 - x - 1 = 0$. So the characteristic property of the number from which much of its usefulness is derived--is the relation

$$d^2 = d + 1$$

i.e. $$d = \frac{1}{d} + 1$$

The golden rectangle is then defined as a rectangle whose sides are in the ratio $\frac{a}{b} = \frac{1}{d}$

Figure 1



golden rectangle

The Greeks regarded the above figure as divinely inspired and held that among all rectangles the golden rectangle was the most pleasing to the eye. [8,p.6]

The following properties are worthy to be noticed, because they explain some properties shared by numerator and denominator of "$\frac{5}{3}$" with the first three nonzero decimal numbers of golden ratio. (The first three nonzero decimal numbers of the golden ratio are 6, 1 and 8)

1.) The third decimal number of the golden ratio namely 8, is the sum of the numerator and denominator of "$\frac{5}{3}$". i.e. $8 = 5 + 3$

2.) The sum of the first three decimal numbers of the golden ratio is equal to the product of the numerator and denominator of "$\frac{5}{3}$". i.e. $6 + 1 + 8 = 5 \cdot 3 = 15$. It is to be noticed that 15 is the magic constant which is explained in the information given before theorem 23 of Chapter 3. 15 is the magic constant of the 3rd order magic square (Figure 8), whose members include 6, 1, 8 and whose middle term is always

$$\frac{6 + 1 + 8}{3} = 5$$

(This is proved in the proof of theorem 23 of Chapter 3)

The third chapter begins with the study of 'certain numbers' which when arranged in the form of a fraction yield values that become approximations to the golden proportion. (Some of them become approximations to $\frac{5}{3}$ also). When these numbers are observed in the light of continued fractions, they yield an interesting property that is associated with the set of natural numbers. This property is explained as the first theorem in third chapter.

In this manner the tool $\frac{5}{3}$ paves the way for the study of some properties in the continued fractions.

The following information is worthy to note.

The difference between the first two odd primes namely 5 and 3 is 2. The sum of the terms in the continued fraction of $\frac{5}{3}$ is 1 + 1 + 2 = 4. Consider the least pair of primes whose difference is equal to the sum of the terms in the continued fraction of $\frac{5}{3}$. Such a pair is 7 and 11.

It is somewhat interesting to note that the two pairs of primes (3,5) and (7,11) are related with each other through the continued fractions.

Consider $\frac{5}{3} + \frac{11}{7}$

$$\frac{5}{3} + \frac{11}{7} = \frac{35 + 33}{21} = \frac{68}{21}$$

Now $\frac{68}{21} = 3 + \frac{5}{21}$

$$= 3 + \cfrac{1}{4 + \cfrac{1}{5}}$$

So $\frac{5}{3} + \frac{11}{7} = (3,4,5)$

The following facts are noted from the above information.

1.) The number of terms in the continued fraction of $(\frac{5}{3} + \frac{11}{7})$ is the same as the number of terms in the continued fraction of $\frac{5}{3}$.

2.) The first and last terms in the continued fraction of $\frac{5}{3} + \frac{11}{7}$ are 3 and 5, and the middle term is equal to the average of 3 and 5.

3.) The importance of the terms in the continued fraction of $\frac{5}{3} + \frac{11}{7}$ is already mentioned in the beginning of the section 2.2. They were well known to the ancient Greek mathematicians because of the relation $3^2 + 4^2 = 5^2$.

4.) $\frac{5}{3} \cdot \frac{11}{7} = \frac{55}{21} = (2,1,1,1,1,1,2)$

The number of terms in the continued fraction of $\frac{5}{3} \cdot \frac{11}{7}$ is equal to the sum of the numbers3 and 4 where 3 is the number of terms in the continued fraction of $\frac{5}{3}$ and 4 being the number of terms in the continued fraction of $\frac{11}{7}$.

In this way the fraction $\frac{5}{3}$ is strangely associated with the fraction $\frac{11}{7}$. Besides, the history of each fraction is equally old. Some mathematical giants pointed out that $\frac{11}{7}$ is an approximation to $\frac{\pi}{2}$, i.e. to the ratio of the semicircumference of a circle to its diameter. They claim that the designers of Pyramids had chosen this proportion because they viewed the semicircle as a figure unexcelled in beauty. [5,p.50]

It is already explained in section 1.1, how important was the fraction $\frac{5}{3}$ to Greek mathematicians for they used it as an approximation to 3. It is an approximation to the golden proportion also.

It is interesting to know how these fractions of ancient history are related with each other through the theory of continued fractions, as explained above.

2.3 The fraction $\frac{5}{3}$ and the Pascal Triangle

Pascal was a great mathematician. He introduced the following triangle which became famous in the world of mathematics. It is called the Pascal Triangle.

Figure 2

```
                        1
                      1   1
                    1   2   1
                  1   3   3   1
                1   4   6   4   1
              1   5  10  10   5   1
            1   6  15  20  15   6   1
          1   7  21  35  35  21   7   1
        .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
      .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
```

Explanation of Figure 2

The above figure is called Pascal's triangle in which each number is the sum of the two immediately above it. [9,p.272]

The numbers in the Figure 2 could also be arranged in the following way.

Figure 3

1	1	1	1	1	1	. . .
1	2	3	4	5	6	. . .
1	3	6	10	15	21	. . .
1	4	10	20	35	56	. . .
1	5	15	35	70	126	. . .
1	6	21	56	126	252	. . .
1	7	28	84	210	462	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .

The mathematician, Leibnitz, called the above numbers in Figure 3, as 'combinatory numbers.' [4,p.32]

Because each horizontal line is formed from the one above it by making every number in it equal to the sum of those above and to the left of it in the row immediately above it. For example the fourth number in the fourth line, namely 20, is equal to 1 + 3 + 6 + 10. [2,p.284]

The interesting fraction $\frac{5}{3}$ can also be obtained as a product of two series of numbers given below.

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \frac{1}{126} + \frac{1}{210} + \cdots \qquad (A)$$

$$\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \frac{1}{126} + \frac{1}{252} + \frac{1}{462} + \cdots \qquad (B)$$

The denominators of the fractions of the series (A) and (B) are given below.

1, 5, 15, 35, 70, 126, 210, . . . (C)

1, 6, 21, 56, 126, 252, 462, . . . (D)

These numbers in (C) and (D) are the fifth and sixth horizontal rows of Figure 3.

Claim 1:

$$\frac{5}{3} = \left(\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \cdots\right)\left(\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \cdots\right)$$

Proof of the Claim

Part 1: To show that $\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \cdots = \frac{4}{3}$. Consider the denominators of the above fractions. They are

1	5	15	35	70	126	......

The successive orders of differences are

4	10	20	35	56	.....
6	10	15	21		.....
4	5	6			.....
1	1				.....

Clearly terms in the fourth order of difference are equal. So according to the theorem namely, "if the terms in the p^{th} order of differences are equal, the n^{th} term of the series is a rational integral function of n of p dimensions." [7,p.327] the n^{th} term of the series

1,5,15,35,70,126,...

can be assumed as

$$U_n = A + Bn + Cn^2 + Dn^3 + En^4$$

Then

$$U_1 = 1 = A + B + C + D + E$$

$$U_2 = 5 = A + 2B + 4C + 8D + 16E$$

$$U_3 = 15 = A + 3B + 9C + 27D + 81E$$

$$U_4 = 35 = A + 4B + 16C + 64D + 256E$$

$$U_5 = 70 = A + 5B + 25C + 125D + 625E$$

When the above equations are solved the following values are obtained.

$$A = 0,\ B = \frac{1}{4},\ C = \frac{11}{24},\ D = \frac{1}{4},\ E = \frac{1}{24}$$

$$U_n = 0 + \frac{n}{4} + \frac{11n^2}{24} + \frac{n^3}{4} + \frac{n^4}{24}$$

$$= \frac{n^4 + 6n^3 + 11n^2 + 6n}{24}$$

So the n^{th} term, T_n, of the series $\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \ldots$ is given by

$$Tn = \frac{24}{n^4 + 6n^3 + 11n^2 + 6n}$$

$$= \frac{4!}{n(n + 1)(n + 2)(n + 3)}$$

In order to find the sum of n terms, S_n, of the series under consideration, the following rule could be followed. [7,p.316]

The rule

Write down the n^{th} term, strike off a factor from the beginning, divide by the number of factors so diminished and by the common difference, change the sign and add a constant.

So
$$S_n = C - \frac{1 \cdot 24}{3 \cdot \not{n}(n+1)(n+2)(n+3)}$$

$$S_1 = 1 = C - \frac{1 \cdot 24}{3 \cdot 2 \cdot 3 \cdot 4}$$

$$1 = C - \frac{1}{3}$$

$$C = 1 + \frac{1}{3} = \frac{4}{3}$$

$$S_n = \frac{4}{3} - \frac{1 \cdot 24}{3 \cdot (n+1)(n+2)(n+3)}$$

If the symbol ∞ stands for infinity then

$$S_\infty = \lim_{n \to \infty} \left[\frac{4}{3} - \frac{8}{(n+1)(n+2)(n+3)} \right]$$

$$= \frac{4}{3} - 8 \cdot \lim_{n \to \infty} \left[\frac{1}{(n+1)(n+2)(n+3)} \right]$$

$$= \frac{4}{3} - 8.0$$

$$= \frac{4}{3}$$

So $\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \frac{1}{126} + \frac{1}{210} + \ldots = \frac{4}{3}$

Part 2:

Following the above procedure exactly it could be shown that

$\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \frac{1}{126} + \frac{1}{252} + \frac{1}{462} + \ldots = \frac{5}{4}$

Hence

$$\left(\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \ldots\right)\left(\frac{1}{1} + \frac{1}{6} + \frac{1}{21} + \frac{1}{56} + \ldots\right)$$

$$= \frac{4}{3} \cdot \frac{5}{4} = \frac{5}{3}$$

So the claim is proved.

Claim 2: $\frac{5}{3}$ can also be obtained due to the division of a series by another series where the two series are related with the Pascal Triangle.

Proof of the claim: Form a finite arithmetic series with the first term and the common difference being 5. Form another finite arithmetic series with the first term and the common difference being equal to 3. Let S_1 and S_2 represent the sums of those series.

It is interesting to note that

$$\frac{S_1}{S_2} = \frac{5}{3}$$

$$S_1 = 5 + 10 + 15 + 20 + 35 + \ldots + 5n$$

$$S2 = 3 + 6 + 9 + 12 + 15 + \ldots + 3n$$

$$\frac{S_1}{S_2} = \frac{5\,\frac{n(n+1)}{2}}{3\,\frac{n(n+1)}{2}} = \frac{5}{3}$$

note: $S_1 = 5(1 + 2 + 3 + 4 + 5 + \ldots + n)$

$S_2 = 3(1 + 2 + 3 + 4 + 5 + \ldots + n)$

The numbers 1, 2, 3, 4, 5, 6,...,n are the first n terms of the second oblique column of numbers in Pascal Triangle.

2.4 The fraction $\frac{5}{3}$ and some related theorems

The difference between the numerator and the denominator of "$\frac{5}{3}$" is 2 and the number of terms in the continued fraction of $\frac{5}{3}$ is 3 for $\frac{5}{3}$ = (1,1,2). This helps to formulate the following theorem.

Theorem 1: Consider the fraction $\frac{M}{N}$ where N is an odd number and the difference between M and N is 2. Prove that the number of terms in the continued fraction of $\frac{M}{N}$ is 3.

Proof: Under the given conditions N is of the form $2n + 1$ where n is a natural number. Consequently $M = (2n + 1) + 2$

Now
$$\frac{M}{N} = \frac{(2n+1)+2}{2n+1}$$

$$= 1 + \frac{2}{2n+1}$$

$$= 1 + \cfrac{1}{\cfrac{2n+1}{2}}$$

$$= 1 + \cfrac{1}{n + \cfrac{1}{2}}$$

$$\frac{M}{N} = (1,n,2)$$

The number of terms in the continued fraction of $\frac{M}{N}$ is 3.

It is to be noticed that the above theorem still holds, even if M is 2 more than a multiple of N.

Information for Theorem 2

Consider $\frac{"4"}{3}$ where 4 is the sum of the terms in the continued fraction of $\frac{5}{3}$. 4 is 1 more than 3 and $\frac{4}{3}$ has two terms in its continued fraction expansion. This helps to formulate the following theorem.

Theorem 2: Consider $\frac{M}{N}$ where M is 1 more than a multiple of N. Prove that the number of terms in the continued fraction of $\frac{M}{N}$ is the same as the number of terms in the continued fraction of $\frac{M_1}{N_1}$ where $N_1 = 2$ and M_1 being an odd number.

Proof: Under the given conditions

$$\frac{M}{N} = \frac{k \cdot N + 1}{N} \qquad \text{where } k \text{ is a natural number}$$

$$= k + \frac{1}{N}$$

$\frac{M}{N} = (k,N)$. $\frac{M}{N}$ has 2 terms in its continued fraction expansion.

Now $\frac{M_1}{N_1} = \frac{2n + 1}{2}$ where n is a natural number

$\frac{M_1}{N_1} = (n,2)$. $\frac{M_1}{N_1}$ has 2 terms in its continued fraction expansion. Hence the theorem is proved.

Information for Theorem 3

The difference between 5 and 3 is 2. Now consider all the fractions of the similar type.

Table 1

Fraction (F)	number of terms in the continued fraction (n)
$\frac{5}{3}$	3
$\frac{7}{5}$	3
$\frac{9}{7}$	3
$\frac{11}{9}$	3
. . . .	. . . .
. . . .	. . . .

Theorem 3: Consider all fractions $\frac{M}{N}$, given in column 1 of Table 1. The difference between M and N is 2. In Theorem 1, it is obtained that $\frac{M}{N} = (1,n,2)$. Collect all the middle terms in the continued fraction of $\frac{M}{N}$. Prove that the collection is the set of natural numbers.

Proof: From the given information one has

$$\frac{M}{N} = \frac{(2n + 1) + 2}{(2n + 1)} = 1 + \cfrac{1}{1 + \cfrac{1}{2}} = (1,n,2)$$

Let $f(n) = \frac{(2n + 2) + 2}{(2n + 1)} = (1,n,2)$

By giving values to n, one obtains the following table.

Table 2

n	f(n)	(1,n,2)
1	f(1)	(1,1,2)
2	f(2)	(1,2,2)
3	f(3)	(1,3,2)
4	f(4)	(1,4,2)
. . .	. . .	. . .
. . .	. . .	. . .

This is a 1-1 correspondence between the natural numbers and the fractions of the desired form.

<u>A small result</u>: It is obtained in Theorem 1 that $\frac{M}{N} = (1,n,2)$. Given the fraction $\frac{M}{N}$, can one give a formula that gives the middle term, namely n ?

<u>Answer</u>: Yes. There are three formulas

(1) Middle term = $\frac{1}{2}\left[\frac{\text{numerator} + \text{denominator}}{2}\right] - 1$

(2) Middle term = $\frac{\text{numerator} - 3}{2}$

(3) Middle term = $\frac{\text{denominator} - 1}{2}$

<u>Theorem 4</u>: Consider any fraction of column 1 of Table 1. Let that fraction be represented by $\frac{M_2}{N_2}$. Prove that the number of terms in the continued fraction of $\frac{M_2 \pm kN_2}{N_2}$ is 3. (k is a natural number).

Proof: Let M_2 and N_2 represent the numerator and denominator of any fraction of column 1 of Table 1. The denominator N_2 is of the form 2n + 1, where n is a natural number. Also $M_2 = N_2 + 2$

Now
$$\frac{M_2 \pm kN_2}{N_2} = \frac{(N_2 + 2) \pm kN_2}{N_2}$$
$$= \frac{N_2\,(1 \pm k) + 2}{N_2}$$
$$= (1 \pm k) + \frac{2}{N_2}$$
$$= (1 \pm k) + \frac{1}{\frac{N_2}{2}}$$
$$= (1 \pm k) + \frac{1}{\frac{2n+1}{2}} \qquad \text{Since } N_2 = 2n + 1$$
$$= (1 \pm k) + \frac{1}{n + \frac{1}{2}}$$

$$\frac{M_2 \pm kN_2}{N_2} = ((1 \pm k),n,2)$$

The number of terms in the continued fraction of $\frac{M_2 \pm kN_2}{N_2}$ is 3.

Theorem 5: In Theorem 4, it is obtained that $\frac{M_2 \pm kN_2}{N_2} = (1 \pm k,n,2)$. Collect all the middle terms, namely n. Prove that the collection is the set of natural numbers.

Proof. This proof is similar to the proof of Theorem 3.

Theorem 6: Consider any fraction of column 1 of Table 1. Let that fraction be represented by $\frac{M_2}{N_2}$. Show that the number of terms in the continued fraction expansion of $\frac{M_2 \cdot N_2}{M_2 + N_2}$ is 3.

Proof: Let M_2 and N_2 represent the numerator and denominator of any fraction of column 1 of Table 1. It is to be noticed that N_2 and M_2 are of the following form.

$N_2 = 2n + 1$ where n is a natural number

$M_2 = (2n + 1) + 2$

$$\frac{M_2 \cdot N_2}{M_2 + N_2} = \frac{((2n + 1) + 2) \cdot (2n + 1)}{((2n + 1) + 2) + (2n + 1)}$$

$$= \frac{(2n + 1)^2 + 2(2n + 1)}{4(n + 1)}$$

$$= \frac{4n^2 + 8n + 3}{4(n + 1)} = \frac{4n(n + 1) + 4n + 3}{4(n + 1)}$$

$$= n + \frac{4n + 3}{4n + 4}$$

$$= n + \cfrac{1}{\cfrac{4n + 4}{4n + 3}}$$

$$= n + \cfrac{1}{1 + \cfrac{1}{4n + 3}}$$

$$\frac{M_2 \cdot N_2}{M_2 + N_2} = (n, 1, (4n + 3))$$

The number of terms in the continued fraction expansion of $\frac{M_2 \cdot N_2}{M_2 + N_2}$ is 3.

Theorem 7: In Theorem 6 it is proved that $\frac{M_2 \cdot N_2}{M_2 + N_2} = (n,1,(4n + 3))$

Collect all the first terms in the continued fraction of $\frac{M_2 \cdot N_2}{M_2 + N_2}$.

Prove that the collection is the set of natural numbers.

Proof: The proof is very similar to the proof of Theorem 3.

Theorem 8: In Theorem 6 it is proved that $\frac{M_2 \cdot N_2}{M_2 + N_2} = (n,1,(4n + 3))$.

Collect all the third terms in the continued fraction of $\frac{M_2 \cdot N_2}{M_2 + N_2}$.

Then the following set S_1 is obtained.

$S_1 = \{4n + 3 \mid n \text{ is a natural number}\}$

$S_1 = \{7,11,15,19,23,...\}$

Let S_1 be represented by the set $\{a_1,a_2,a_3,...\}$ where $a_1 = 7$, $a_2 = 11$, $a_3 = 5$,..., and so on.

Prove the following

1.) The number of terms in the continued fraction of $\frac{a_i + 1}{a_i}$ is equal to the difference $a_{i+1} - a_i$.

2.) The collection of all second terms in the continued fraction of $\frac{a_i + 1}{a_i}$ is the set of natural numbers.

1.) Proof: It is to be noticed that every element in the set S_1 is an odd number and the difference between two successive elements of S_1 is 4.

So if a_i is of the form $2n + 1$, then a_{i+1} is $(2n + 1) + 4$

$$\frac{a_{i+1}}{a_i} = \frac{(7 + (n - 1)4) + 4}{7 + (n - 1)4}$$

$$= \frac{(4n + 3) + 4}{4n + 3} = 1 + \frac{4}{4n + 3}$$

$$= 1 + \cfrac{1}{\cfrac{4n + 3}{4}}$$

$$= 1 + \cfrac{1}{n + \cfrac{3}{4}}$$

$$= 1 + \cfrac{1}{n + \cfrac{1}{1 + \cfrac{1}{3}}}$$

$$\frac{a_{i+1}}{a_i} = (1,n,1,3)$$

The number of terms in the continued fraction of $\frac{a_{i+1}}{a_i}$ is equal to 4. Since $a_{i+1} - a_i = 4$, the theorem is proved.

2.) Proof : The proof is very similar to the proof of Theorem 3.

Information for Theorem 9

In the fraction $\frac{5}{3}$, 3 does not divide 5 but 3 divides 5 + 1. This helps to formulate the following theorem.

Theorem 9 : Let $\frac{M}{N}$ be a fraction satisfying the following conditions.

1.) $N > 2$

2.) $M > N$ and N divides $M + 1$

Then the number of terms in the continued fraction of $\frac{M}{N}$ is 3.

Proof : Under the given conditions it is obvious that M is $(N - 1)$

more than a multiple of N.

So $\frac{M}{N} = \frac{N \cdot n + N - 1}{N}$ n is a natural number such that $M = N \cdot n + N - 1$

$$= n + \frac{N - 1}{N}$$

$$= n + \cfrac{1}{\cfrac{N}{N - 1}}$$

$$= n + \cfrac{1}{\cfrac{N - 1 + 1}{N - 1}}$$

$$= n + \cfrac{1}{1 + \cfrac{1}{N - 1}}$$

$$\frac{M}{N} = (n, 1, N - 1)$$

The number of terms in the continued fraction of $\frac{M}{N}$ is 3.

As it is proved before, it is easy to verify that, the collection of all the first terms in the continued fraction of $\frac{M}{N}$ is the set of natural numbers. In the same way if we collect all the third terms in the continued fraction of $\frac{M}{N}$ and include 1 in that collection that set would also turn out to be the set of natural numbers.

Information for Theorem 10

Consider the squares listed below

0 1 4 9 16 15 . . .

First order of difference are

1 3 5 7 9 . . .

Note: It is to be noted that the numbers 3 and 5 are occurring in the first order of difference.

Consider the sum of numbers in the first order of difference

i.e. $1 + 3 + 5 + 7 + 9 = 25 - 0 = 25$

$3 + 5 + 7 + 9 = 25 - 1 = 24$

Also the number of terms in the continued fraction expansion of $\frac{25}{24}$ is equal to the common difference of the series of numbers that are given as the 'first order of difference' above. [4,p.31]

This helps to formulate the following theorem.

Theorem 10: Consider the series

$1, 3, 5, 7, 9, 11, 13, \ldots$

Let $S_n = 1 + 3 + 5 + 7 + 9 + 11 + 13 + \ldots$

Prove that the number of terms in the continued fraction expansion of $\frac{S_n}{S_n - 1}$ is equal to the common difference of the given arithmetic progression.

Proof: $S_n = \frac{n}{2}[2a_1 + (n - 1)d]$ (formula)

$S_n = \frac{n}{2}[2 \cdot 1 + (n - 1)2]$

$= \frac{n}{2}[2 + 2n - 2] = n^2$

Now $\frac{S_n}{S_n - 1} = \frac{n^2}{n^2 - 1}$

S_n is 1 more than $S_n - 1$. So by Theorem 2 of Section 2.2. $\frac{S_n}{S_n - 1}$ has two terms in its continued fraction. Since the common difference of the given arithmetic progression is 2, the theorem is proved.

Introduction to Theorem 11:

It is obvious that $\frac{5}{2} + 5 \neq \frac{5}{2} \cdot 5$. In other words the numbers $\frac{5}{2}$ and 5 are not qualified to be called escalator numbers. (Escalator numbers are studied in the fourth chapter). Yet the numbers $(\frac{5}{2} + 5)$ and $(\frac{5}{2} \cdot 5)$ share a common property in the light of continued fractions.

$$\frac{5}{2} + \frac{5}{1} = \frac{5 + 10}{2} = \frac{15}{2} = (7,.2)$$

$$\frac{5}{2} \cdot 5 = \frac{25}{2} = (12,2)$$

The number of terms in the continued fraction expansion of $(\frac{5}{2} + 5)$ and $(\frac{5}{2} \cdot 5)$ is same. This helps to formulate the following theorem.

Theorem 11: Consider $\frac{M}{N}$ where M is 1 more than a multiple of N. Show that for m being the natural number, the number of terms in the continued fraction expansion of

$$\frac{mN + 1}{N} + (mN + 1) \quad \text{and} \quad \frac{mN + 1}{N} \cdot (mN + 1)$$

is the same.

Proof: $\frac{mN + 1}{N} + (mN + 1) = \frac{mN + 1 + mN^2 + N}{N}$

$$= \frac{N(m + mN + 1) + 1}{N}$$

$$= ((m + mN + 1),N)$$

$$\frac{(mN + 1)}{N} \cdot (mN + 1) = \frac{m^2N^2 + 2mN + 1}{N}$$

$$= \frac{N(m^2N + 2m) + 1}{N}$$

$$= (m^2N + 2m) + \frac{1}{N}$$

$$= ((m^2N + 2m), N)$$

The number of terms in the continued fraction expansion of

$$\left(\frac{mN+1}{N} + mN + 1\right) \quad \text{and} \quad \left(\frac{mN+N}{N} \cdot (mN+1)\right)$$

is 2. Hence the theorem is proved.

Theorem 12: Let $a = bq + r$ where $r < b < a$ and r,b and a are natural numbers, greater than 1. If the number of terms in the continued fraction of $\frac{a}{b}$ is k then the number of terms in the continued fraction of $\frac{b}{r}$ is $k - 1$.

Proof: Given $a = bq + r$

$$\frac{a}{b} = q + \frac{r}{b}$$

$$= q + \cfrac{1}{\frac{b}{r}}$$

Since the natural number b is greater than r, let $\frac{b}{r}$ have m terms in its continued fraction expansion. Since the number of terms in the continued fraction expansion of $\frac{a}{b}$ is given to be k,

$$1 + m = k$$

$$\text{i.e.} \quad m = k - 1$$

Hence $\frac{b}{r}$ has $(k - 1)$ terms in its continued fraction expansion.

Theorem 13: Consider the following series $\frac{2}{1}, \frac{4}{3}, \frac{8}{7}, \frac{16}{15}, \ldots$. Prove that for $n > 1$, n^{th} term has two terms in its continued fraction expansion.

Proof: For $n > 1$, the numerator of each fraction of the given series of numbers is 1 more than the denominator. Hence by Theorem 2 of this section n^{th} term has two terms in its continued fraction expansion.

Theorem 14: Consider the series given by

$$S = \frac{2}{3} + \frac{3}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \frac{2}{3^5} + \frac{3}{3^6} + \ldots$$

Prove that S has two terms in its continued fraction expansion.

Proof: $S = \left(\frac{2}{3} + \frac{2}{3^3} + \frac{2}{3^5} + \ldots \right) + \left(\frac{3}{3^2} + \frac{3}{3^4} + \frac{3}{3^6} + \ldots \right)$

$$= \frac{\frac{2}{3}}{1 - \left(\frac{1}{3^2}\right)} + \frac{\frac{2}{3^2}}{1 - \left(\frac{1}{3^2}\right)}$$

$$= \frac{2}{3} \cdot \frac{9}{8} + \frac{3}{9} \cdot \frac{9}{8}$$

$$= \frac{3}{4} + \frac{3}{8} = \frac{6 + 3}{8} = \frac{9}{8}$$

The numerator of the fraction given by S is 1 more than its denominator. So by Theorem 2 of this section S has two terms in its continued fraction expansion.

Theorem 15: Prove that $\frac{5}{3}$ is the only fraction of the form $\frac{n + 2}{n}$ when expanded in the form of a continued fraction has terms in its expansion whose sum is equal to the average of n + 2 and n.

Proof: Assume that there exists a pair of numbers (n + 2,n), other than the pair (5,3) such that $\frac{n + 2}{n}$ has terms in its continued fraction expansion whose sum is equal to

$$\frac{n + n + 2}{2} = (n + 1)$$

Now $$\frac{n+2}{n} = 1 + \frac{2}{n} = 1 + \cfrac{1}{\frac{n}{2}}$$

$$= 1 + \cfrac{1}{\frac{(n-1)}{2} + \frac{1}{2}}$$

$$= \left(1, \frac{n-1}{2}, 2\right)$$

Now $$n + 1 = 1 + \frac{n-1}{2} + 2 = \frac{2 + n - 1 + 4}{2} = \frac{n+5}{2}$$

i.e. $n + 5 = 2n + 2$

$2n - n = 5 - 2$

i.e. $n = 3$ so $n + 2 = 3 + 2 = 5$

This is a contradiction, so the assumption is wrong. So (5,3) is the only such pair.

Chapter III: 25 Interesting Theorems

Introduction to Theorem 1: The interesting fraction $\frac{5}{3}$ can be expressed in the following manner.

$$\frac{5}{3} = 1 + \frac{2}{3}$$

The symbols that are involved in the right hand side of the equation are shown below.

1 2 3	First three natural numbers
+	The addition operation

When the binary operation, + , takes place among the numbers 1, 2 and 3 the result could also be expressed, by means of the same numbers and another binary operation, namely multiplication.

It is obvious that

$$1 + 2 + 3 = 1 \times 2 \times 3 = 6$$

6 is called a perfect number because the number 6 is equal to the sum of its proper factors.

The symbols 1, 2, 3 and + could be arranged in the following way.

Figure 4

+

1

2 3

Note that the numbers 1, 2 and 3 are arranged above in the form of a triangle. Call 1, 2 as side 1 and call 1, 3 as side 2. Let the binary operation, addition, + , take place on the numbers of the side 1 and on the numbers of the side 2. These resulting numbers, namely 1 + 2 = 3 and 1 + 3 = 4 are arranged on the corresponding sides as shown below.

Figure 5

+

1

Side 1 2 3 Side 2

3 4

Let the binary operation take place between this resulting number and the one above it. These new resulting numbers are arranged in the same way as 3 and 4 are arranged. When this process is continued one obtains the following triangle.

Figure 6

+

1

2 3

3 4

Side 1 5 7 Side 2

8 11

13 18

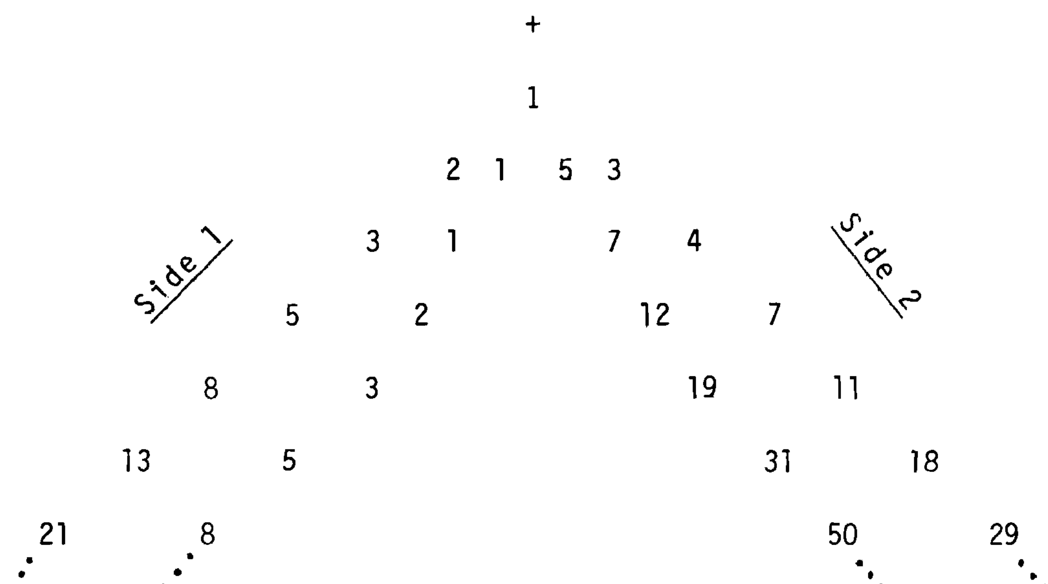
. .

. .

. .

Let x represent the difference between any two numbers of the above figure, which are placed horizontally. Let z represent the sum of any two numbers of the above figure which are placed horizontally. Let x = y. One could arrange the numbers y and z along the sides 1 and 2 of Figure 6 as shown below.

Figure 7



The newly obtained series of numbers, namely 1, 1, 2, 3, 5, 8, ... is well known in the world of mathematics as Fibonacci numbers. It is quite obvious that the set of Fibonacci numbers is the set of numbers on the side 1 of Figure 7. The other series of numbers, 1, 5, 7, 12, 19, ..., and 2, 3, 4, 7, 11, ..., share a common property with respect to the set of natural numbers. How they share it, is explained by means of continued fractions in the subsequent theorems.

Consider the set of numbers on the side 1 of Figure 7. In other words consider the set of Fibonacci numbers. The following theorem could be formulated.

Theorem 1: Let $s = u_1, u_2, u_3, u_4, \ldots$

where $u_1 = 1$

$u_2 = 2$ and

$u_n = u_{n-1}$ for $n > 2$

Let $\frac{u_{i+1}}{u_i} = (c_1, c_2, c_3, \ldots)$ where $(c_1, c_2, c_3, \ldots)$ is the continued fraction expansion of $\frac{u_{i+1}}{u_i}$. (Note: Some C_i may be zero). Note the number of terms in the continued fraction expansion of $\frac{u_{i+1}}{u_i}$ and call it n. Prove that the set of all n's, say M is the set of all natural numbers.

Proof: $S = \{1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

The following table is formed with the help of the elements of S

Table 3

Fraction	Continued Fraction Expansion	Number of terms in the Continued Fraction Expansion
$\frac{2}{1}$	(2)	1
$\frac{3}{2}$	(1,2)	2
$\frac{5}{3}$	(1,1,2)	3
$\frac{8}{5}$	(1,1,1,2)	4
$\frac{13}{8}$	(1,1,1,1,2)	5
$\frac{21}{13}$	(1,1,1,1,1,2)	6

. . . .

From the table it is suggested that $\frac{U_{n+1}}{U_n}$ has n nonzero components in its continued fraction expansion. For n = 1, one gets

$$\frac{U_{1+1}}{U_1} = \frac{U_2}{U_1} = \frac{2}{1} = (2,0,0,\ldots)$$ (There is one nonzero term in the continued fraction)

Now for $n = k \geq 2$, assume $\frac{U_{k+1}}{U_k}$ has k nonzero components in its continued fraction expansion.

$$\frac{U_{k+1}}{U_k} = (\overbrace{1, 1, \ldots C_k}^{k \text{ terms}}, 0, 0, 0, \ldots)$$

Then $$\frac{U_{k+2}}{U_k} = \frac{U_{k+1} + U_k}{U_{k+1}} = 1 + \frac{U_k}{U_{k+1}}$$

$$= 1 + \frac{1}{\frac{U_{k+1}}{U_k}}$$

So $$\frac{U_{k+2}}{U_{k+1}} = (\overbrace{1,1,1,\ldots,C_k}^{1 + k \text{ terms}},0,0,\ldots)$$

Hence by finite mathematical induction one concludes that, the set M which consists of the number of terms in the continued fraction expansion of each $\frac{U_{n+1}}{U_n}$ includes all the natural numbers.

<u>Note</u>: It is to be noted that, a natural number in the third column of Table 3 represents the position of 2 in the corresponding expression of the second column. Also the same natural number is 1 less than the sum of those terms of the corresponding expression of column 2.

<u>Theorem 2</u> : Consider the numbers that are on the side 2 of Figure 7. The set of those numbers is $S = \{1, 3, 4, 7, 11, \ldots\}$. Define S

the following way.

$S = u_1, u_2, u_3, u_4, \ldots$

$u_1 = 1$

$u_2 = 3$ and

$u_n = u_{n-1} + u_{n-2}$ for $n > 2$

Let $\frac{U_{i+1}}{U_i} = (c_1, c_2, c_3, \ldots)$ where $(c_1, c_2, c_3, \ldots)$ is the continued fraction expansion of $\frac{U_{i+1}}{U_i}$. (Note: Some c_i may be zero). Note the number of terms in the continued fraction expansion of $\frac{U_{i+1}}{U_i}$ and call it n.

Prove that the set of all n's, say M, is the set of all natural numbers.

<u>Proof</u>: $S = \{1, 3, 4, 7, 11, \ldots\}$

The following table is formed with the help of the elements of S.

Table 4

Fraction	Continued Fraction Expansion	Number of terms in the Continued Fraction expansion
$\frac{3}{1}$	(3)	1
$\frac{4}{3}$	(1,3)	2
$\frac{7}{4}$	(1,1,3)	3
$\frac{11}{7}$	(1,1,1,3)	4
$\frac{18}{11}$	(1,1,1,1,3)	5
⋮	⋮	⋮

The rest of the proof is very similar to the proof of Theorem 1.

Note: It is to be noted that, a natural number in the third column of Table 4, represents the position of 3 in the corresponding expression of the second column. Also the same natural number is 2 less than the sum of those terms of the corresponding expression of column 2.

Theorem 3: Consider the numbers that are on the side 2 of Figure 5. The set of those numbers is $S = \{1, 5, 7, 12, 19, 31, 50, \ldots\}$. Define S in the following way.

$$S = \{u_1, u_2, u_3, u_4, \ldots\}$$

$$u_1 = 1$$

$$u_2 = 5$$

$$u_3 = 7$$

and $$u_n = u_{n-1} + u_{n-2} \quad \text{for } n > 3$$

Let $\frac{u_{i+1}}{u_i} = (c_1, c_2, c_3, \ldots)$ where $(c_1, c_2, c_3, \ldots)$ is the continued fraction expansion of $\frac{u_{i+1}}{u_i}$. (Note: Some c_i may be zero). Note the number of terms in the continued fraction expansion of $\frac{u_{i+1}}{u_i}$ and call it n. Prove that the set of all n's, say M includes all the natural numbers greater than 2, and 1. In other words prove that $M = \{1, 3, 4, 5, 6, \ldots\}$

Proof: The proof is very similar to the proof of Theorem 1.

Theorem 4: Consider the set S in Theorem 1. The number of terms in the continued fraction of $\frac{u_{n+1}}{u_n}$ is equal to the number of divisions in finding (u_{n+1},u_n) by Euclidean Algorithm.

Note: (u_{n+1},u_n) is the greatest common divisor of u_{n+1} and u_n.

Proof :

First Part : This part consists of the proof that the number of terms in the continued fraction of $\frac{u_{n+1}}{u_n}$ is n.

The proof is exactly the same as that of Theorem 1.

Second Part : The following proof shows that the number of divisions in finding (u_{n+1},u_n) by the Euclidean Algorithm is n.

For n = 1, one computes

$$(u_{n+1},u_n) = (u_{1+1},u_1) = (u_2,u_1) = (2,1)$$

$$2 = 1 \cdot 2 + 0 \quad \text{(1 division)}$$

For n = 2

$$(u_{n+1},u_n) = (u_{2+1}u_2) = (u_3,u_2) = (3,2)$$

$$3 = 2 \cdot 1 + 1$$

$$2 = 1 \cdot 1 + 1 \quad \text{(2 divisions)}$$

For n = k > 2 Suppose that, in finding (u_{k+1},u_k) by the Euclidean Algorithm, k divisions are necessary.

$$u_{(k+1)+1} = 1 \cdot u_{k+1} + u_k$$

$$\left.\begin{array}{l} u_{k+1} = 1 \cdot u_k + u_{k-1} \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ u_3 = u_2 + u_1 \end{array}\right\} \quad \text{k divisions}$$

(u_{k+2},u_{k+1}) requires k + 1 divisions

By finite mathematical induction one concludes that, in finding (u_{n+1}, u_n) by Euclidean Algorithm, n divisions are necessary.

Hence the theorem is proved.

Note: The following information is useful in understanding the Theorem 5 that follows.

A note on Theon diameters: Theon of Smyrna (C.125) was a noteworthy writer who contributed a lot to arithmetic. One of his propositions is listed below.

Consider two groups of numbers arranged as follows

$n_1 = 1 + 0 \qquad d_1 = 1 + 0 = 1$

$n_2 = 1 + 1 \qquad d_2 = 2 + 1 = 3$

$n_3 = 2 + 3 \qquad d_3 = 4 + 3 = 7$

$n_4 = 5 + 7 \qquad d_4 = 10 + 7 = 17$

. . . .

. . . .

$n_r = n_{r-1} + d_{r-1} \qquad d_r = 2n_{r-1} + d_{r-1}$

Then d^2 is of the form $2n^2 \pm 1$. For example $d_1^2 = 1 = 2n_1^2 - 1$, $d_2^2 = 9 = 2n_2^2 + 1$. The numbers d_r were called Theon diameters. [17,p.5-6]

Consider the following table. This table is associated with the information given above.

Table 5

i	d_i	n_i	$\frac{d_i}{n_i} = (c_1,c_2,c_3,\ldots)$	Number of terms in column 4
1	1	1	(1)	1
2	3	2	(1,2)	2
3	7	5	(1,2,2)	3
4	17	12	(1,2,2,2)	4
5	41	29	(1,2,2,2,2)	5
6	99	70	(1,2,2,2,2,2)	6
7	239	169	(1,2,2,2,2,2,2)	7
.	.	.	. . .	.
.	.	.	. . .	.

Theorem 5: Consider the numbers in the third column of Table 5. They are 1, 2, 5, 12, 29, 70, 169,... . From this series of numbers one could define the following set.

Let $S_1 = u_1,u_2,u_3,u_4,\ldots$ where

$u_1 = 1$

$u_2 = 2$ and

$u_n = 2 \cdot u_{n-1} + u_{n-2}$ for n 2

Let $\frac{u_{i+1}}{u_i} = (c_1,c_2,c_3,\ldots)$ where $(c_1,c_2,c_3,\ldots)$ is the continued fraction expansion of $\frac{u_{i+1}}{u_i}$. (Note: Some c_i may be zero). Note the number of terms in the continued fraction expansion of $\frac{u_{i+1}}{u_i}$ and call it n. Prove that the set of all n's, say M, is the set of all natural numbers.

Proof: One obtains the set S_1 as shown below.

$$S_1 = \{1, 2, 5, 12, 29, 70, 169, \ldots\}$$

Consider the following table.

Table 6

Fraction	Continued Fraction Expansion	Number of Terms in the Continued Fraction Expansion
$\frac{2}{1}$	(2)	1
$\frac{5}{2}$	(2,2)	2
$\frac{12}{5}$	(2,2,2)	3
$\frac{29}{12}$	(2,2,2,2)	4
$\frac{70}{29}$	(2,2,2,2,2)	5
. . .	. . .	. . .
. . .	. . .	. . .

From the above Table 6 one notices that $\frac{u_{n+1}}{u_n}$ has n nonzero components in its continued fraction expansion.

For n = 1, one gets

$$\frac{u_{1+1}}{u_1} = \frac{u_2}{u_1} = \frac{2}{1} = (2,0,0,0,\ldots)$$

(There is one nonzero term in the continued fraction expansion)

For $n = k \geqslant 2$, assume $\frac{u_{k-1}}{u_k}$ has k nonzero components in its continued fraction expansion.

$$\text{i.e.} \quad \frac{u_{k+1}}{u_k} = (\overbrace{2,2,2,\ldots,2}^{k \text{ terms}},0,0,\ldots)$$

$$\text{Then} \quad \frac{u_{k+2}}{u_{k+1}} = \frac{2 \cdot u_{k+1} + u_k}{u_{k+1}}$$

$$= 2 + \frac{u_k}{u_k + 1}$$

$$= 2 + \frac{1}{\frac{u_{k+1}}{u_k}}$$

$$\text{So} \quad \frac{u_{k+2}}{u_{k+1}} = (\overbrace{2,2,2,\ldots,2}^{1 + k \text{ terms}},0,0,\ldots)$$

Hence by finite mathematical induction one concludes that the set M, which consists of the number of terms in the continued fraction expansion of each $\frac{u_{n+1}}{u_n}$ includes all the natural numbers.

<u>An interesting result</u>: If one tries to form a set, whose elements are the sum of the terms in each expression of column 4 of Table 5, the following set, say A, is obtained.

$$A = \{1,\ 1+2,\ 1+2+2,\ 1+2+2+2,\ 1+2+2+2+2, \ldots\}$$

$$\text{i.e.} \quad A = \{1,\ 3,\ 5,\ 7,\ 9, \ldots\}$$

If one tries to form a set, whose elements are the sum of the terms in each expression of column 2 of Table 6, the following set, say B, is obtained.

$$B = \{2,\ 2+2, 2+2+2,\ 2+2+2+2, \ldots\}$$

i.e. B $= \{2, 4, 6, 8, 10, \ldots\}$

It is obvious that A union B is the set of natural numbers.

Theorem 6: Consider Theon diameters. Let $\frac{d_i}{n_i} = (c_1, c_2, c_3, \ldots)$ where $(c_1, c_2, c_3, \ldots)$ is the continued fraction expansion of $\frac{d_i}{n_i}$. (Note: Some c_i may be zero). Note the number of terms in the continued fraction expansion of $\frac{d_i}{n_i}$ and call it n. Prove that the set of all n's is the set of natural numbers.

Proof: Consider the Table 5. From the table one notices that $\frac{d_i}{n_i}$ has i nonzero components in its continued fraction expansion.

One small result namely $\frac{d_i}{n_i} = 1 + \frac{n_{i-1}}{n_i}$ needs to be established.

It is given that $d_i = 2n_{i-1} + d_{i-1}$ and $n_i = n_{i-1} + d_{i-1}$

$\therefore \quad d_i - n_i = n_{i-1}$ i.e. $d_i = n_i + n_{i-1}$

So $\frac{d_i}{n_i} = 1 + \frac{n_{i-1}}{n_i}$

or $\frac{d_k}{n_k} = 1 + \frac{n_{k-1}}{n_k}$

$= 1 + \cfrac{1}{\cfrac{n_k}{n_{k-1}}}$ $(k > 1)$ → (A)

From Table 5

For k = 1,

$\frac{d_1}{n_1} = \frac{1}{1} = (1,0,0,\ldots)$ (There is one nonzero term in the continued fraction expansion).

For k = 2,

$$\frac{n_k}{n_{k-1}} = \frac{n_2}{n_{2-1}} = \frac{n_2}{n_1} = \frac{2}{1} = 2$$

So for k = 2,

$\frac{n_k}{n_{k-1}}$ has 1 term in its continued fraction expansion. Using (A) one finds that $\frac{d_k}{n_k}$ has 2 terms in its continued fraction expansion.

For k = 2

$$\frac{n_k}{n_{k-1}} = \frac{n_3}{n_{3-1}} = \frac{n_3}{n_2} = \frac{5}{2} = 2 + \frac{1}{2} = (2,2)$$

$\frac{n_k}{n_{k-1}}$ has 2 terms in its continued fraction expansion. Consequently, because of (A), $\frac{d_k}{n_k}$ has 3 terms in its continued fraction expansion.

The above result may be summed up in the following table.

Table 7

k	$\frac{n_k}{n_{k-1}}$	$\frac{d_k}{n_k}$
1		1
2	1	2
3	2	3

Now assume for k = n $\geq$ 4 $\frac{n_m}{n_{m-1}}$ has (m-1) terms in its continued fraction expansion. Because of (A) one obtains

$$\frac{d_m}{n_m} = 1 + \frac{1}{\frac{n_m}{n_{m-1}}}$$

So $\frac{d_m}{n_m}$ has (m - 1)+ 1 = m terms in its continued fraction expansion.

Hence by finite mathematical induction, one concludes that the set which consists the number of terms in the continued fraction expansion of each $\frac{d_i}{n_i}$ includes all the natural numbers.

Theorem 7 Consider the numbers in the second column of the Table 5. They can be written in the form of a series as

1, 3, 7, 17, 41, 99, 239,...

The following pattern is noticed.

Table 8

Fraction	Continued Fraction Expansion	Number of Terms in the Continued Fraction Expansion
$\frac{3}{1}$	(3)	1
$\frac{7}{3}$	(2,3)	2
$\frac{17}{7}$	(2,2,3)	3
$\frac{41}{17}$	(2,2,2,3)	4
$\frac{99}{41}$	(2,2,2,2,3)	5
. . . .	. . . . .	. . . .
. . . .	. . . . .	. . . .

The set S which consists the numbers in the above series could be defined in the following way.

$$S = \{ u_1, u_2, u_3, u_4, \ldots \} \text{ where}$$

$$u_1 = 1$$

$$u_2 = 3$$

$$u_n = 2u_{n-1} + u_{n-2} \text{ for } n > 2$$

Let $\frac{u_{i+1}}{u_i} = (c_1, c_2, c_3, \ldots)$ where $(c_1, c_2, c_2, \ldots)$ is the continued fraction expansion of $\frac{u_{i+1}}{u_i}$. (Note: Some c_i may be zero). Note the number of terms in the continued fraction expansion of $\frac{u_{i+1}}{u_i}$ and call it n. Prove that the set of all n's, say M is the set of all natural numbers.

<u>Proof</u>: The proof is very similar to the proof of Theorem 3 of this chapter.

<u>An interesting result</u>: Compute the sums of the terms of each expression in the Table 8. The following set, say D is obtained.

$$D = \{ 3,\ 2+3,\ 2+2+3,\ 2+2+2+3, \ldots \}$$

$$= \{ 3,\ 5,\ 7,\ 9,\ 11,\ 13, \ldots \}$$

It is to be noticed that the first two elements of D are 3 and 5, the numbers with which this thesis work started.

<u>Theorem 8</u>: Consider the series 1, 2, 4, 8, 16, 32, 64,... . The sum of the n terms of the series could be given by the formula $S_n = a \cdot \frac{q^n - 1}{q-1}$ where a = 1 and q = 2. Prove that the number of terms in the continued

fraction of $\frac{S_{n+1}}{S_n}$ (for $n > 1$) is equal to the common ratio of the above geometric series.

Proof: The common ratio of the geometric series, 1, 2, 4, 8, 16, 32,... is 2. The series $S_1, S_2, S_3, S_4, \ldots$ is given by 1, 3, 7, 15, 31, 63,... for $n > 1$, it needs to be proved that $\frac{S_{n+1}}{S_n}$ has 2 terms in its continued fraction expansion.

One could find the n^{th} term of the series 3, 7, 15, 31, 63,... by the following method.

The series is

3   7   15   31   63   . . .

The first order of difference is

4   8   16   32   . . .

It is to be noticed that the series of numbers in the first order of difference are in geometric progression. In order to find the n^{th} term of the given series one could follow the following rule. [7,p.330]

The rule: "If the first few terms of a series are given, and if the p^{th} order of differences of these terms form a geometrical progression whose common ratio is r, then it could be assumed that the general term of the given series is $a\, r^{n-1} + f(n)$ where f(n) is a rational integral function of n of p - 1 dimentions."

So assume that the n^{th} term is

$u_n = a \cdot 2^{n-1} + b$ where the constants a and b are to be determined.

$u_1 = 3 = a + b$

$u_2 = 7 = a \cdot 2 + b$

Now $a + b = 3$

$2a + b = 7$

$a = 4$ and $b = -1$

Hence $u_n = 4 \cdot 2^{n-1} - 1$

For $n > 1$, put $S_{n+1} = u_n$ and $S_n = u_{n-1}$

$$\text{So } \frac{S_{n+1}}{S_n} = \frac{u_n}{u_{n-1}} = \frac{4 \cdot 2^{n-1} - 1}{4 \cdot 2^{n-1-1} - 1} = \frac{2^{n-1+2} - 1}{2^{n-1-1+2} - 1}$$

$$= \frac{2^{n+1} - 1}{2^n - 1}$$

$$\frac{S_{n+1}}{S_n} = 2 + \frac{1}{2^n - 1}$$

$$\text{So } \frac{S_{n+1}}{S_n} = (2, 2^n - 1)$$

For $n > 1$, $\frac{S_{n+1}}{S_n}$ has 2 terms in its continued fraction expansion.

Theorem 9: Consider $\frac{M}{N}$ where M is square of an integer and N = 3. If M is not a multiple of 3, then prove that the number of terms in the continued fraction of $\frac{M}{3}$ is the same as the number of terms in the continued fraction of $\frac{M}{4}$ where M is not a multiple of 4.

Proof :

Part 1 : Let a be any integer. Use the division algorithm with $b = 3$. There are three possible cases namely $a = 3q$, $a = 3q + 1$ and $a = 3q + 2$. Corresponding to these cases there are three possible values of a^2.

$(3q)^2 = 9q^2 = 3 \cdot (3q^2)$

$(3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1$

$(3q + 2)^2 = 9q^2 + 12q + 4 = 3 \cdot (3q2 + 4q + 1) + 1$

In the first case $a^2 = M$ is a multiple of 3. In the second and third cases $a^2 = M$ is one more than a multiple of 3. Now if $a^2 = M$ is not a multiple of 3, then M is of the form 3Q + 1

$$\frac{M}{3} = \frac{3Q + 1}{3} \quad \text{where Q is an integer.}$$

$$= Q + \frac{1}{3}$$

So $\frac{M}{3} = (Q,3)$

So if M is not a multiple of 3 then the number of terms in the continued fraction of $\frac{M}{3}$ is 2.

Part 2: To prove that the number of terms in the continued fraction expansion of $\frac{M}{4}$ is also 2, provided M is not a multiple of 4.

Let a be any integer. Use the division algorithm with b = 4. There are 4 possible cases; a = 4q, a = 4q + 1, a = 4q + 2 and a = 4q + 3. Corresponding to these cases are 4 possible values of a^2.

$(4q)^2 = 16q^2 = (4q^2)4$

$(4q + 1)^2 = 16q^2 + 8q + 1 = (4q^2 + 2q)4 + 1$

$(4q + 2)^2 = 16q^2 + 16q + 4 = (4q^2 + 4q + 1)4$

$(4q + 3)^2 = 16q^2 + 24q + 9 = (4q^2 + 6q + 2)4 + 1$

In the first and third cases $a^2 = M$ is a multiple of 4. In the second and fourth cases, $a^2 = M$ is 1 more than a multiple of 4. Now if

$a^2 = M$ is not a multiple of 4 then M is of the form $4Q + 1$ where Q is an integer.

Now $\frac{M}{4} = \frac{4Q + 1}{4} = Q + \frac{1}{4}$

So $\frac{M}{4} = (Q,4)$. $\frac{M}{4}$ has 2 terms in its continued fraction expansion.

Hence the theorem is proved.

<u>Theorem 10</u>: Consider $\frac{n^3}{7}$ where n is an integer greater than 7. If n^3 is not a multiple of 7 then the number of terms in the continued fraction of $\frac{n^3}{7}$ is either 2 or 3.

<u>Proof</u>: If n is greater than 7, then the division algorithm could be used with b = 7, n has one of the following forms.

$n = 7x + 1$

$n = 7x + 2$

$n = 7x + 3$

$n = 7x + 4$

$n = 7x + 5$

$n = 7x + 6$

<u>Case 1</u>: <u>Let $n = 7x + 1$</u>

$$n^3 = (7x + 1)^3 = 343x^3 + 1 + 3 \cdot 7x \cdot 1(7x + 1)$$
$$= 343x^3 + 147x^2 + 21x + 1$$
$$= 7(49x^3 + 21x^2 + 3x) + 1$$

<u>Case 2</u>: <u>Let $n = 7x + 2$</u>

$$n^3 = (7x + 2)^3 = 343x^3 + 8 + 42x(7x + 2)$$

$$= 343x^3 + 7 \cdot 42x^2 + 2 \cdot 42x + 8$$

$$= 7(49x^3 + 42x^2 + 12x + 1) + 1$$

Case 3: Let $n = 7x + 3$

$$n^3 = (7x + 3)^3 = (7x)^3 + 27 + 3 \cdot 7x \cdot 3(7x + 3)$$

$$= (7x)^3 + 7 \cdot 9x(7x+3) + 27 + 1 - 1$$

$$= 7(7^2x^3 + 9x(7x + 3) + 4) - 1$$

$$= 7(49x^3 + 63x^2 + 27x + 4) - 1$$

Case 4: Let $n = 7x + 4$

$$n^3 = (7x + 4)^3 = (7x)^3 + (4)^3 + 3 \cdot 7x \cdot 4(7x + 4)$$

$$= 7(7^2x^3 + 9 + 12x (7x + 4)) + 1$$

$$= 7(49x^3 + 84x^2 + 48x + 9) + 1$$

Case 5: Let $n = 7x + 5$

$$n^3 = (7x + 5)^3 = (7x)^3 + (5)^3 + 3 \cdot 7x \cdot 5(7x + 5)$$

$$= 7(7^2x^3 + 17 + 15x (7x + 5)) + 6 + 1 - 1$$

$$= 7(49x3 + 105x^2 + 75x + 18) - 1$$

Case 6: Let $n = 7x + 6$

$$n^3 = (7x + 6)^3 = (7x)^3 + (6)^3 + 3 \cdot 7x \cdot 6(7x + 6)$$

$$= 7(7^2x^3 + 30 + 18x (7x + 6)) + 6 + 1 - 1$$

$$= 7(49x^3 + 126x^2 + 108x + 31) - 1$$

From the preceding information it is obvious that, when n is not a multiple of 7, n^3 is of the form $7Q + 1$ or $7Q - 1$ where Q is an integer.

Case 1: Let $n^3 = 7Q + 1$

Then $\frac{n^3}{7} = \frac{7Q + 1}{7} = Q + \frac{1}{7}$

$$\frac{n^3}{7} = (Q,7)$$

So $\frac{n^3}{7}$ has 2 terms in its continued fraction expansion.

Case 2: Let $n^3 = 7Q - 1$

Then $\frac{n^3}{7} = \frac{7Q - 1}{7} = \frac{7Q - 1 - 6 + 6}{7}$

$$= \frac{7(Q - 1) + 6}{7}$$

$$= (Q - 1) + \frac{6}{7}$$

$$= (Q - 1) + \frac{1}{\frac{7}{6}}$$

$$= (Q - 1) + \frac{1}{1 + \frac{1}{6}}$$

$$\frac{n^3}{7} = ((Q - 1),1,6)$$

$\frac{n^3}{7}$ has 3 terms in its continued fraction. So the number of terms in the continued fraction expansion of $\frac{n^3}{7}$ is either 2 or 3.

Theorem 11: Consider $\frac{n^3}{9}$ where n is an integer greater than 9. If n^3 is not a multiple of 9, then the number of terms in the continued fraction of $\frac{n^3}{9}$ is either 2 or 3.

Proof: The proof is very similar to the proof of Theorem 10.

Theorem 12: Consider $\frac{n^4}{5}$ where n is an integer greater than 5. $\frac{n^4}{5}$ has always 2 terms in its continued fraction expansion.

Proof: The integer n takes one of the following forms because of the division algorithm.

$5x + 1$

$5x + 2$

$5x + 3$

$5x + 4$

If $n = 5x + 1$ then $n^4 = (5x + 1)^4 = 5(125x^4 + 100x^3 + 30x^2 + 4x) + 1$

If $n = 5x + 2$ then $n^4 = (5x + 2)^4 = 5(125x^4 + 200x^3 + 120x^2 + 32x + 3) + 1$

If $n = 5x + 3$ then

$$n^4 = (5x + 3)^4 = 5(125x^4 + 300x^3 + 270x^2 + 54x + 16) + 1$$

If $n = 5x + 4$ then

$$n^4 = (5x + 4)^4 = 5(125x^4 + 400x^3 + 480x^2 + 64x + 51) + 1$$

In all the above cases n4 is of the form $5Q + 1$, where Q is an integer.

So $\frac{n^4}{5} = \frac{5Q + 1}{5} = Q + \frac{1}{5}$

$$\frac{n^4}{5} = (Q,5)$$

So $\frac{n^4}{5}$ has 2 terms in its continued fraction expansion.

Theorem 13: Consider $\frac{M^3}{N}$ where N is an odd positive integer and M being 1 more than a multiple of N. Prove that $\frac{M3}{N}$ has 2 terms in its continued fraction expansion.

Proof: Since N is an odd positive integer N is of the form 2n + 1 where n is a positive integer. So $M = (2n + 1)x + 1$ where x is an integer.

$$\text{Now } M^3 = ((2n + 1)x + 1)^3$$
$$= (2n + 1)^3x^3 + 1 + 3(2n + 1)x((2n + 1)x + 1)$$
$$= (2n + 1)[(2n + 1)^2x^3 + 3x((2n + 1)x + 1)] + 1$$

So M^3 is of the form NQ + 1 where Q is an integer.

$$\text{So } \frac{M^3}{N} = \frac{NQ + 1}{N}$$
$$= Q + \frac{1}{N}$$

$$\frac{M^3}{N} = (Q,N)$$

$\frac{M^3}{N}$ has 2 terms in its continued fraction expansion.

Theorem 14: Consider $\frac{M}{P}$ where p is prime and M is greater than p. Let $\frac{M}{P}$ has n terms in its continued fraction expansion, i.e. $\frac{M}{P} = (c_1,c_2,c_3,\ldots c_n)$. Prove that $\frac{aP - a + M}{P}$ also has n terms in its continued fraction expansion, no matter whatever the integer a may be.

Proof: The following Fermat's theorem is used in this proof.

Fermat's Theorem: If p is a prime number, then the difference $a^P - a$ is, for any integer a, divisible by p. [15,p.56]

Hence there exists an integer x such that $a^P - a = xp$.

$$\frac{M}{P} = (c_1,c_2,c_3,\ldots c_n) \text{ implies}$$

$$\frac{M}{P} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{\ddots}{+ \cfrac{1}{c_n}}}}$$

Now $$\frac{a^P - a + M}{P} = \frac{xp + M}{P} = x + \frac{M}{P} = x + c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{\ddots}{+ \cfrac{1}{c_n}}}}$$

$$= (x + c_1) + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{\ddots}{+ \cfrac{1}{c_n}}}}$$

i.e. $$\frac{a^P - a + M}{P} = (x + c_1, c_2, c_{3P}, \ldots, c_n)$$

$\frac{a^P - a + M}{P}$ has n terms in its continued fraction expansion.

Information for Theorem 15: A modern version of one of Zeno's paradoxes is as follows.

Achilles can run 1000 yards a minute while a turtle can run 100 yards a minute. The turtle is placed 1000 yards ahead of Achilles. Zeno's argument states that Achilles can never overtake the turtle for when Achilles has advanced 1000 yards, the turtle is still 100 yards ahead of him. By the time Achilles has covered these 100 yards, the turtle is still ahead of him and so on, ad inifinitum, as the accompanying table shows. [10,p.293-294]

Table 9

Postion	Achilles	Tortoise
1	0	1000
2	1000	1100
3	1100	1110
4	1110	1111
5	1111	1111 · 1
6	1111 · 1	1111 · 11
7	1111 · 11	1111 · 111
	etc.	etc.

Theorem 15: Consider the Table 9. Let M and N represent the distances covered by the turtle and Achilles respectively. Prove that $\frac{M}{N}$, $(N \neq 0)$ has 2 terms in its continued fraction expansion.

Proof: For $N \neq 0$, M is 1 more than a multiple of N. So by Theorem 2 of Section 2.4, $\frac{M}{N}$ has 2 terms in its continued fraction expansion.

Theorem 16: Consider the series

$$\sum_{i=0}^{1} x \cdot 10^i, \quad \sum_{i=0}^{2} x \cdot 10^i, \sum_{i=0}^{3} x \cdot 10^i, \ldots, \sum_{i=0}^{n-1} x \cdot 10^i$$

where x is a natural number less than or equal to 9. Prove that for n 1, the number of terms in the continued fraction of

$$\frac{n^{th} \text{ term of the series}}{(n-1)^{th} \text{ term of the series}} \text{ is } 2.$$

Proof: $\frac{n^{th} \text{ term}}{(n-1)^{th} \text{ term}}$ = $\frac{\sum_{i=0}^{n-1} x \cdot 10^i}{\sum_{i=0}^{n=2} x \cdot 10^i}$

$$\frac{x \cdot \sum_{i=0}^{n-1} 10^i}{x \cdot \sum_{i=0}^{n-2} 10i}$$

$$\frac{\sum_{i=0}^{n-i} 10^i}{\sum_{i=0}^{n-2} 10^i}$$

$$\frac{10^{\circ} + \sum_{i=1}^{n-1} 10^i}{\sum_{i=0}^{n-2} 10^i}$$

$$\frac{\sum_{i=1}^{n-1} 10 \cdot 10^{i-1} + 1}{\sum_{i=0}^{n-2} 10^i}$$

$$\frac{10 \cdot \sum_{i=1}^{n-2} 10^{i-1} + 1}{\sum_{i=0}^{n-2} 10^i}$$

Now the last term of $\sum_{i=1}^{n-1} 10^{i-1}$ is 10^{n-1-1}, i.e. 10^{n-2}. Hence $\sum_{i=1}^{n-1} 10^{i-1}$ can be written as $\sum_{i=0}^{n-2} 10^i$.

So $$\frac{n^{th} \text{ term}}{(n-1)^{th} \text{ term}} = \frac{10 \cdot \sum_{i=0}^{n-2} 10^i + 1}{\sum_{i=0}^{n-2} 10^i}$$

$$10 + \frac{1}{\sum_{i=0}^{n-2} 10^i}$$

$$\frac{n^{th} \text{ term of the series}}{(n-1)^{th} \text{ term of the series}} = (10, \sum_{i=0}^{n-2} 10^i)$$

$\frac{n^{th} \text{ term}}{(n-1)^{th} \text{ term}}$ has 2 terms in its continued fraction expansion.

Theorem 17: Consider the relation

$$s = 1 + r + r^2 + r^3 + \ldots + r^n + \ldots$$

where the common ratio r is of the form $\frac{1}{r_1}$, (the natural number r_1 is greater than 2). Prove that the continued fraction of S has 2 terms.

Proof: $$S = \frac{1}{1 - \frac{1}{r_1}}$$

$$= \frac{1}{\frac{r_1 - 1}{r^1}}$$

$$= \frac{r_1}{r_1 - 1}$$

$$= \frac{r_1 - 1 + 1}{r_1 - 1}$$

$$= 1 + \frac{1}{r_1 - 1}$$

$S = (1, r_1 - 1)$

So S has 2 terms in its continued fraction expansion.

Theorem 18: Consider the following equation.

$S_n = 1 + 2 + 3 + 4 + 5 + \ldots + n$

Prove the following:

(i) For $n > 1$, $\frac{(n+1)^{th} \text{ term}}{n^{th} \text{ term}}$ has 2 terms in its continued fraction expansion.

(ii) For $n > 2$, prove that $\frac{S_n + 1}{S_n}$ has at most 3 terms in its continued fraction expansion.

(iii) Prove that $\frac{\sum_{k=1}^{n} k^2}{\sum_{k=1}^{n} k}$ has at most 3 terms in its continued fraction expansion.

Proof: (i) For $n > 1$, let $(n+1)^{th}$ term and n^{th} term be represented by M and N respectively it is to be noticed that M is 1 more than N. So by Theorem 2 of Section 2.4, $\frac{M}{N}$ has 2 terms in its continued fraction expansion.

(ii) $S_n = \frac{n(n+1)}{2}$

$$S_{n+1} = \frac{(n+1)(n+2)}{2}$$

$$\frac{S_{n+1}}{S_n} = \frac{\frac{(n+1)(n+2)}{2}}{\frac{n(n+1)}{2}}$$

$$= \frac{n + 2}{n}$$

$$= 1 + (2/n)$$

$$= 1 + (1)/(n/2)$$

Case 1: n is even

So $\frac{S_{n+1}}{S_n} = 1 + \frac{1}{n_1}$ Since $n = 2n_1$ for some integer n_1

Therefore $\frac{S_{n+1}}{S_n} = (1, n_1)$

Case 2: n is odd

$$\frac{S_{n+1}}{S_n} = 1 + \cfrac{1}{\cfrac{2n_1 + 1}{2}}$$

$$= 1 + \frac{1}{n_1 + (1/2)}$$

Therefore $\frac{S_{n+1}}{S_n} = (1, n_1 , 2)$

So $\frac{S_{n+1}}{S_n}$ has at most 3 terms in its continued fraction expansion.

(iii) $$\frac{\sum_{K=1}^{n} K^2}{\sum_{K=1}^{n} K} = \frac{\frac{1}{3}(1+2n) \cdot \sum_{K=1}^{n} K}{\sum_{K=1}^{n} K} = \frac{2n+1}{3} = \frac{M}{3}$$

where $M = 2n + 1$

Now use the division algorithm with $b = 3$. So M is one of the following forms:

$M = 3q$

$M = 3q + 1$ where q is an integer.

$M = 3q + 2$

If M = 3q then $\frac{M}{3} = \frac{3q}{3} = q$

If M = 3q+1 then $\frac{M}{3} = \frac{3q+1}{3} = q + \frac{1}{3}$

If M = 3q+2 then $\frac{M}{3} = \frac{3q+2}{3} = q + \frac{2}{3} = q + \cfrac{1}{1 + \cfrac{1}{2}}$

In this case $\frac{M}{3} = (q, 1, 2)$.

It is obvious that $\frac{M}{3}$ has at most 3 terms in its continued fraction expansion. In other words $\dfrac{\sum_{k=1}^{n} k^2}{\sum_{k=1}^{n} k}$ has at most 3 terms in its continued fraction expansion.

<u>Theorem 19</u>: Let $\frac{M}{N}$ be a fraction of natural numbers where M and N satisfy the following conditions.

(i) N is a positive odd number other than 1.

(ii) M is (N - 1) more than a multiple of N.

(iii) Then $\frac{M}{N}$ has 3 terms in its continued fraction expansion.

<u>Proof</u>: Let N = 2m + 1 where m is a positive integer.

$$\frac{M}{N} = \frac{n(2m+1) + ((2m+1) - 1)}{2m+1}, \quad n \text{ is a positive integer}$$

$$= n + \frac{(2m+1) - 1}{2m+1}$$

$$= n + \frac{2m}{2m+1}$$

$$= n + \cfrac{1}{1+\cfrac{1}{2m}}$$

$$\frac{M}{N} = (n, 1, 2m)$$

Therefore $\frac{M}{N}$ has 3 terms in its continued fraction expansion.

Theorem 20: In Theorem 19, the following result is obtained.

$$\frac{M}{N} = (n, 1, 2m).$$

Let the natural numbers n and m be greater than 1 and S_n, S_{2m} stand for the sums of n terms and 2m terms respectively. Prove that $\frac{Sn}{S_{n-1}}$ has at most 3 terms in their continued fraction expansion.

Proof:

$$\frac{Sn}{S_{n-1}} = \frac{\frac{n(n+1)}{2}}{\frac{(n-1)\cdot n}{2}} = \frac{n+1}{n-1} = \frac{(n-1)+2}{n-1}$$

S_n is 2 more than S_{n-1} . By Theorem 1 of Section 2.9, $\frac{S_n}{S_{n-1}}$ has at most 3 terms in its continued fraction expansion.

The sum of k terms of an arithmetic progression is given by $S_k = \frac{k}{2}\ [2a_1 + (k-1)d]$ where a_1 is the first term and d is the common difference.

Now $S_{2m} = \frac{2m}{2}[2 \cdot 2 + (2m-1)2]$

$$= m(4 + 4m - 2) = m(4m + 2)$$

$$= 2m(2m + 1)$$

$$\frac{S_{2m}}{S_{2m-1}} = \frac{2m(2m+1)}{(2m-1)(2m-1+1)} \qquad \frac{2m(2m+1)}{(2m-1)(2m)}$$

$$= \frac{2m + 1}{2m - 1}$$

$$= \frac{(2m-1) + 2}{2m - 1}$$

S_{2m} is 2 more than S_{2m-1} . So by Theorem 1 of Section 2.4, $\frac{S_{2m}}{S_{2m-1}}$ has at most 3 terms in its continued fraction expansion.

Information for Theorem 21: Let the symbol ∞, stand for infinity.

Theorem 21: Let $S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

Prove that $\frac{1}{S}$ has 2 terms in its continued fraction expansion.

Proof: $S = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots$

$$S = \frac{1}{1 - (-\frac{1}{2})} = \frac{1}{1 + \frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$\frac{1}{S} = \frac{1}{\frac{2}{3}} = \frac{3}{2} = 1 + \frac{1}{2} = (1,2)$$

So $\frac{1}{S}$ has 2 terms in its continued fraction expansion.

Theorem 22: Consider the fraction $\frac{M_i}{N}$, $i = 1, 2, 3, \ldots$ where $N > 1$ and $M_i > N$. If N divides into each M_i, leaving the same remainder, then the number of terms in the continued fraction of each $\frac{M_i}{N}$ is the same.

Proof: Let r_i represent the remainder when N divides into M_i. Let $\frac{N}{r_i}$ have k terms in its continued fraction expansion. Now $\frac{M_i}{N} = n + \frac{r_i}{N}$ where n is a natural number. Now $\frac{M_i}{N} = n + \frac{1}{\frac{N}{r_i}}$

Since $\frac{N}{r_i}$ has k terms in its continued fraction expansion $\frac{M_i}{N}$ has (k + 1) terms in its continued fraction expansion.

Information for Theorem 23:

Magic Square: Magic square is a square divided into n^2 cells in which numbers from 1 to n^2 are placed in such a manner that the sums of the rows, columns and both diagonals are identical.

The order of a magic square is the number of rows or columns. Hence a magic square of n^2 cells has order n. If the order, n, can be expressed as 2m+1, where m is a natural number, one has a magic square of odd order. [20,p. 59-60] Call the sum of each row, column and diagonal by the name M.C. (M.C. stands for "magic constant").

The contribution of the fraction $\frac{5}{3}$ to the theory of continued fractions is studied in earlier sections. The numbers 5 and 3 appear in magic squares of the following type, (Table 8).

It is to be noticed that the central number of the 3rd order magic square must be 5. 3 and 5 appear in the magic square, (Figure 8), and the magic constant of this magic square is equal to $3 \cdot 5 = 15$.

Figure 8

8	1	6
3	5	7
4	9	2

<u>Theorem 23</u>: Consider any magic square of order 3. Let M.C. represent the magic constant of the magic square. Let n be any number in the magic square.

Prove that the number of terms in the continued fraction expansion of $\frac{M.C.}{n}$ is less than or equal to the order of the magic square.

<u>Proof</u>: The order of the magic square is 3.

<u>Claim</u>: The central number in the magic square is 5.

<u>Proof of the Claim</u>: Denote the elements of the square as shown below.

Figure 9

a	b	c
d	e	f
g	h	i

If one adds the row, column, and the diagonals containing the center element, e, one finds

$$(a + e + i) + (g + e + c) + d + e + f) + (b + e + h)$$

$$= 3e + (a + b + c + d + e + f + g + h + i)$$

But the sum of all the elements in the square must be the sum of the first nine natural numbers.

$$a + b + c + \ldots + i = 1 + 2 + 3 + \ldots + 9$$

$$= \frac{9\ (9 + 1)}{2}$$

$$= 45$$

Hence

$$3e + (a + b + c + \ldots + i) = 3e + 45$$

The magic constant is 15, hence the sum of the row, column and two diagonals containing e must be $4 \cdot 15 = 60$.

Then

$$3e + 45 = 60$$

$$e = 5$$

So the claim is proved.

Here, the magic constant = $\dfrac{\text{sum of the elements in the square}}{\text{order of the square}}$

$$= \frac{1 + 2 + 3 + \ldots + 9}{3}$$

$$= \frac{9\,(9 + 1)/2}{3}$$

$$= \frac{45}{3}$$

$$= 15$$

Let T represent the number of terms in the continued fraction expansion of $\frac{M.C.}{n}$.

Now $$\frac{15}{8} = 1 + \frac{7}{8}$$

$$= 1 + \cfrac{1}{\frac{8}{7}}$$

$$= 1 + \cfrac{1}{1 + \frac{1}{7}} = (1, 1, 7)$$

Here T = 3. T is equal to the order of the magic square., i.e. 3 = 3.

$\frac{15}{1} = (15)$. $T = 1$ and $T < 3$

$\frac{15}{6} = \frac{5}{2} = 2 + \frac{1}{2}$

$= (2,2)$. $T = 2$ and $T < 3$

$\frac{15}{3} = 5 = (5)$. $T = 1$ and $T < 3$

$\frac{15}{3} = 3 = (3)$. $T = 1$ and $T < 3$

$\frac{15}{7} = 2 + \frac{1}{7} = (2, 7)$. $T = 2$ and $T < 3$

$\frac{15}{4} = 3 + \frac{3}{4} = 1 + \cfrac{1}{1+\cfrac{1}{3}} = (1, 1, 3)$. $T = 3$ and $T = 3$

$\frac{15}{9} = \frac{5}{3} = 1 + \frac{2}{3} = 1 + \cfrac{1}{1+\cfrac{1}{2}} = (1, 1, 2)$. $T = 3$ and $T = 3$

$\frac{15}{2} = 7 + \frac{1}{2} = (7, 2)$. $T = 2$ and $T < 3$

Hence the number of terms in the continued fraction expansion of $\frac{M.C.}{n}$ is less than or equal to the order of the magic square.

Theorem 24: Let $A(n) = 2^{2n+1} + 1$

Prove that $\frac{A(n + 1)}{A(n)}$ has exactly three terms in its continued fraction expansion.

Proof: $\frac{A(n + 1)}{A(n)} = \frac{2^{2(n+1)+1} + 1}{2^{2n+1} + 1}$

$$= \frac{2^{2n+3} + 1}{2^{2n+1} + 1}$$

$$= \frac{8\cdot 2^{2n} + 1}{2\cdot 2^{2n} + 1}$$

$$= 3 + \frac{2\cdot 2^{2n} - 2}{2\cdot 2^{2n} + 1}$$

$$= 3 + \cfrac{1}{\cfrac{2\cdot 2^{2n} + 1}{2\cdot 2^{2n} - 2}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{3}{2\cdot 2^{2n} - 2}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{2(2^{2n} - 1)}{3}}}$$

$$\frac{A(n + 1)}{A(n)} = 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{2((2^2)^n - (1)^n)}{2^2 - 1}}} \qquad \ldots\ldots\ldots (B)$$

Now consider $(2^2)^n - (1)^n$

$$(2^2)^n - (1)^n = (2^2 - 1)[(2^2)^{n-1} + (2^2)^{n-2} + (2^2)^{n-3} + \ldots (1)^{n-1}]$$

$$= (2^2 - 1)N \quad \text{where}$$

$$N = [(2^2)^{n-1} + (2^2)^{n-2} + (2^2)^{n-3} + \ldots + (1)^{n-1}]$$

So the equation (B) could be written as follows:

$$\frac{A(n + 1)}{A(n)} = 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{2\cdot(2^2 - 1)N}{2^2 - 1}}} \qquad \text{(N is given above)}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{2N}}$$

Therefore $\frac{A(n+1)}{A(n)} = (3,1,2N)$

$\frac{A(n+1)}{A(n)}$ has exactly 3 terms in its continued fraction expansion.

<u>Theorem 25</u>: Let $A(n) = 2^{2n+2} + 1$. Prove that $\frac{A(n+1)}{A(n)}$ has exactly 5 terms in its continued fraction expansion.

<u>Proof</u>:

$$\frac{A(n+1)}{A(n)} = \frac{2^{2(n+1)+2} + 1}{2^{2n+2} + 1}$$

$$= \frac{16 \cdot 2^{2n} + 1}{4 \cdot 2^{2n} + 1}$$

$$= 3 + \frac{4 \cdot 2^{2n} - 2}{4 \cdot 2^{2n} + 1}$$

$$= 3 + \cfrac{1}{\cfrac{4 \cdot 2^{2n} + 1}{4 \cdot 2^{2n} - 2}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{3}{4 \cdot 2^{2n} - 2}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4 \cdot 2^{2n} - 2}{3}}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4 \cdot 2^{2n} - 2 - 2 + 2}{3}}}$$

$$\frac{A(n+1)}{A(n)} = 3 + \cfrac{1}{1 + \cfrac{1}{\frac{4(2^{2n} - 1) + 2}{3}}} \qquad ....(C)$$

Consider $2^{2n} - 1$

$$2^{2n} - 1 = (2^2)^n - (1)^n$$

$$=(2^2 - 1)[(2^2)^{n-1} + (2^2)^{n-2} \cdot 1 + (2^2)^{n-3} \cdot 1^2 + \dots$$

$$+ (2^2)^{n-n} \cdot (1)^{n-1}]$$

$$= 3\ N \text{ where } N = [(2^2)^{n-1} + (2^2)^{n-2} + (2^2)^{n-3} + \dots + 1]$$

using this result in (C), one gets

$$\frac{A(n+1)}{A(n)} = 3 + \cfrac{1}{1 + \cfrac{1}{\frac{4 \cdot 3N + 2}{3}}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{4N + \frac{2}{3}}}$$

$$= 3 + \cfrac{1}{1 + \cfrac{1}{4N + \cfrac{1}{1 + \frac{1}{2}}}}$$

So $\frac{A(n+1)}{A(n)} = (3, 1, 4N, 1, 2)$

Hence $\frac{A(n+1)}{A(n)}$ has exactly 5 terms in its continued fraction expansion.

Chapter IV: Observations and Suggestions For Further Study

4.1 Combinatory Numbers

Consider the combinatory numbers of the Figure 3. The numbers in the fifth row are 1, 5, 15, 35, 70, ... The sum of the fractions where denominators are the above numbers is given below.

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \frac{1}{126} + \cdots$$

By a tedious procedure it is shown in the Section 2.3 that

$$\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \frac{1}{126} + \cdots = \frac{4}{3}$$

By adopting the same procedure, one could show the following results. (It is to be noticed that the deonminators of the fractions in each of the following series are the combinatory numbers taken from Figure 3).

Table 10

Series	Sum of the Series
$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \cdots =$	$\frac{2}{1}$
$\frac{1}{1} + \frac{1}{4} + \frac{1}{10} + \frac{1}{20} + \frac{1}{35} + \cdots =$	$\frac{3}{2}$
$\frac{1}{1} + \frac{1}{5} + \frac{1}{15} + \frac{1}{35} + \frac{1}{70} + \cdots =$	$\frac{4}{3}$

[4, p.50]

Now the question is, is there any other easy way to get the same results that are in the second column of the above Table 10, without going through the tedious procedure for calculating them? (This tedious procedure is already explained in Section 2.3).

The answer is yes, and the theory of continued fractions serves as an easy way. This easy way is shown below. The combinatory numbers from the Figure 3 are given in the first column of the following Table 11.

Table 11

Combinatory Numbers	nth term
1, 1, 1, 1, 1, 1, ...	1
1, 2, 3, 4, 5, ...	n
1, 3, 6, 10, 15, 21, ...	$\frac{n(n+1)}{2!}$
1, 4, 10, 20, 35, 56, ...	$\frac{n(n+1)(n+2)}{3!} = \frac{n(n+1)}{2!} \cdot \frac{(n+2)}{3}$
1, 5, 15, 35, 70, 126, ...	$\frac{n(n+1)(n+2)(n+3)}{4!} = \frac{n(n+1)(n+2)}{3!} \cdot \frac{n+3}{4}$
1, 6, 21, 56, 126, 252, ...	$\frac{n(n+1)(n+2)(n+3)(n+4)}{5!} = \frac{n(n+1)(n+2)(n+3)}{4!} \cdot \frac{(n+4)}{5}$
and so on	

Consider the numbers in the second column of the Table 11. They are listed in the second column of the Table 12. Now one could use the theory of continued fractions as shown below.

It is obvious that the 4th column of the Table 12 and the second column of the Table 10 represent the same numbers.

Table 12

i	numbers	Terms in the continued fraction of m_{i+1}/m_i	For $n = 2$, the third column becomes
1	1	(n)	(2)
2	n	$(\frac{n+1}{2})$	$(\frac{3}{2})$
3	$\frac{n(n+1)}{2!}$	$(\frac{n+2}{3})$	$(\frac{4}{3})$
4	$\frac{n(n+1)}{2!} \cdot \frac{(n+2)}{3}$	$(\frac{n+3}{4})$	$(\frac{5}{4})$
5	$\frac{n(n+1)(n+2)}{3!} \quad \frac{(n+3)}{4}$	$(\frac{n+4}{5})$	$(\frac{6}{5})$
and so on			

For $n = 2$, the numbers that are obtained in the fourth column of the Table 12 are $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$. These numbers are the sums of the series where the denominators of the fractions of the series are associated with the famous Pascal triangle and Leibnitz's combinatory numbers.

Suggestions for Further Study: It might be interesting to investigate the properties of numbers obtained in the same way for natural numbers greater than 2.

4.2 Triangular Numbers

The numbers in the fourth column of Table 12 are $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$

$\ldots, \frac{n+1}{n}, \ldots$.

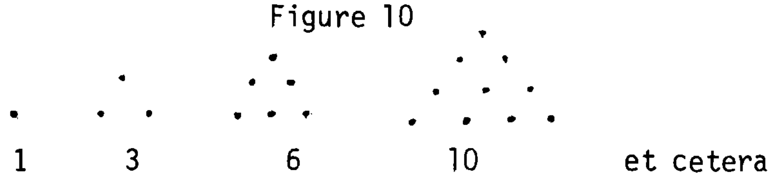
In the above fractions, if one multiplies the numerator by the denominator and finds the average, the following numbers are obtained.

$$\frac{2\cdot 1}{2}, \frac{3\cdot 2}{2}, \frac{4\cdot 3}{2}, \frac{5\cdot 4}{2}, \frac{6\cdot 5}{2}, \ldots, \frac{(n+1)n}{2}, \ldots$$

i.e. $1, 3, 6, 10, 15, \ldots \frac{n(n+1)}{2}, \ldots$ (B)

The numbers in the series (B) are wellknown in the mathematical world as triangular numbers, because the objects represented by those numbers could be arranged in the form, namely triangular form.

Figure 10



These triangular numbers, 1, 3, 6, 10, ... could be found to be the third oblique series of numbers in Pascal's triangle (Figure 2).

By adopting the proof of Theorem 18(ii), it could be very easily shown that the number of terms in the continued fraction expansion of

$\frac{(n+1)\text{th term}}{n\text{th term}}$ of the triangular numbers is at most 3.

Now consider the series $a_1, a_2, a_3, a_4, \ldots$ (C)

where the number of terms in the continued fraction expansion of

$\frac{a_{i+1}}{a_i}$ is at most 3.

Suggestions for Further Study: It might be interesting to investigate whether the numbers in the series (C) behave like triangular numbers;

in other words, could the objects that represent the numbers in (C), be arranged in a good looking figurative representation? Does such a series (C) exist? If yes, what is it?

4.3 Escalator Numbers

Introduction: Amazingly enough, the numbers in the fourth column of Table 12 also satisfy the requirements of certain numbers called escalator numbers. Every number in the fourth column of Table 12 is of the form $\frac{n+1}{n}$ where n is a natural number. By Theorem 2 of Section 2.4, $\frac{n+1}{n}$ has 2 terms in its continued fraction expansion. (n + 1) and $\frac{n+1}{n}$ satisfy the requirements of escalator numbers whose definition is given below.

Definition: Certain numbers are called escalator numbers or simply escalators, because they can be climbed in n steps by summation or in one step of n factors by multiplication.

An escalator number A_n could be defined by the relation $A_n = \sum a_n = \prod a_n$ where a_n is the gradual sum of any number n of rational summands a_n, whose sum at any point must equal to their gradual product. Such as for instance

$$A_4 = 3 + \frac{3}{2} + \frac{9}{7} + \frac{81}{67} = 3 \cdot \frac{3}{2} \cdot \frac{9}{7} \cdot \frac{81}{67} = \frac{6561}{938}$$

It is easy to see that for any arbitrary $A_1 = a_1 \neq 1$ one can get a_2 by the requirement $A_2 = a_1 + a_2 = a_1 a_2$.

Hence

$$a_1a_2 - a_2 = a_1$$

$$a_2 = \frac{a_1}{a_1-1} = \frac{A_1}{A_1-1}$$

Similarly

$$a_3 = \frac{A_2}{A_2-1}$$

and in general $$a_{n+1} = \frac{A_n}{A_n-1}$$

This is a recurrence formula which permits one to compute consecutively as many summand-factors a_n as one pleases, whose gradual sums or products are the successive escalators A_n . [14, p. 91]

Every number in the fourth column of Table 12 is of the form $\frac{n+1}{n}$ and it contributes to the theory of 'escalator numbers' for

$$\frac{n+1}{n} + (n+1) = \frac{n+1}{n} \cdot (n+1)$$

In this way the theory of continued fraction leads one to notice how certain numbers possess certain qualities.

<u>Suggestions for Further Study</u>:

(1) Does every fraction $\frac{M}{N}$ that has 2 terms in its continued fraction expansion serve as an escalator number? The answer is no. Because $\frac{5}{2}$ has 2 terms in its continued fraction expansion and

$$\frac{5}{2} \cdot 5 \neq \frac{5}{2} + 5 \text{ .}$$

Then is there any other fraction that has 2 terms in its continued fraction expansion and which is not of the type $\frac{n+1}{n}$, serve as an escalator number? If yes what are those escalator numbers? What is their contribution to the theory of continued fractions? Do they share any property with the Leibnitz's combinatory numbers or with the numbers in the Pascal's triangle?

(2) The reader must have noticed above that

$$A_4 = 3 + \frac{3}{2} + \frac{9}{7} + \frac{81}{67} = 3 \cdot \frac{3}{2} \cdot \frac{9}{7} \cdot \frac{81}{67} = \frac{6561}{938}$$

Choose the numbers $\frac{81}{67}$ and $\frac{6561}{938}$. Writing them in the continued fraction expansion one gets $\frac{81}{67} = (1, 4, 1, 3, 1, 2)$ and $\frac{6561}{938} = (6, 1, 186, 1, 1, 2)$.

It is to be noticed that the number of terms in the continued fraction expansion of $\frac{81}{67}$ and $\frac{6561}{935}$ is the same.

Do all escalator numbers possess such type of property; or in other words given escalator numbers

$$a_1 + a_2 + a_3 + \ldots + a_i = a_1 \cdot a_2 \cdot a_3 \cdot \ldots a_i = A$$

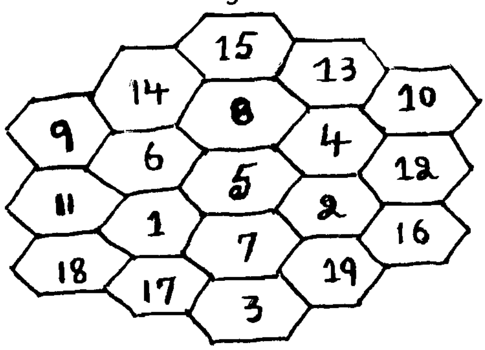
does there always exist an a_i such that the number of terms in its continued fraction expansion is the same as the number of terms in the continued fraction expansion of A.

If there exist some, find them and test their behavior in the light of continued fractions.

4.4 Magic Hexagon

A hexagonal array of the numbers 1 through k into k cells, such that all of the rows sum to the same number, is called a magic hexagon. The number of cells in a shortest row is called the order of the hexagon. A magic hexagon of order 3 is shown below. It is proved that it is the only hexagon of any order that exists. [12,p.116]

Figure 11



Consider the above hexagon of order 3. Is there any way that one can get a magic square of order 3, from the magic hexagon of order 3? The answer is yes and the theory of continued fractions helps. The procedure is given below.

Let n represent the order of the magic hexagon, i.e. $n = 3$. Let N represent any number in the magic hexagon. Let m represent the sum of the terms in the continued fraction expansion of $\frac{N}{3}$. Now consider the following table.

Chapter V: Summary

The material so far introduced emphasizes the significance of the numbers $\frac{5}{3}$, 5 and 3 in the world of mathematics. It also tells about the contribution made by eminent mathematicians to the theory of continued fractions. It is made clear in this little book how certain numbers such as Fibonacci numbers, Theon diameters, et cetera, share a common property with the natural numbers when they are studied in the light of continued fractions. It is interesting to know the procedure of obtaining a magic square of 3rd order from the magic hexagon of 3rd order.

Table 13

N	Continued Fraction Expansion of $\frac{N}{3}$	m
1	(0, 3)	3
2	(0, 1, 2)	3
3	(0, 1)	1
4	(1, 3)	4
5	(1, 1, 2)	4
6	(2)	2
7	(2, 3)	5
8	(2, 1, 2)	5
9	(3)	3
10	(3, 3)	6
11	(3, 1, 2)	6
12	(4)	4
13	(4, 3)	7
14	(4, 1, 2)	7
15	(5)	5
16	(5, 3)	8
17	(5, 1, 2)	8
18	(6)	6
19	(6, 3)	9

Now consider the set M,

M = m m is a number in the third column of Table 13

i.e. M = 1, 2, 3, 4, 5, 6, 7, 8, 9

In order to find the magic square of order 3, one could use the elements of the set M and also the 'trial and error process.' One might get the figure that looks like Figure 8. (In fact Figure 8 is a magic square of order 3).

<u>Suggestions for Further Study</u>: Consider the figures of the above type, i.e. magic hexagon, magic square, et cetera. The reader might employ the procedure suggested above and he could try to derive one figure from the other.

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