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Analytic functions are among the most important functions considered in complex function theory. The purpose of this thesis is to examine some basic properties and theorems of analytic functions, and describe the space of analytic functions. In this space a metric is defined, and is used to examine convergence, equicontinuity, boundedness, and compactness.

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THE SPACE OF ANALYTIC FUNCTIONS – A(U)

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TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Let r and θ be polar coordinates of the point (x,y) which corresponds to a nonzero complex number $z = x + iy$. Since $x = r\cdot\cos\theta$ and $y = r\cdot\sin\theta$, z can be expressed in polar form as $z = r(\cos\theta + i\sin\theta)$. If we define $e^{i\theta} = \cos\theta + i\sin\theta$, then we can represent any nonzero complex number z in exponential form: $z = re^{i\theta}$.

Analytic Functions

DEFINITION 1.1: Let $f: A \mapsto \mathbb{C}$, where $A \subset \mathbb{C}$ is an open set. Then f is differentiable (in the complex sense) at $z_0 \in A$ if $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists. This limit is denoted by $f'(z_0)$, or $\frac{df}{dz}(z_0)$.

DEFINITION 1.2: A function f is analytic on A if f is differentiable at each $z_0 \in A$. A function is said to be analytic at a point z_0 if it is analytic on a neighborhood of z_0.

The term "analytic" is synonymous with the term "holomorphic". A function that is defined and analytic on the whole complex plane $\mathbb{C}$ is called an entire function.

Differentiation and Integration

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, that is $f(z) = f(x + iy) = u + iv$. This can be written as $f(z) = u(x,y) + iv(x,y)$, where $u(x,y)$ and $v(x,y)$ are real-valued functions of the real variables x and y. We often denote u as $\mathrm{Re}\, w$ and v as $\mathrm{Im}\, w$.

DEFINITION 1.3: Let $w(t) = u(t) + iv(t)$ be a complex-valued function of a real variable t over a given interval $a \leq t \leq b$, where u and v are real-valued piecewise continuous functions of t and $a,b \in \mathbb{R}$. Then $\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$. It follows that $\mathrm{Re} \int_a^b w(t)dt = \int_a^b \mathrm{Re}\, [w(t)]dt$.

THEOREM 1.1: Given $\int_a^b w(t)dt$ in definition 1.3, $| \int_a^b w(t)dt | \leq \int_a^b |w(t)|dt$. [3]

PROOF: Assume $a < b$, and that $\int_a^b w(t)dt$ is a nonzero complex number $r_0 e^{i\theta_0}$. So $r_0 = \int_a^b e^{-i\theta_0} w(t)dt = \mathrm{Re} \int_a^b e^{-i\theta_0} w(t)dt = \int_a^b \mathrm{Re}(e^{-i\theta_0} w(t))dt$. But $\mathrm{Re}(e^{-i\theta_0} w(t)) \leq |e^{-i\theta_0} w(t)| = | e^{-i\theta_0} ||w(t)| = |w(t)|$. So $r_0 \leq \int_a^b |w(t)|dt$. Since $r_0 = |r_0 e^{i\theta_0}| = | \int_a^b w(t)dt |$, then $| \int_a^b w(t)dt | \leq \int_a^b |w(t)|dt$.

DEFINITION 1.4: A path in a region $G \subset \mathbb{C}$ is a continuous function $\gamma:[a,b] \mapsto G$ for some interval $[a,b]$ in $\mathbb{R}$. If $\gamma'(t)$ exists for each t in $[a,b]$ and $\gamma':[a,b] \mapsto \mathbb{C}$ is continuous, then γ is a smooth path. If there is a partition of $[a,b]$, $a = t_0 < t_1 < \dots < t_n = b$, such that γ is smooth on each subinterval $[t_{j-1}, t_j]$, $1 \leq j \leq n$, then γ is a piecewise smooth path, or a contour. When only the initial and final values of $z(t)$ are the same, a contour C is called a simple closed contour. In this thesis, the image of $\gamma[a,b]$ will be denoted by γ when this will not cause any confusion.

DEFINITION 1.5: Suppose that the equation $z = z(t)$ $(a \leq t \leq b)$ represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. Let $f(z) = u(x,y) + iv(x,y)$ be piecewise continuous on a contour C; that is, if $z(t) = x(t) + iy(t)$, the function $f[z(t)] = u[x(t), y(t)] + iv[x(t), y(t)]$ is piecewise continuous on the interval $a \leq t \leq b$. We define the line integral, or contour integral, of f along C as follows: $\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$.

Note that since C is a contour, $z'(t)$ is also piecewise continuous on the interval $a \leq t \leq b$, and the existence of $\int_C f(z)dz$ is ensured.

THEOREM 1.2: $|\int_C f(z)dz| \leq M \cdot L$, where $M = \max\{|f(z)| : z \in C\}$ and L = length of the contour C ($L = \int_a^b |z'(t)|dt$). [3]

PROOF: By theorem 1.1 and definition 1.5, $|\int_C f(z)dz| = |\int_a^b f[z(t)]\, z'(t)dt| \leq \int_a^b |f[z(t)]\, z'(t)|\, dt = \int_a^b |f[z(t)]| \cdot |z'(t)|\, dt \leq M \cdot \int_a^b |z'(t)|\, dt = M \cdot L$.

THEOREM 1.3 (Fundamental Theorem of Calculus for Contour Integrals): Suppose $\gamma : [a,b] \mapsto \mathbb{C}$ is a piecewise smooth path and that F is a function defined and analytic on an open set G containing γ. Then $\int_\gamma F'(z)dz = F(\gamma(a)) - F(\gamma(b))$. In particular, if $\gamma(0) = \gamma(1)$, then $\int_\gamma F'(z)dz = 0$. [4]

DEFINITION 1.6: A set A is called convex if it contains the straight line segment between every pair of its points. That is, if z_0 and z_1 are in A, then so is $sz_1 + (1-s)z_0$ for every number s between 0 and 1.

THEOREM 1.4 (Cauchy's Theorem): Given an open set $U \subset \mathbb{C}$, let $f: U \mapsto \mathbb{C}$ be continuous on U and analytic on $U - \{z_0\}$, where z_0 is some fixed point of U. If U is convex, then $\int_\gamma f(z)dz = 0$ for every closed path $\gamma \subset U$. [2]

LEMMA 1.1: Let Γ be the circle $| z - z_0 | = r$. Then $\int_\Gamma \frac{dz}{z-z_0} = 2\pi i$. [6]

PROOF: We parametrize Γ by $z = \Gamma(\theta) = z_0 + re^{i\theta}$, $0 \le \theta \le 2\pi$. So by the chain rule, $\Gamma'(\theta) = ire^{i\theta}d\theta = dz$, and

$$\int_\Gamma \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{z_0 + re^{i\theta} - z_0} = i\int_0^{2\pi} d\theta = 2\pi i.$$

$D(z_0,r)$ will be used to represent the open disk with center z_0 and radius r. $\overline{D}(z_0,r)$ will be used to represent the closed disk with center z_0 and radius r.

THEOREM 1.5 (Cauchy's Integral Formula): Let f be analytic on an open set U containing the circle

$C = \{z : |z-z_0| = r\}$ and its interior. Then for any $z \in D(z_0,r)$, $f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}\,dw$. [2]

PROOF: Let $g(w) = \frac{f(w)-f(z)}{w-z}$ if $w \in U$ and $w \neq z$, and $g(z) = f'(z)$. Since f is analytic on U, g is continuous on U and analytic on $U-\{z\}$, and by theorem 1.4, $\int_C g(w)dw = 0$; that is, $\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = \frac{f(z)}{2\pi i}\int_C \frac{dw}{w-z}$. By lemma 1.1, the integral on the right is equal to $2\pi i$, and we have $\frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw = f(z)$.

THEOREM 1.6 (Cauchy's Integral Formula for Derivatives): Let f be analytic on an open set U containing the circle $C = \{ z : |z-z_0| = r\}$ and its interior. Then f has derivatives of all orders for any $z \in D(z_0,r)$, and

$$f^{(K)}(z) = \frac{K!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{K+1}}\,dw\ , \text{ for } K = 1,\ 2,\ 3,\ \ldots\ . \quad [2]$$

PROOF: We will prove this using induction. We first prove it for $k = 1$. We will do this by showing that $\lim_{h\to 0}\left[\frac{f(z+h)\ -\ f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\,dw\right] = 0$. By theorem 1.5,

$$\frac{f(z+h)\ -\ f(z)}{h} = \frac{1}{2\pi i h}\int_C \frac{f(w)}{w-z-h}\,dw - \frac{1}{2\pi i h}\int_C \frac{f(w)}{w-z}\,dw.$$ So

$$\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw = \frac{1}{2\pi i h}\int_C \left[\frac{f(w)}{w-z-h} - \frac{f(w)}{w-z} - \frac{hf(w)}{(w-z)^2}\right]dw$$

$$= \frac{h}{2\pi i}\int_C \frac{f(w)}{(w-z)^2(w-z-h)}\,dw.$$ We know f is bounded on C; and if

$w \in C$, z is a fixed element of $D(z_0,r)$, and h is small enough so that $\overline{D}(z,h) \subset D(z_0,r)$, then $|(w-z)^2(w-z-h)|$ is bounded away from 0. So $\frac{h}{2\pi i}\int_C \frac{f(w)}{(w-z)^2(w-z-h)}\,dw \to 0$ as $h \to 0$.

Assume the theorem is true for $n = k$. Then

$$\frac{f^{(K)}(z+h) - f^{(K)}(z)}{h} = \frac{K!}{2\pi i h}\int_C \left[\frac{f(w)}{(w-z-h)^{K+1}} - \frac{f(w)}{(w-z)^{K+1}}\right] dw, \text{ and}$$

$$\frac{f^{(K)}(z+h) - f^{(K)}(z)}{h} - \frac{(K+1)!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{K+2}}\, dw =$$

$$\frac{K!}{2\pi i h}\int_C f(w)\cdot\left[\frac{1}{(w-z-h)^{K+1}} - \frac{1}{(w-z)^{K+1}} - \frac{h(k+1)}{(w-z)^{K+2}}\right] dw.$$

CLAIM: The expression in brackets is of order h^2, so the integral approaches 0 as $h \to 0$.

PROOF OF CLAIM: The expression in brackets, when combined into a single fraction by use of the common denominator, becomes

$$\frac{(w-z)^{K+2} - (w-z)(w-z-h)^{K+1} - h(k+1)(w-z-h)^{K+1}}{(w-z-h)^{K+1}(w-z)^{K+2}}.$$

By use of the binomial theorem, the numerator becomes

$$(w-z)^{K+2} - (w-z)\left[(w-z)^{K+1} + \binom{K+1}{1}(-h)(w-z)^{K} + \ldots + (-h)^{K+1}\right] -$$

$$h(k+1)\left[(w-z)^{K+1} + \binom{K+1}{1}(-h)(w-z)^{K} + \ldots + (-h)^{K+1}\right],$$ and the fraction can be simplified to

$$\frac{h^2(k+1)^2(w-z)^{K} + [z-w-h(k+1)]\left[\binom{K+1}{2}(-h)^2(w-z)^{K-1} + \ldots + (-h)^{K+1}\right]}{(w-z-h)^{K+1}(w-z)^{K+2}}.$$

The expression in brackets is of order h^2, and so the integral approaches 0 as $h \to 0$. Thus the formula holds for $n = k + 1$, and the theorem holds for any $K = 1, 2, \ldots$.

DEFINITION 1.7: A subset K of a metric space $(\mathbb{X},d)$ is compact if for every collection $\mathbb{G}$ of open sets in $(\mathbb{X},d)$ with the property $K \subset \{ G : G \in \mathbb{G} \}$, there is a finite number of sets $G_1, \ldots, G_n$ in $\mathbb{G}$ such that $K \subset G_1 \cup G_2 \cup \ldots \cup G_n$.

THEOREM 1.7: Let γ be a path, and let $g:\gamma \mapsto \mathbb{C}$ be a continuous map. Define, for each $z \in \mathbb{C}-\gamma$, $f(z)=\frac{1}{2\pi i}\int_\gamma \frac{g(w)}{w-z}dw$. (i) Then all derivatives of f exist at z, and $f^{(K)}(z) = \frac{K!}{2\pi i}\int_\gamma \frac{g(w)}{(w-z)^{K+1}}dw$. Thus f and all its derivatives are analytic on $\mathbb{C}-\gamma$. (ii) Furthermore, $f^{(K)}(z) \to 0$ as $z \to \infty$ for each K. [2]

PROOF: The argument for part (i) is the same argument used in theorem 1.6, except that f is replaced by g in the integrands, and will not be repeated. The only time we used the fact that f was analytic (and that C was a circle) was when we used theorem 1.5 to express f(z) in terms of f(w), $w \in C$. For part (i), this step is provided in the hypothesis. To prove (ii), we have γ a compact set, so γ is bounded. That is, $|z| \leq r$ for any $z \in \gamma$ and for r some nonnegative real number. So $\gamma \subset \overline{D}(0,r)$. Since g is continuous, $|g(z)| \leq M$ for any $z \in \gamma$ and for M some nonnegative real number, so by theorem 1.2, for $|z| > r$ then $|f^{(K)}(z)| \leq \frac{K!}{2\pi} \frac{M}{(|z|-r)^{K+1}} \cdot (\text{length of } \gamma) \to 0$ as $z \to \infty$.

THEOREM 1.8 (Analytic Convergence Theorem): (i) Let $f_1, f_2, \ldots$ be analytic on the open set $U \subset \mathbb{C}$, and assume $f_n \to f$, uniformly on compact subsets. Then f is analytic on U. (ii) For each $p = 1, 2, \ldots, f_n^{(p)} \to f^{(p)}$ on U, uniformly on compact subsets. [2]

PROOF: (i) Let $z \in D(z_0,r)$ where $\overline{D}(z_0,r) \subset U$. Let $\Gamma = \{z' : |z'-z_0| = r\}$. By theorem 1.5, $f_n(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f_n(w)}{w-z}dw$. Since Γ is compact and $|w-z|$ is bounded away from 0 on Γ, $\frac{f_n(w)}{w-z} \to \frac{f(w)}{w-z}$ uniformly on Γ. So $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw$. (Note that f is continuous on Γ because of the uniform convergence of $\{f_n\}$, so that the integral exists.) By theorem 1.7, f is analytic at z, and since z is an arbitrary point of U, f is analytic on U.

(ii) By theorem 1.6, $f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{p+1}}dw$, for

$z \in D(z_0,r)$ and $f_n^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{f_n(w)}{(w-z)^{p+1}}dw$. So

$f_n^{(p)}(z)-f^{(p)}(z) = \frac{p!}{2\pi i}\int_\Gamma \frac{f_n(w)-f(w)}{(w-z)^{p+1}}dw$. If $z \in \overline{D}(z_0,r_1)$,

for $r_1 < r$, then by theorem 1.2, $|f_n^{(p)}(z) - f^{(p)}(z)| \leq$
$\frac{p!}{2\pi}\max_{w\in\Gamma}|f_n(w)-f(w)| \cdot \frac{2\pi r}{(r-r_1)^{p+1}} \to 0$ as $n \to \infty$. Therefore,

$f_n^{(p)} \to f^{(p)}$ uniformly on any closed subdisk of $D(z_0,r)$. Since z_0 is an arbitrary point of U, $f_n^{(p)} \to f^{(p)}$ uniformly on compact subsets of U.

<u>Power Series</u>

DEFINITION 1.8: The power series $\sum_{n=0}^{\infty} a_n w^n$, with a_n and w complex, is said to converge at the point w_0 if and only if

$\sum_{k=0}^{n} a_k w_0^k$ approaches a complex number B as $n \to \infty$. The series converges absolutely at w_0 if and only if $\sum_{n=0}^{\infty} |a_n w_0^n| < \infty$. The series converges uniformly on the set S if and only if there is a function $B: S \mapsto \mathbb{C}$ with the property that for each $\epsilon > 0$ there is an integer N such that $| \sum_{k=0}^{n} a_k w^k - B(w) | < \epsilon$ for all $w \in S$ and all $n \geq N$.

THEOREM 1.9: If $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at the point z, where $|z-z_0| = r$, the series converges absolutely on $D(z_0,r)$, uniformly on each closed subdisk of $D(z_0,r)$, hence uniformly on compact subsets of $D(z_0,r)$. [2]

PROOF: $|a_n (z'-z_0)^n| = |a_n (z-z_0)^n| \cdot | \frac{z'-z_0}{z-z_0} |^n$. Assume $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges at the point z, where $|z-z_0| = r$. Then $a_n (z-z_0)^n \to 0$, and $\{a_n (z-z_0)^n\}_{n=0}^{\infty}$ is bounded. Let $M = \sup\{|a_n (z-z_0)^n|\}_{n=0}^{\infty}$. If $|z'-z_0| \leq r' < r$, then $|\frac{z'-z_0}{z-z_0}| \leq \frac{r'}{r} < 1$. Since $\sum_{n=0}^{\infty} M (\frac{r'}{r})^n$ is a convergent geometric series, we use the comparison test for power series, and $| a_n (z'-z_0)^n | = |a_n (z-z_0)^n| \cdot | \frac{z'-z_0}{z-z_0} |^n \leq M (\frac{r'}{r})^n$ tells us that $\sum_{n=0}^{\infty} a_n (z'-z_0)^n$ converges absolutely . For $z' \in \overline{D}(z_0,r')$, the Weierstrass M-test [4] tells us that $\sum_{n=0}^{\infty} a_n (z'-z_0)^n$ converges uniformly on that closed disk, and hence uniformly on compact subsets of $D(z_0,r)$.

THEOREM 1.10 (Taylor's Theorem): Let f be analytic on $D(z_0,r)$. Then $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$, for $z \in D(z_0,r)$. The series converges absolutely on $D(z_0,r)$, and converges uniformly on compact subsets of $D(z_0,r)$. [2]

PROOF: Let $|z-z_0| < r_1 < r$, $\Gamma = \{z : |z-z_0| = r_1\}$. By theorem 1.5, $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z_0}\left[\frac{1}{1-\frac{z-z_0}{w-z_0}}\right]dw$

$= \frac{1}{2\pi i}\int_\Gamma \sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}dw$. There exists an M such that

$|f(w)| \leq M$ for all $w \in \Gamma$, since f is continuous and Γ is compact. So if $|z-z_0| \leq r_2 < r_1$, $\frac{|f(w)(z-z_0)^n|}{|w-z_0|^{n+1}} \leq \frac{M}{r_1}\left(\frac{r_2}{r_1}\right)^n$.

Then $\sum_{n=0}^{\infty} f(w)\frac{(z-z_0)^n}{(w-z_0)^{n+1}}$ converges uniformly for $w \in \Gamma$. We

obtain $f(z) = \sum_{n=0}^{\infty}(z-z_0)^n \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z_0)^{n+1}}dw = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$,

by theorem 1.6. By theorem 1.9, the series converges absolutely on $D(z_0,r)$, and converges uniformly on compact subsets of $D(z_0,r)$.

THEOREM 1.11: If a function g can be represented by a power series expansion $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ in a neighborhood of z_0, then g is analytic at z_0. [2]

PROOF: Assume a function g can be represented by a power series expansion $\sum_{n=0}^{\infty}a_n(z-z_0)^n$ in a neighborhood of z_0. Let $g_n(z) = \sum_{k=0}^{n}a_k(z-z_0)^k$. Since $\sum_{k=0}^{n}a_k(z-z_0)^k$ is a polynomial, it is analytic on $\mathbb{C}$. Then $\{g_n(z)\}$ converges to $g(z)$ on some disk $D(z_0,r)$, and by theorem 1.9, it converges uniformly on compact subsets of $D(z_0,r)$. By theorem 1.8, g is analytic on $D(z_0,r)$.

DEFINITION 1.9: Let f be analytic at z_0 with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$. Then f is said to have a zero of order m at z_0 if and only if $a_n = 0$ for $n < m$ and $a_m \neq 0$. ("Multiplicity" is sometimes used as a synonym for "order".) This means that $f(z) = (z-z_0)^m g(z)$, where (by theorem 1.11) g is analytic at z_0 and $g(z_0) \neq 0$. A zero of order 1 is sometimes called a simple zero.

DEFINITION 1.10: A space U is connected if and only if the only subsets of U that are both open and closed in U are the empty set and U itself.

THEOREM 1.12: Let f be analytic on the open connected set $U \subset \mathbb{C}$. Suppose that f has a limit point of zeros in U, that is, there is a point $z_0 \in U$ and a sequence of points $z_n \in U$, $z_n \neq z_0$, such that $z_n \to z_0$ and $f(z_n) = 0$ for all n (hence $f(z_0) = 0$). Then f is identically 0 on U. [2]

PROOF: By theorem 1.10, we can expand f in a Taylor series about z_0, say $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, $|z-z_0| < r$. We show that all $a_n = 0$ by assuming this is not so. Let m be the smallest integer such that $a_m \neq 0$. By definition 1.9, $f(z) = (z-z_0)^m g(z)$, where g is analytic at z_0 and $g(z_0) \neq 0$. By continuity, g is nonzero in a neighborhood of z_0, contradicting the fact that z_0 is a limit point of zeros.

Let $A = \{z \in U$: there is a sequence of points $z_n \in U$, $z_n \neq z$, $z_n \to z$, with $f(z_n) = 0$ for all $n\}$. Since $z_0 \in A$ by hypothesis, A is not empty. If $z \in A$, then by the above argument f is zero on a disk $D(z,\varepsilon)$ and it follows that $D(z,\varepsilon) \subset A$. Thus A is open (in $\mathbb{C}$, and so in U). If we can show that A is also closed in U, the fact that U is connected gives $A = U$, and we will be done.

Let $\{w_n\} \subset A$, such that $w_n \to w$ and $w \in U$. If any of the $w_n = w$, then $w \in A$ and we're done; so assume $w_n \neq w$, $n = 1, 2, \ldots$. But since $w_n \in A$ and $f(w_n) = 0$, then $w \in A$ by definition of A. Thus A is closed in U.

THEOREM 1.13 (Identity Theorem): Let f and g be analytic on the open connected set $U \subset \mathbb{C}$. Let S be a subset of U having a limit point in U. If f and g agree on S, they agree everywhere on U. [2]

PROOF: Let $h = f - g$. Then the hypothesis of theorem 1.12 is satisfied, and h is identically 0 on U, which says that $f = g$ everywhere on U.

CHAPTER 2

THE SPACE OF ANALYTIC FUNCTIONS: A(U)

DEFINITION 2.1: Let $\mathbb{C}$ be the set of complex numbers and U be an open subset of $\mathbb{C}$. Let A(U) be the set of all analytic functions on U, and C(U) the set of all continuous functions from U to $\mathbb{C}$.

Construction of a Metric on A(U)

We will define a metric d on C(U), prove that it is a metric, and thus show that (C(U),d) is a metric space. It follows that (A(U),d) is also a metric space.

DEFINITION 2.2: Let $K_n = \{z \in \mathbb{C} : |z| \leq n \text{ and } |z-w| \geq 1/n \text{ for all } w \in \mathbb{C}-U\}$.

The K_n are compact sets, since they are closed and bounded. Moreover, $K_n \subset K_{n+1} \subset \ldots$, and $\bigcup_{n=1}^{\infty} K_n = U$. For any compact subset K of U, K has a positive Euclidean distance from $\mathbb{C}-U$. To make $K \subset K_n$, we only have to make n sufficiently large.

DEFINITION 2.3: Let $\|f-g\|_{K_n} = \sup \{|f(z)-g(z)| : z \in K_n\}$. We define $d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f-g \|_{K_n}}{1 + \| f-g \|_{K_n}}$ for $f,g \in C(U)$.

Note: since $\| f-g \|_{K_n}$ is a continuous real-valued function on a compact set, the function has a maximum value.

To show that d is a metric on C(U), we need to show that d satisfies the following properties:

(1) $d(f,g) \geq 0$ for all $f,g \in C(U)$; $d(f,g) = 0$ if and only if $f = g$.

(2) $d(f,g) = d(g,f)$ for all $f,g \in C(U)$.

(3) $d(f,g) + d(g,h) \geq d(f,h)$ for all $f,g,h \in C(U)$.

(i) Since $| f(z)-g(z) | \geq 0$ for all $z \in U$, then $d(f,g) \geq 0$ for all $f,g \in C(U)$. (ii) Since $| f(z)-g(z) | = | g(z)-f(z) |$ for all $z \in U$, the property that $d(f,g) = d(g,f)$ is fairly easy to show, and will not be proved here. (iii) It takes a bit more work to show that d satisfies the third property. We will use the fact that for p,q nonnegative real numbers and $p \leq q$, then $\frac{p}{1+p} \leq \frac{q}{1+q}$.

$$d(f,h) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f-h \|_{K_n}}{1 + \| f-h \|_{K_n}} = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f-g+g-h \|_{K_n}}{1 + \| f-g+g-h \|_{K_n}}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f-g \|_{K_n} + \| g-h \|_{K_n}}{1 + \| f-g \|_{K_n} + \| g-h \|_{K_n}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f-g \|_{K_n}}{1 + \| f-g \|_{K_n} + \| g-h \|_{K_n}} +$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| g-h \|_{K_n}}{1 + \| f-g \|_{K_n} + \| g-h \|_{K_n}}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f-g \|_{K_n}}{1 + \| f-g \|_{K_n}} + \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| g-h \|_{K_n}}{1 + \| g-h \|_{K_n}}$$

$$= d(f,g) + d(g,h).$$

We have shown that d is a metric, and therefore (C(U),d) is a metric space. We use the fact that subspaces of metric spaces are also metric spaces, and have immediately that (A(U),d), a subspace of (C(U),d), is also a metric space. When we discuss (C(U),d) and (A(U),d) in the remainder of this thesis, we will represent them as C(U) and A(U), respectively.

<u>Convergence</u>

DEFINITION 2.4: A sequence of functions $\{f_n\}$ is said to converge uniformly on U if there is a function $f:U \mapsto \mathbb{C}$ such that for each $\varepsilon > 0$, there is a J such that for each positive integer j, if $j \geq J$ then $| f_j(z)-f(z) | < \varepsilon$ for all $z \in U$. We denote this by "$f_j \to f$ uniformly on U".

THEOREM 2.1: If $f_1, f_2, \ldots, f \in C(U)$, then $d(f_j,f) \to 0$ if and only if $f_j \to f$ uniformly on compact subsets of U. [2]

PROOF: Assume $d(f_j,f) \to 0$. Then $\| f_j-f \|_{K_n} \to 0$ as $j \to \infty$ for each n. If K is a compact subset of U, then K is contained in some K_n, and $\| f_j-f \|_K \leq \| f_j-f \|_{K_n} \to 0$. Then as $j \to \infty$, $\| f_j-f \|_{K_n} \to 0$, which implies that $| f_j(z)-f(z) | \to 0$ for $z \in K$, and $f_j \to f$ uniformly on K.

We assume that $f_j \to f$ uniformly on compact subsets of U. Given $\epsilon > 0$, choose N so that $\sum_{n=N+1}^{\infty} 2^{-n} < \epsilon/2$. To find this N, we do the following:

$$\sum_{n=1}^{\infty} 2^{-n} = \sum_{n=1}^{N} 2^{-n} + \sum_{n=N+1}^{\infty} 2^{-n} = 1 - 2^{-N} + \sum_{n=N+1}^{\infty} 2^{-n} = 1.$$

So to make $\sum_{n=N+1}^{\infty} 2^{-n} < \epsilon/2$, choose N so that $2^{-N} < \epsilon/2$. We

choose J so that if $j \geq J$, then $\dfrac{\| f_j - f \|_{K_N}}{1 + \| f_j - f \|_{K_N}} < \epsilon/2$, or that

$\| f_j - f \|_{K_N} < \dfrac{\epsilon}{2 - \epsilon}$. This is possible since $f_j \to f$ uniformly on compact subsets of U. (The previous inequality actually holds for all natural numbers $n < N$, since $K_i \subset K_j$ for $i < j$, and we will use this fact to finish the proof.) So given $\epsilon > 0$, and having chosen the appropriate N and J, we have $d(f_j, f) =$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\| f_j - f \|_{K_n}}{1 + \| f_j - f \|_{K_n}} = \sum_{n=1}^{N} \frac{1}{2^n} \frac{\| f_j - f \|_{K_n}}{1 + \| f_j - f \|_{K_n}} +$$

$$\sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\| f_j - f \|_{K_n}}{1 + \| f_j - f \|_{K_n}} < \sum_{n=1}^{N} \frac{1}{2^n} \frac{\epsilon}{2} + \sum_{n=N+1}^{\infty} \frac{1}{2^n} \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon .$$

DEFINITION 2.5: Let $(\mathbf{X}, d)$ be a metric space. A sequence $\{x_j\}_{j=0}^{\infty}$ of points of $\mathbf{X}$ is a Cauchy sequence in $(\mathbf{X}, d)$ if given $\epsilon > 0$, there is an integer N such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

DEFINITION 2.6: The metric space $(\mathbf{X},d)$ is complete if every Cauchy sequence in $\mathbf{X}$ converges.

An argument similar to that used to prove Theorem 2.1 could be used to show that $\{f_n\}$ is a Cauchy sequence in $C(U)$ if and only if $f_i - f_j \to 0$ uniformly on compact subsets of U. We will use this result to prove the following theorem.

THEOREM 2.2: (i) $C(U)$ is a complete metric space; (ii) $A(U)$ is a closed subset of $C(U)$; (iii) $A(U)$ is a complete metric space. [2]

PROOF: (i) Let $\{f_1, f_2, \ldots\}$ be a Cauchy sequence in $C(U)$. Then $d(f_i,f_j) < \varepsilon$ whenever $i,j \geq$ some J, or $d(f_i,f_j) \to 0$ as $i,j \to \infty$. Then $\sup\{ \mid f_i(z)-f_j(z) \mid : z \in K_n \} \to 0$. Since this is true for any n, then for each $z \in U$, $\{f_1(z),f_2(z),\ldots\}$ is a Cauchy sequence in the complete metric space of complex numbers. So $f_j(z)$ approaches a limit $f(z)$. We need to show that $f_i \to f$ uniformly on K. Given any compact subset K of U and $\varepsilon/2 > 0$, then for some J, $\mid f_i(z)-f_j(z) \mid < \varepsilon/2$ for all $z \in K$ when $i,j \geq J$. If z is arbitrary in K but fixed, then there is an integer $j \geq J$ so that $\mid f_j(z)-f(z) \mid < \varepsilon/2$. But then $\mid f_i(z)-f(z) \mid \leq \mid f_i(z)-f_j(z) \mid + \mid f_j(z)-f(z) \mid < \varepsilon$ for all $z \in K$ when $i \geq J$. Thus $f_i \to f$ uniformly on K. By the uniform limit theorem [5], f is a continuous function. Thus $f \in C(U)$, and $C(U)$ is complete.

(ii) By theorem 1.8, if a sequence of analytic functions on U converges to a function f uniformly on compact subsets, then f is analytic on U. Therefore, $A(U)$ is closed in $C(U)$.

(iii) $A(U)$ is complete since it is a closed subset of a complete space .

THEOREM 2.3 (Hurwitz' Theorem): Let $f_1, f_2, \ldots \in A(U)$, and let $f_n \to f$ uniformly on compact subsets of U. Suppose that $\overline{D}(z_0,r)$ is a subset of U, and f is not zero on $\{z : | z-z_0 | = r\}$. Then there is a positive integer N such that if $n \geq N$, then f_n and f have the same number of zeros in $D(z_0,r)$. [2]

PROOF: Let $\varepsilon = \min \{ |f(z)| : | z-z_0 | = r \}$. This is possible since f is continuous on a compact set. Then for n large enough, $| f(z)-f_n(z) | < \varepsilon$ for all $z \in \overline{D}(z_0,r)$, since $f_n \to f$ uniformly on compact subsets of U. Now $| f(z)-f_n(z) | < \varepsilon \leq |f(z)|$ for $| z-z_0 | = r$. Rouche's Theorem [1] states that given a simple closed curve γ with f and g analytic inside and on γ, with γ passing through no zeros of g, and assuming $| f(z)-g(z) | < | f(z) |$ on γ, then f and g have the same number of zeros inside γ. We use this theorem to conclude that f_n and f have the same number of zeros in $D(z_0,r)$.

THEOREM 2.4: Let $f_1, f_2, \ldots \in A(U)$, and let $f_n \to f$ uniformly on compact subsets of U. If U is connected and the f_n are never 0 on U, then f is either never 0 or identically 0 on U. [2]

PROOF: We will prove this by proving the contrapositive. Let U be a connected subset of $\mathbb{C}$. Assume $f(z_0) = 0$ and f is not identically 0 on U. We will use the contrapositive of theorem 1.12. It states: let f be analytic on the open connected set $U \subset \mathbb{C}$; suppose there is a point $z_0 \in U$ and a sequence of points $z_n \in U$, $z_n \neq z_0$, such that $z_n \to z_0$ and $f(z_n) = 0$ for all n (hence $f(z_0) = 0$); then f is identically 0 on U. Since we're assuming f is not identically 0 on U, then there doesn't exist a sequence of points converging to z_0 such that $f(z_n) = 0$ for all n. This implies that there exists a closed disk $\overline{D}(z_0,r) \subset U$ such that f is not 0 on $\{ z : | z-z_0 | = r \}$. From theorem 2.3, there is a positive integer N such that for $n \geq N$, f_n and f have the same number of zeros in $D(z_0,r)$. Since f has a zero in $D(z_0,r)$, the f_n must have a zero in $D(z_0,r)$.

THEOREM 2.5: Let $f_1, f_2, \ldots \in A(U)$, and let $f_n \to f$ uniformly on compact subsets of U. If U is connected and all the f_n are one-to-one, then f is either one-to-one or identically constant on U. [2]

PROOF: Let $z_0 \in U$, and set $g_n(z) = f_n(z) - f_n(z_0)$. Since $f_n \in A(U)$ and since the difference of two analytic functions is analytic, $g_n \in A(U-\{z_0\})$. Also, $g_n \to f - f(z_0)$ uniformly on compact subsets of $U-\{z_0\}$, and $U-\{z_0\}$ is connected. For all n, g_n is never 0 on $U-\{z_0\}$ (since f_n is one-to-one). So by theorem 2.4, $f - f(z_0)$ is either always 0 or never 0 on $U-\{z_0\}$. Since z_0 is arbitrary, f is either identically constant or one-to-one.

<u>Equicontinuity and Boundedness</u>

DEFINITION 2.7: The functions in a family $\mathbf{F}$ of functions are said to be equicontinuous on a set $U \subset \mathbb{C}$ if and only if, for each $\epsilon > 0$, there exists a $\delta > 0$ such that $| f(z)-f(z_0) | < \epsilon$ whenever $| z-z_0 | < \delta$ and $z_0, z \in U$, simultaneously for all functions $f \in \mathbf{F}$.

DEFINITION 2.8: The set $\mathbf{F} \subset C(U)$ is bounded if and only if for each compact $K \subset U$, $\sup\{ \|f\|_K : f \in \mathbf{F} \} < \infty$.

THEOREM 2.6: Let $\mathbf{F}$ be a bounded subset of $A(U)$. Then $\mathbf{F}$ is equicontinuous at each point $z_0 \in U$. [2]

PROOF: Let $\overline{D}(z_0,r) \subset U$. If $z \in D(z_0,r/2)$ and $f \in \mathbf{F}$, then theorem 1.5 (Cauchy Integral Formula) says that

$$f(z)-f(z_0) = \frac{1}{2\pi i}\int_\Gamma f(w)\left[\frac{1}{w-z} - \frac{1}{w-z_0}\right]dw = \frac{1}{2\pi i}(z-z_0)\int_\Gamma \frac{f(w)dw}{(w-z)(w-z_0)}$$

where $\Gamma = \{ z : | z-z_0 | = r \}$. Let $M = \sup\{ \|f\|_\Gamma : f \in \mathbf{F} \}$. M exists since Γ is a compact subset of U, and by hypothesis, $M < \infty$. By theorem 1.2, $| \int_\Gamma f(z)dz | \leq M(\Gamma) \cdot \text{length } \Gamma$, where $M(\Gamma) = \max\{ | f(z) | : z \in \Gamma \}$. From this, we have $| f(z)-f(z_0) | \leq \frac{1}{2\pi} \cdot | z-z_0 | \cdot [M/(\frac{r}{2})^2] \cdot 2\pi r$. Simplifying, we get $| f(z)-f(z_0) | \leq | z-z_0 | \cdot 4M/r$. If we have $\delta < \min\{r/2, (r\cdot\epsilon)/4M\}$, then for $| z-z_0 | < \delta$, $| f(z)-f(z_0) | < \epsilon$ for all $f \in \mathbf{F}$.

DEFINITION 2.9: Suppose $\{f_n\}_{n=0}^{\infty}$ is a sequence of functions and that U is a subset of $\mathbb{C}$ such that U lies in the domain of f_n for each integer $n \geq 0$. If $\{f_n(z)\}_{n=0}^{\infty}$ converges for each $z \in U$, then $\{f_n\}_{n=0}^{\infty}$ converges pointwise on U. If $\{f_n\}_{n=0}^{\infty}$ converges pointwise on U, we define $f: U \mapsto \mathbb{C}$ by $f(z) = \lim_{n\to\infty} f_n(z)$ for each $z \in U$. We say that f_n converges pointwise to f on U.

THEOREM 2.7: Let $F \subset C(U)$, and assume F is equicontinuous at each point of U. If $f_n \in F$ and f_n converges pointwise to f on U, then $f \in C(U)$ and f_n converges to f uniformly on compact subsets of U. [2]

PROOF: Choose $\epsilon > 0$. Let K be any compact subset of U. If $z \in K$, there is a disk $D(z,\delta) \subset U$ such that if $z' \in D(z,\delta)$, then $|\, g(z')-g(z)\, | < \epsilon/5$ for all $g \in F$, since F is equicontinuous. In particular, $|\, f_n(z')-f_n(z)\, | < \epsilon/5$ for $n = 1, 2, \ldots$. Thus $|f(z)-f(z')| = |\, f(z) - f_n(z) + f_n(z) - f_n(z') + f_n(z') - f(z')\, | \leq |\, f(z)-f_n(z)\, | + |\, f_n(z)-f_n(z')\, | + |\, f_n(z')-f(z')\, |$. Since f_n converges pointwise to f, there exist N_1 and N_2 such that for $n \geq N_1$, $|\, f(z)-f_n(z)\, | < \epsilon/5$, and $|\, f_n(z')-f(z')\, | < \epsilon/5$ for $n > N_2$. Choose $N = \max\{N_1, N_2\}$. Then $|\, f(z)-f(z')\, | < \frac{3}{5}\epsilon$.

We have proved that f is continuous. Let K be any compact subset of U. So $K \subset \bigcup_{j=1}^{m} D(z_j,\delta_j)$ for some $z_1, z_2, \ldots, z_m$ and for $\delta_j < \delta$. Also, $|\, f(z)-f_n(z)\, | \leq |\, f(z)-f(z_j)\, | + |\, f(z_j)-f_n(z_j)\, | + |\, f_n(z_j)-f_n(z)\, |$. If $z \in K$, then $z \in D(z_j,\delta_j)$ for some j. By continuity, the first term on the right (above) is less than $\frac{3}{5}\epsilon$. By equicontinuity, the third term is less than $\epsilon/5$. By pointwise convergence, for each $D(z_j,\delta_j)$ the second term is less than $\epsilon/5$ for sufficiently large n, say $n \geq n_j$. If we choose $N = \max\{n_j,\ j = 1, 2, \dots, m\}$, then for $n \geq N$, $|\ f(z_j)-f_n(z_j)\ | < \epsilon/5$ for all z_j, and also $|\ f(z)-f_n(z)\ | < \epsilon$ for all $z \in K$. We have proved that f_n converges to f uniformly on compact subsets of U.

THEOREM 2.8: Let $F \subset C(U)$, and assume F is equicontinuous at each point of U. If $f_n \in F$ and $f_n(z)$ converges to a limit function f only for z in a dense subset of U, then f_n converges to a limit function f for all $z \in U$. Also, $f \in C(U)$, and f_n converges to f uniformly on compact subsets of U. [2]

PROOF: Let S be a dense subset of U. We assume f_n converges pointwise on S. Choose $\epsilon > 0$, and any $z \in U$. There is a disk $D(z,\delta) \subset U$ such that if $z' \in D(z,\delta)$, then $|\ g(z')-g(z)\ | < \epsilon/3$ for all $g \in F$. Since S is dense, we can find $w \in \{S \cap D(z,\delta)\}$. Hence $|\ f_m(z)-f_n(z)\ | \leq |f_m(z)-f_m(w)| + |f_m(w)-f_n(w)| + |f_n(w)-f_n(z)| < \epsilon/3 + |f_m(w)-f_n(w)| + \epsilon/3 = \epsilon$ for n and m sufficiently large (since a convergent sequence in a metric space must be a Cauchy sequence). Because $\{f_n(z)\}$ is a Cauchy sequence for any $z \in U$, therefore $\{f_n\}$ converges pointwise on U. Using theorem 2.7, we find that $\{f_n\}$ converges to a continuous function f on U, and f_n converges to f uniformly on compact subsets of U.

THEOREM 2.9 (Montel's Theorem): Let F be a bounded subset of A(U). If $\{f_n\}$ is a sequence of functions in F, there is a subsequence $\{f_{n_k}, k = 1, 2, \ldots\}$ converging uniformly on compact subsets to $f \in A(U)$. [4]

PROOF: Let $\{z_1, z_2, \ldots\}$ be a countable dense subset of U (for example, all points of U whose real and imaginary parts are rational numbers). Let F be a bounded subset of A(U), and choose $\{f_1, f_2, \ldots\} \subset F$. Since F is bounded, $\{f_n(z_1)\}$ is a bounded sequence of complex numbers, and by the Bolzano-Weierstrass theorem [1] it has a convergent subsequence. So we can extract a subsequence $\{f_{11}(z_1), f_{12}(z_1), \ldots\}$ converging to w_1. Then $\{f_{11}(z_2), f_{12}(z_2), \ldots\}$ is a bounded sequence of complex numbers, and we can extract a subsequence $\{f_{21}(z_2), f_{22}(z_2), \ldots\}$ converging to w_2. Continuing this process, we produce an array:

$$\begin{array}{lllll} \{f_{11}(z_1), & f_{12}(z_1), & f_{13}(z_1), & \ldots & \} \rightarrow w_1 \\ \{f_{21}(z_2), & f_{22}(z_2), & f_{23}(z_2), & \ldots & \} \rightarrow w_2 \\ \{f_{31}(z_3), & f_{32}(z_3), & f_{33}(z_3), & \ldots & \} \rightarrow w_3 \\ \vdots & \vdots & \vdots & \vdots & \qquad \vdots \end{array}$$

in which the k^{th} horizontal row converges to some complex number w_k and the functions used in each row are selected from those in the row above. Let $g_n = f_{nn}$ for $n = 1, 2, \ldots$. $\{g_n\}$ is a subsequence of the original sequence of functions, and $\{g_n\}_{n=k}^{\infty}$ is a subsequence of $\{f_{k1}, f_{k2}, \ldots\}$. So $\lim_{j\to\infty} g_j(z_k) = w_k$ for each $k = 1, 2, \ldots$. By theorem 2.6, F is equicontinuous at each point of U. And by theorem 2.8,

$\{g_n\}$ converges uniformly on compact subsets of U. By theorem 1.8, $\{g_n\}$ converges to $f \in A(U)$.

Compactness

Compactness for subsets of A(U) is the next property we will investigate. By using our notion of boundedness, we will show that (like the real number line) a subset of A(U) is compact if and only if it is closed and bounded.

DEFINITION 2.10: Y is sequentially compact if every sequence in Y has a convergent subsequence.

THEOREM 2.10: Let $F \subset A(U)$. F is compact if and only if F is closed and bounded. [2]

PROOF: Let $F \subset A(U)$, and let F be closed and bounded. If $\{f_n\} \subset F$, then by theorem 2.9, there is a subsequence converging to $f \in A(U)$, uniformly on compact subsets. Since F is closed, $f \in F$. So F is sequentially compact, and since F is a metric space it follows that F is compact. To prove the converse, assume F is compact. Any compact subset of a metric space is closed, so all we need to show is boundedness. We will first prove that the map $f \mapsto \|f\|_K$ from C(U) to the reals (for K a fixed compact subset of U) is continuous. If $d(f_n,f) \to 0$, then by theorem 2.1, $f_n \to f$ uniformly on K. This says that for each $\varepsilon > 0$, $|f_n(z)-f(z)| < \varepsilon$ for every $n \geq$ some N and for all $z \in K$.

This implies that $\| f_n - f \|_K \to 0$. From the property that $| |a|-|b| | \leq |a-b|$ for $a,b \in \mathbb{R}$, $| \|f_n\|_K - \|f\|_K | \leq \| f_n - f \|_K \to 0$, and the map is continuous. Since F is compact, the image $\{ \|f\|_K : f \in F \}$ under the above continuous map is compact, and in this case must be bounded. Thus $\sup\{\|f\|_K : f \in F\} < \infty$, which proves that F is bounded.

THEOREM 2.11: Let F be a nonempty compact subset of A(U). If $z_0 \in U$, there is a function g such that $| g'(z_0) | \geq | f'(z_0) |$ for all $f \in F$. [2]

PROOF: By theorem 1.8, if a sequence of analytic functions $f_1, f_2, \ldots$ (on the open set $U \subset \mathbb{C}$) converges to f uniformly on compact subsets, then f is analytic, and $f'_n \to f'$ on U, uniformly on compact subsets. We need to show that the map $f \mapsto | f'(z_0) |$ of F into the reals is continuous. By theorem 2.1, if $\{f_n\} \subset F$ and $d(f_n,f) \to 0$, then $f_n \to f$ and (by theorem 1.8) $f'_n \to f'$, uniformly on compact subsets of U. For $\epsilon > 0$, there exists a δ such that for $| f_n - f | < \delta$, $|f'_n(z_0) - f'(z_0)| < \epsilon$ for $z_0 \in K$ (compact) $\subset U$. So our map is continuous. Since the continuous image of a compact set is compact, $\{ | f'(z_0) | : f \in F \}$ is compact. Since a compact subset of $\mathbb{R}$ is closed and bounded, $\{ |f'(z_0)| : f \in F \}$ contains its supremum, which we call $| g'(z_0) |$. Therefore, there exists a function $g \in F$ such that $| g'(z_0) | \geq | f'(z_0) |$.

THEOREM 2.12: Let U be a connected and open subset of $\mathbb{C}$. Let $F = \{ f \in A(U) :$ f is a one-to-one map of U into

$\overline{D}(0,1)$ and $|\ f'(z_0)\ | \geq b$ }, where b is a fixed positive real number and z_0 is some fixed point of U. Then F is compact. [2]

PROOF: We will prove F is closed and bounded. F is clearly bounded, since $\sup\{\|f\|_K : f \in F\} \leq 1$ for each compact $K \subset U$. If $\{f_n\} \subset F$ and $d(f_n,f) \to 0$, we need to show that $f \in F$, which would mean that F is closed. Since $d(f_n,f) \to 0$, $f_n \to f$ uniformly on compact subsets of U by theorem 2.1, and $|f(z)| \leq 1$ for all $z \in U$. By theorem 1.8, $f \in A(U)$ and also $f'_n \to f'$ uniformly on compact subsets of U. It follows from this that $|\ f'(z_0)\ | \geq b$ (since $f'_n(z_0) \geq b$). From theorem 2.5, we know that f is either identically constant or one-to-one. Since $0 < b \leq |\ f'(z_0)\ |$, f can't be identically constant. So f is one-to-one, and $f \in F$. We now know that F is closed and bounded, and by theorem 2.10, F is compact.

THEOREM 2.13: Let $\{f_n\}$ be a bounded sequence in A(U). If $\{f_n\}$ is not convergent relative to d, then there are two subsequences of $\{f_n\}$ converging relative to d to different limit functions. [2]

PROOF: If $\{f_n\}$ does not converge uniformly on K, then there exists an $\epsilon > 0$ such that for all n, there is an $m > n$ such that $\|\ f_n - f_m\ \|_K \geq \epsilon$. Choose any positive integer n_1; choose $m_1 > n_1$ such that $\|\ f_{n_1} - f_{m_1}\ \|_K \geq \epsilon$. Now pick an $n_2 > m_1$, and choose $m_2 > n_2$ such that $\|\ f_{n_2} - f_{m_2}\ \|_K \geq \epsilon$. Continue in this manner to obtain subsequences $\{\ f_{n_j}\ \}$ and

$\{ f_{m_j} \}$ such that $\| f_{n_j} - f_{m_j} \|_K \geq \epsilon$ for all j.

By applying theorem 2.9 to $\{ f_{n_j} \}$ and $\{ f_{m_j} \}$, we can obtain subsequences $\{ f_{r_i} \}$ of $\{ f_{n_j} \}$ and $\{ f_{s_i} \}$ of $\{ f_{m_j} \}$ such that $\| f_{r_i} - f_{s_i} \|_K \geq \epsilon$ for all i; and $\{ f_{r_i} \}$ converges to some $f \in A(U)$ and $\{ f_{s_i} \}$ converges to some $g \in A(U)$ relative to d. Now $\epsilon \leq \| f_{r_i} - f_{s_i} \|_K \leq \| f_{r_i} - f \|_K + \| f - g \|_K + \| g - f_{s_i} \|_K$, and so $\| f - g \|_K \geq \epsilon$. This says that f does not equal g, and we have two subsequences of $\{f_n\}$ converging to different limit functions.

THEOREM 2.14 (Vitali's Theorem): Let $\{f_n\}$ be a bounded sequence in $A(U)$, where U is connected. Suppose that the sequence converges pointwise on $S \subset U$, where S has a limit point in U. Then the sequence converges uniformly on compact subsets of U. [2]

PROOF: Given the hypotheses, assume that $\{f_n\}$ does not converge uniformly on compact subsets of U. By theorem 2.13, there exist two subsequences of $\{f_n\}$ converging to different limit functions f and g. Since $\{f_n\}$ converges pointwise on S, each subsequence of $\{f_n\}$ must converge at each point of S to the same limit, so f and g must agree on S. Theorem 1.13 (Identity Theorem) tells us that f and g must agree everywhere on U. We have arrived at a contradiction, and can conclude that $\{f_n\}$ must converge uniformly on compact subsets of U.

DEFINITION 2.11: A family $\mathbf{F}$ of functions is normal in

U if every sequence $\{f_n\}$ of functions $f_n \in \mathbf{F}$ contains a subsequence which converges uniformly on every compact subset of U. This definition does not require the limit functions of the convergent subsequences to be members of $\mathbf{F}$.

DEFINITION 2.12: The family $\mathbf{F} \subset C(U)$ is relatively compact if and only if the closure $\overline{\mathbf{F}}$ is compact.

<u>**Problems**</u>

PROBLEM 2.1: A family $\mathbf{F} \subset C(U)$ is normal if and only if $\mathbf{F}$ is relatively compact.

SOLUTION: Assume that $\mathbf{F}$ is normal. Then the convergent subsequence converges to a limit in $\overline{\mathbf{F}}$. So $\overline{\mathbf{F}}$ is sequentially compact. Since $\overline{\mathbf{F}}$ is a metric space (subspace of C(U)), $\overline{\mathbf{F}}$ is compact, and $\mathbf{F}$ is relatively compact.

Assume that $\mathbf{F}$ is relatively compact. This says that $\overline{\mathbf{F}}$ is compact. So $\overline{\mathbf{F}}$ is sequentially compact, and each sequence in $\mathbf{F} \subset \overline{\mathbf{F}}$ has a convergent subsequence. Therefore, $\mathbf{F}$ is normal.

PROBLEM 2.2: If $\mathbf{F} \subset A(U)$, $\mathbf{F}$ is relatively compact if and only if $\mathbf{F}$ is bounded.

SOLUTION: Assume $\mathbf{F}$ is relatively compact. Then $\overline{\mathbf{F}}$ is compact, and by theorem 2.10, $\overline{\mathbf{F}}$ is bounded. So $\mathbf{F}$ must be bounded, since $\mathbf{F} \subset \overline{\mathbf{F}}$.

Assume $\mathbf{F}$ is bounded. By theorems 2.9 and 2.1, $\mathbf{F}$ is sequentially compact, so $\mathbf{F}$ is compact. Then $\overline{\mathbf{F}}$ is compact, and $\mathbf{F}$ is relatively compact.

PROBLEM 2.3: Let L be a multiplicative linear functional on A(U); that is, $L:A(U) \to \mathbb{C}$, $L(af+bg) = aL(f) + bL(g)$, and $L(fg) = L(f)L(g)$ for all $a,b \in \mathbb{C}$, and $f,g \in A(U)$. (Exclude the case $L \equiv 0$.) Then L is a point evaluation; that is, $L(f) = f(z_0)$ for some $z_0 \in U$.

SOLUTION: Let $f \equiv k$. Then $L(f\cdot g) = L(kg) = kL(g)$. Also, $L(f\cdot g) = L(f)\cdot L(g)$, and so for $f \equiv k$, $L(f) = k$. Let $z_0 = L(I)$, where I is the identity map on U. Assume $z_0 \notin U$. Then if $g(z) = \frac{1}{z-z_0}$, then $g \in A(U)$. Then $g(z)(z-z_0) = 1 = g\cdot(I-z_0)$. And $L(g)(L(I)-L(z_0)) = L(g)(z_0-z_0) = L(1) = 1$. We have $L(g)(0) = 0 = 1$, a contradiction, and thus $z_0 \in U$. Let $f \in A(U)$, and let $z_0 = L(I)$. Define $g(z) = \frac{f(z)-f(z_0)}{z-z_0}$ for $z \neq z_0$, and $g(z_0) = f'(z_0)$. Then $g \in A(U)$, and $g(z)(I-z_0) = f(z)-f(z_0)$. So $L(g)(L(I)-L(z_0)) = 0 = L(f)-L(f(z_0)) = L(f)-f(z_0)$, and $L(f) = f(z_0)$, where $z_0 \in U$.

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