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Title: AN INVESTIGATION OF A STABLE NUMERICAL ALGORITHM FOR THE EVALUATION OF FRACTIONAL ORDER BESSEL FUNCTIONS

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For the evaluation of Bessel functions of integer orders, many good algorithms have been proposed. However, the computation of Bessel functions of fractional orders presents difficulties. In Numerical Recipes [10], there is an efficient routine for the numerical approximation of Bessel functions of fractional orders. This thesis will analyze this algorithm. The method uses continued fractions to evaluate Bessel functions of fractional orders. Chapter 1 is the introduction to Bessel functions. Chapter 2 describes a specific application of Bessel functions of fractional orders. Chapter 3 provides background material the student of continued fractions and Chapter 4 analyzes the algorithm.

AN INVESTIGATION OF A STABLE NUMERICAL ALGORITHM FOR THE EVALUATION OF FRACTIONAL ORDER BESSEL FUNCTIONS

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Introduction

Bessel equation of order $\nu(\geq 0)$ is one of the most important differential equations in applied mathematics. This equation has the form:

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Many good algorithms have been proposed for the evaluation of Bessel functions of integer orders. However, the computing Bessel functions of fractional orders presents special difficulities. Recently a stable and efficient algorithm for the evaluation of bessel functions of fractional orders appears in Numerical Recipes [10]. A brief description of the analysis for this method is given but this thesis will a more details explanation of the mathematical basis for the algorithm.

In Chapter 1, the basic notation and definitions in the study of Bessel functions, are presented. The method of deriving two linearly independent power series solutions is described. Chapter 2 provides a details description of one particular application of fractional order Bessel functions, namely in the theory of absorption and scattering by small particles. Since the algorithm under discussion use technique that involve the evaluation of continued fractions, Chapter 3 provides back background material on this subject. The final chapter, Chapter 4 presents a detailed description of the algorithm. Several of the more technical details appears in Appendices A through E.

Chapter 1

Bessel's equation

There are certain forms of differential equations that are useful to describe a variety of physical phenomena. Bessel's equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{1.1}$$

is one such type. This equation occurs in the study of electricity, heat conduction, and stress tests [9]. An application involving the light scattering properties and interstellar dust will be particularly addressed later. This work is specifically interested in methods of approximating the solution to (1.1) when ν is a non-integer, positive rational number.

Before discussing the solution of Bessel's equation, it is necessary to mention some basic definitions. Bessel's equation of order ν is a special case of a general second order linear differential equation with form :

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \tag{1.2}$$

Definitions:

1. A point x_0 such that $P(x_0) \neq 0$ is called an *ordinary point* of (1.2).
2. A point x_0 such that $P(x_0) = 0$ is called the *singular point* of (1.2).
3. A singular point x_0 is called a *regular singular point* of (1.2) if

   $$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} \text{ is finite.}$$

   and

   $$\lim_{x \to x_0} (x - x_0)^2\frac{R(x)}{P(x)} \text{ is finite.}$$

Now consider the series solution of Bessel's equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \ x > 0 \tag{1.3}$$

where $P(x) = x^2, Q(x) = x, R(x) = x^2 - \nu^2$.
Since

$$\lim_{x\to 0}(x-0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{x}{x^2} = 1 \text{ is finite}$$

and

$$\lim_{x\to 0}(x-0)^2\frac{R(x)}{P(x)} = \lim_{x\to 0} x^2\frac{(x^2-\nu^2)}{x^2} = -\nu^2 \text{ is finite}$$

$x = 0$ is a regular singular point. Hence the method of Frobenius (see [9]) is an appropriate method for determining a series solution which we describe in section 1.1.

1.1 Bessel function of the first kind of order ν

Suppose

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

is a series solution of (1.3), then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \qquad \text{and} \\ y'' &= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} \end{aligned}$$

Substituting in (1.3), yields

$$\begin{aligned} & x^2y'' + xy' + (x^2-\nu^2)y \\ = & \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} + \\ & \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + (x^2-\nu^2)\sum_{n=0}^{\infty} a_n x^{n+r} \\ = & \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)-\nu^2]a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} \\ = & (r^2-\nu^2)a_0 x^r + ((1+r)^2-\nu^2)a_1 x^{1+r} \\ & + \sum_{n=2}^{\infty}((n+r)(n+r-1)+(n+r)-\nu^2)a_n x^{n+r} \\ & + \sum_{k=2}^{\infty} a_{k-2}x^{k+r} \qquad \text{with } k = n+2 \end{aligned}$$

$$
\begin{aligned}
&= (r^2-\nu^2)a_0x^r + [(1+r)^2-\nu^2]a_1x^{1+r} \\
&\quad + \sum_{n=2}^{\infty}\left\{[(n+r)^2-\nu^2]a_n + a_{n-2}\right\}x^{n+r} \qquad (1.4)\\
&= 0
\end{aligned}
$$

Requiring y to be a solution of (1.3) leads to

$$(r^2-\nu^2)a_0 = 0 \qquad (1.5)$$

$$[(1+r)^2-\nu^2]a_1 = 0 \qquad (1.6)$$

$$\text{and} \quad [(n+r)^2-\nu^2]a_n + a_{n-2} = 0 \ \text{, for } n \geq 2 \qquad (1.7)$$

(1.5) is called the indicial equation. Suppose $a_0 \neq 0$, then $r^2-\nu^2=0$. The roots of the indicial equation are ν and $-\nu$. Since equation(1.5), (1.6), one can not have a solution for both $r^2=\nu^2$ and $(1+r)^2=\nu^2$, hence set $a_1=0$. Then (1.7) gives

$$a_n = \frac{-a_{n-2}}{(n+r)^2-\nu^2} = \frac{-a_{n-2}}{(n+r-\nu)(n+r+\nu)}.$$

Thus, it follows

$$a_1 = a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$$

and let $n=2m$, then

$$a_{2m} = \frac{-a_{2m-2}}{(2m+r-\nu)(2m+r+\nu)}\ , \quad m \geq 1$$

Consider the case $r=\nu$. The recurrence relation for the coefficients for the series solution is

$$a_{2m} = \frac{-a_{2m-2}}{(2m+\nu-\nu)(2m+\nu+\nu)} = \frac{-a_{m-2}}{2^2m(m+\nu)}$$

hence

$$
\begin{aligned}
a_2 &= \frac{-a_0}{2^2\cdot 1\cdot(1+\nu)} \\
a_4 &= \frac{-a_2}{2^2\cdot 2\cdot(2+\nu)} = \frac{(-1)^2a_0}{2^4\cdot 2\cdot 1\cdot(1+\nu)(2+\nu)} \\
a_6 &= \frac{-a_4}{2^3\cdot 3\cdot(3+\nu)} = \frac{(-1)^3a_0}{2^6\cdot 3\cdot 2\cdot 1\cdot(1+\nu)(2+\nu)(3+\nu)} \\
&\vdots \\
a_{2m} &= \frac{(-1)^m a_0}{2^{2m}\cdot m!(m+\nu)(m-1+\nu)\cdots(2+\nu)(1+\nu)}.
\end{aligned}
$$

Therefore, corresponding to the case when $r=\nu$, one solution is

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\nu}$$

$$
\begin{aligned}
&= a_0x^\nu + a_2x^{2+\nu} + a_4x^{4+\nu} + \cdots + a_{2m}x^{2m+\nu} + \cdots \\
&= a_0x^\nu + \sum_{m=1}^{\infty} a_{2m}x^{2m+\nu} \\
&= a_0x^\nu + \sum_{m=1}^{\infty} \frac{(-1)^m a_0}{m!(m+\nu)(m+\nu-1)\cdots(2+\nu)(1+\nu)}(\frac{x}{2})^{2m} \cdot x^\nu \\
&= a_0x^\nu[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m+\nu)(m+\nu-1)\cdots(1+\nu)}(\frac{x}{2})^{2m}] \qquad (1.8)
\end{aligned}
$$

Note $y_1(x)$ is a power series, whose convergence is seen by applying the ratio test:

$$
\begin{aligned}
& \lim_{m\to\infty} \left| \frac{a_{2m}x^{2m+\nu}}{a_{2(m-1)}x^{2m-2+\nu}} \right| \\
= & \lim_{m\to\infty} \left| \frac{(-1)^m a_0/2^{2m}m!(m+\nu)(m-1+\nu)\cdots(1+\nu)}{(-1)^{m-1}a_0/2^{2m-2}(m-1)!(m-1+\nu)\cdots(1+\nu)} \right| \cdot \left| \frac{x^{2m+\nu}}{x^{2m-2+\nu}} \right| \\
= & \lim_{m\to\infty} \left| \frac{1}{2^2m(m+\nu)} \right| \cdot | x^2 | \\
= & \lim_{m\to\infty} \frac{1}{2^2m(m+\nu)}x^2 \\
= & \ 0 \qquad\qquad , \text{ for } x > 0.
\end{aligned}
$$

Thus the series converges for all x. Thus, $y_1(x)$ is a solution of Bessel's equation.

For simplicity, the gamma function will be used to rewrite $y_1(x)$. Since $\Gamma(x+1) = x\Gamma(x)$,

$$
\begin{aligned}
& (m+\nu)(m+\nu-1)\cdots(2+\nu)(1+\nu) \\
= & (m+\nu)(m+\nu-1)\cdots(2+\nu)(1+\nu)\frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} \\
= & \frac{\Gamma(m+\nu+1)}{\Gamma(\nu+1)}
\end{aligned}
$$

Let $a_0 = \frac{1}{2^\nu\Gamma(\nu+1)}$.
Then

$$
\begin{aligned}
y_1(x) &= \sum_{m=0}^{\infty} a_{2m}x^{2m+\nu} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m!(m+\nu)\cdots(2+\nu)(1+\nu)}(\frac{x}{2})^{2m}x^\nu \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{m!\Gamma(m+\nu+1)/\Gamma(\nu+1)}(\frac{x}{2})^{2m}x^\nu \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)}(\frac{x}{2})^{2m+\nu}
\end{aligned}
$$

which is known as the Bessel function of the first kind of order ν. It is denoted by $J_\nu(x)$.

Corresponding to the other root $-\nu$ of the indicial equation, if ν is not an integer, the second independent solution can be found by replacing ν by $-\nu$ in (1.8).

$$y_2(x) = a_0 x^{-\nu}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m-\nu)(m-\nu-1)\cdots(2-\nu)(1-\nu)}(\frac{x}{2})^{2m}]$$

which is denoted by $J_{-\nu}(x)$, with $a_0 = \frac{1}{2^\nu(1-\nu)}$

$$J_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m-\nu+1)}(\frac{x}{2})^{2m-\nu}$$

Thus, if $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent, the general solution of Bessel's equation is

$$y = AJ_\nu(x) + BJ_{-\nu}(x) \tag{1.9}$$

where A and B are arbitrary constants.

However, there are some limitations for using (1.8) as the general solution of Bessel's equation. If ν is zero, $J_\nu(x)$ and $J_{-\nu}(x)$ are not distinct. Furthermore, if ν is a positive integer, since $\frac{1}{\Gamma(-n)}$ is zero if n is zero or a positive integer, then the first m items of $J_{-\nu}(x)$ start with zero. In this case a change of variables ($n = m - r$) may be used to show:

$$\begin{aligned} J_{-\nu}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m(\frac{x}{2})^{2m-\nu}}{m!\Gamma(m-\nu+1)} \\ &= \sum_{m=\nu}^{\infty} \frac{(-1)^m(\frac{x}{2})^{2m-\nu}}{m!\Gamma(m-\nu+1)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+\nu}(\frac{x}{2})^{2n+\nu}}{(n+\nu)!\Gamma(n+1)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{n+\nu}(\frac{x}{2})^{2n+\nu}}{n!\Gamma(n+\nu+1)} \\ &= (-1)^\nu J_\nu(x) \end{aligned}$$

This means $J_\nu(x)$ and $J_{-\nu}(x)$ are not linearly independent solutions if ν is an integer.

Depending on the roots of the indicial equation, there are several ways to find the second solution(see [7, p240]). We will give an example to find two linearly independent solutions for the Bessel's equation of order $\frac{3}{2}$

$$x^2y'' + xy' + (x^2 - \frac{9}{4})y = 0\ ,\quad x > 0 \tag{1.10}$$

First, let

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

From equation(1.4) with $r = \frac{3}{2}$,

$$\begin{aligned}
& x^2y'' + xy' + (x^2 - \frac{9}{4})y \\
= \ & [r^2 - \frac{9}{4}]a_0x^r + [(1+r)^2 - \frac{9}{4}]a_1x^{1+r} \\
& + \sum_{n=2}^{\infty} \left\{ [(n+r)^2 - (\frac{3}{2})^2]a_n + a_{n-2} \right\} x^{n+r} = 0
\end{aligned} \tag{1.11}$$

The recurrence formula is

$$a_n = \frac{-a_{n-2}}{(n+r)^2 - (\frac{3}{2})^2}$$

Corresponding to $r = \frac{3}{2}$,

$$a_n = \frac{-a_{n-2}}{(n+3)(n)} \quad , \quad n \geq 2$$

and $a_1 = a_3 = \cdots = a_{2n+1} = \cdots = 0$.
Let $n = 2m$

$$\begin{aligned}
a_{2m} &= \frac{-a_{2m-2}}{(2m+3)(2m)} \quad , \quad m = 1, 2, 3, \cdots \\
&= \frac{-a_{2m-2}}{2^2(m+\frac{3}{2})(m)}
\end{aligned}$$

Hence

$$\begin{aligned}
a_2 &= \frac{-a_0}{2^2(1+\frac{3}{2})} \cdot 1 \\
a_4 &= \frac{-a_2}{2^2(2+\frac{3}{2}) \cdot 2} = \frac{(-1)^2 a_0}{2^4(2+\frac{3}{2})(1+\frac{3}{2}) \cdot 2 \cdot 1} \\
a_6 &= \frac{-a_4}{2^2(3+\frac{3}{2}) \cdot 3} = \frac{(-1)^3 a_0}{2^4(3+\frac{3}{2})(2+\frac{3}{2})(1+\frac{3}{2}) \cdot 3 \cdot 2 \cdot 1} \\
&\vdots \\
a_{2m} &= \frac{(-1)^m a_0}{2^{2m}(m+\frac{3}{2})(m-1+\frac{3}{2}) \cdots (2+\frac{3}{2})(1+\frac{3}{2}) \cdot m!}
\end{aligned}$$

Therefore,

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{n+\frac{3}{2}}$$

$$\begin{aligned}
&= a_0x^{\frac{3}{2}} + a_2x^{2+\frac{3}{2}} + \cdots + a_{2m}x^{2m+\frac{3}{2}} + \cdots \\
&= a_0x^{\frac{3}{2}} + \sum_{m=1}^{\infty} a_{2m}x^{2m+\frac{3}{2}} \\
&= a_0x^{\frac{3}{2}} + \sum_{m=1}^{\infty} \frac{(-1)^m a_0}{(m+\frac{3}{2})(m-1+\frac{3}{2})\cdots(2+\frac{3}{2})(1+\frac{3}{2})\cdot m!}(\frac{x}{2})^{2m} \cdot x^{\frac{3}{2}} \\
&= a_0x^{\frac{3}{2}}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+\frac{3}{2})(m-1+\frac{3}{2})\cdots(2+\frac{3}{2})(1+\frac{3}{2})\cdot m!}(\frac{x}{2})^{2m}]
\end{aligned}$$

Let $a_0 = 1$,

$$y_1(x) = x^{\frac{3}{2}}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m+\frac{3}{2})\cdots(2+\frac{3}{2})(1+\frac{3}{2})m!}(\frac{x}{2})^{2m}]$$

According to (1.11), one has

$$\begin{aligned}
[(r_2)^2 - (\frac{3}{2})^2]a_0 &= 0 \\
((1+r_2)^2 - (\frac{3}{2})^2)a_1 &= 0 \quad (r_2 = -\frac{3}{2})
\end{aligned}$$

and

$$a_n = \frac{-a_{n-2}}{(n+r_2+\frac{3}{2})(n+r_2-\frac{3}{2})} = \frac{-a_{n-2}}{n\cdot(n-3)}$$

so a_0 can be an arbitrary constant and $a_1 = 0$. Let $n = 2m$, the recurrence formula is

$$a_{2m} = \frac{-a_{2m-2}}{2m(2m-3)} = \frac{-a_{2m-2}}{2^2m(m-\frac{3}{2})}$$

and $a_1 = a_3 = \cdots = a_{2n+1} = \cdots = 0$. Similar to $y_1(x)$, it is easy to find

$$\begin{aligned}
y_2(x) &= \sum_{n=0}^{\infty} a_nx^{n-\frac{3}{2}} \\
&= a_0x^{-\frac{3}{2}} + a_2x^{2-\frac{3}{2}} + \cdots + a_{2m}x^{2m-\frac{3}{2}} + \cdots \\
&= a_0x^{-\frac{3}{2}}[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(m-\frac{3}{2})(m-1-\frac{3}{2})\cdots(2-\frac{3}{2})(1-\frac{3}{2})}(\frac{x}{2})^{2m}]
\end{aligned}$$

1.2 Bessel function of the second kind of order ν

As we have previously shown, if ν is an integer, (1.9) cannot be a general solution of Bessel's equation because $J_\nu(x)$ and $J_{-\nu}(x)$ are not linearly independent solutions.

For purposes of obtaining a second linearly independent solution y_2 whose independence is not restricted to certain values of ν, such a solution is defined by

$$Y_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$$

which is called the Bessel function of the second kind of order ν. When ν is not an integer, $Y_\nu(x)$ is a combination of two independent solutions J_ν and $J_{-\nu}$, and still is linearly independent of $J_\nu(x)$. If ν is an integer, $Y_\nu(x)$ becomes an indeterminate form $\frac{0}{0}$. However, it can be shown that the limit exists as $\nu \to n$, n is an integer,([3], p274) and it is defined by

$$Y_n(x) = \lim_{\nu\to n} Y_\nu(x) \quad , \quad n = 0, 1, 2, \cdots .$$

Recurrence formulas There are several useful recurrence relations for the Bessel functions of the first kind of order ν. The derivation can be developed from their series definition, such as

$$\begin{aligned}
\frac{d}{dx}[x^\nu J_\nu(x)] &= \frac{d}{dx}[\sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m+\nu}}{m!\Gamma(m+\nu+1)} x^\nu] \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m (2m+2\nu)}{m!\Gamma(m+\nu+1)2^{2m+\nu}} x^{2m+2\nu-1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m (m+\nu)}{m!(m+\nu)\Gamma(m+\nu)} (\frac{x}{2})^{2m+\nu-1} \cdot x^\nu \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+(\nu-1)+1)} (\frac{x}{2})^{2m+\nu-1} \cdot x^\nu \\
&= J_{\nu-1}(x) \cdot x^\nu. \qquad (1.12)
\end{aligned}$$

Similarly, one finds that

$$\frac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x) \qquad (1.13)$$

Differentiate on the left-hand sides in (1.12) and (1.13) and divide the results by x^ν and $x^{-\nu}$ respectively, to obtain

$$\begin{aligned}
J'_\nu(x) + \frac{\nu}{x} J_\nu(x) &= J_{\nu-1}(x) \qquad (1.14) \\
J'_\nu(x) - \frac{\nu}{x} J_\nu(x) &= -J_{\nu+1}(x) \qquad (1.15)
\end{aligned}$$

The sum and difference of (1.14) and (1.15) yield respectively,

$$\begin{aligned}
2J'_\nu(x) &= J_{\nu-1}(x) - J_{\nu+1}(x) \qquad (1.16) \\
\frac{2\nu}{x} J_\nu(x) &= J_{\nu-1}(x) + J_{\nu+1}(x) \qquad (1.17)
\end{aligned}$$

Bessel function of fractional order Consider the Bessel's equation of order 1/2

$$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 0$$

From (1.7) with $\nu = \frac{1}{2}$, and $r = \frac{1}{2}$ gives

$$n(n+1)a_n + a_{n-2} = 0$$

$$\begin{aligned} a_n &= \frac{-a_{n-2}}{n(n+1)} \quad , \quad n = 2, 4, 6, \cdots \\ a_{2m} &= \frac{-a_{2m-2}}{2m(2m+1)} \quad , \quad m = 1, 2, 3, \cdots \\ &= \frac{a_{2m-4}}{2m(2m+1)(2m-2)(2m-1)} \\ &\vdots \\ &= \frac{(-1)^m a_0}{(2m+1)!} \end{aligned}$$

Hence, by (1.8)

$$\begin{aligned} y_1(x) &= a_0x^\nu + \sum_{m=1}^{\infty} a_{2m}x^{2m+\nu} \\ &= x^{1/2}\sum_{m=0}^{\infty} \frac{(-1)^m a_0 x^{2m}}{(2m+1)!} \\ &= x^{-1/2}\sum_{m=0}^{\infty} \frac{(-1)^m a_0 x^{2m+1}}{(2m+1)!} \\ &= x^{-1/2} a_0 \sin x \end{aligned}$$

Let $a_0 = \frac{1}{2^{1/2}\Gamma(\frac{1}{2}+1)} = \sqrt{\frac{2}{\pi}}$. Thus, the Bessel function of the first kind of order 1/2 is

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad , \quad x > 0$$

Similarly, corresponding to another root of the indicial equation $r = -\frac{1}{2}$, one can find

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

By means of the recurrence formula (1.17), it is possible to find $J_{n+1/2}(x)$, where n is an integer.

Spherical Bessel Functions Related to $J_{n+1/2}(x)$, there is a special form of Bessel functions called spherical Bessel functions which are defined by

$$\begin{aligned} j_n(x) &= \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x) \\ y_n(x) &= \sqrt{\frac{\pi}{2x}}\, Y_{n+1/2}(x) \end{aligned}$$

Similar to the Bessel functions of first kind, spherical Bessel functions also have some useful recurrence formulas, and spherical functions arise in the study of a particular application of Bessel functions which will be treated in the next chapter.

Bessel functions of the third kind Bessel functions of the third kind are also called Hankel functions which are defined by

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + iY_\nu(x) \\ H_\nu^{(2)}(x) &= J_\nu(x) - iY_\nu(x) \end{aligned}$$

These functions are linear combinations of the Bessel function of first and second kinds. The most useful property of the Hankel functions is that they have very a simple asymptotic expression if $\mid x \mid$ is very large (see [8, p108]). Hankel functions will be used in a later section.

Chapter 2

Applications of Bessel functions

Bessel functions appear in many physics and engineering problems. One specific example for the application of Bessel functions is in the theory of absorption and scattering by small particles (see [6] chap.3). When a spherical particle is illuminated by a beam of light with specified characteristics, the electromagnetic field surrounding the particle will be influenced, and the field inside the particle will also be changed. It can be shown that a time-harmonic electromagnetic field $(\vec{E}, \vec{H})$ in a linear, isotropic, homogeneous medium must satify the vector wave equation

$$\vec{\nabla}^2\vec{E} + k^2\vec{E} = 0 \qquad , \qquad \vec{\nabla}^2\vec{H} + k^2\vec{H} = 0$$

where $k^2 = \omega^2\varepsilon\mu$ is a constant, and $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \cdot \vec{H} = 0$. In addition, $\vec{\nabla} \times \vec{E} = i\omega\mu\vec{H}$ and $\vec{h} \times \vec{H} = -i\omega\varepsilon\vec{E}$, where ε is the electric permittivity, μ is the magnetic permeability, ω is the angular frequency.

The solution of the vector wave equation can be found by reducing the equation to a comparatively simpler problem of finding the solution of the scalar wave equation. Suppose that, given a scalar function ψ and an arbitrary constant vector $\vec{c}$, we construct a vector function:

$$\vec{S} = \vec{\nabla} \times (\vec{c}\psi)$$

It is known that the divergence of the curl of any vector vanishes. Thus,

$$\nabla \cdot \vec{S} = \nabla \cdot \nabla \times (\vec{c}\psi) = 0$$

If we use the vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{S}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{S}) - \vec{\nabla}^2\vec{S}.$$

Since $\vec{\nabla} \cdot \vec{S} = 0$, we obtain

$$\begin{aligned}
\vec{\nabla}^2\vec{S} &= -\vec{\nabla} \times (\vec{\nabla} \times \vec{S}) \\
&= -\vec{\nabla} \times (\vec{\nabla} \times (\vec{\nabla} \times (\vec{c}\psi))) \\
&= -\vec{\nabla} \times [\vec{\nabla}(\vec{\nabla} \cdot (\vec{c}\psi) - \vec{\nabla}^2(\vec{c}\psi)] \\
&= -\vec{\nabla} \times \vec{\nabla}[\vec{\nabla} \cdot (\vec{c}\psi)] + \vec{\nabla} \times [\vec{\nabla}^2(\vec{c}\psi)] \qquad (2.1)
\end{aligned}$$

where

$$\begin{aligned}\vec{\nabla}\cdot(\vec{c}\psi) &= \psi\vec{\nabla}\cdot\vec{c}+\vec{c}\cdot\vec{\nabla}\psi \\ &= 0+\vec{c}\cdot\vec{\nabla}\psi \qquad \text{(since } \vec{c} \text{ is a constant vector)}\end{aligned}$$

Since $\vec{c}\cdot\vec{\nabla}\psi$ is a scalar function and the vector identity $\vec{\nabla}\times\vec{\nabla}\phi=\vec{0}$, with ϕ a scalar function, the first right-hand term of (2.1) will vanish. That is

$$\begin{aligned}&\vec{\nabla}\times\vec{\nabla}[\vec{\nabla}\cdot(\vec{c}\psi)] \\ =\ &\vec{\nabla}\times\vec{\nabla}[\vec{c}\cdot\vec{\nabla}\psi]=0\end{aligned}$$

Thus, (2.1) becomes

$$\vec{\nabla}^2\vec{S}=\vec{\nabla}\times[\vec{\nabla}^2(\vec{c}\psi)].$$

Therefore, one has

$$\begin{aligned}\vec{\nabla}^2\vec{S}+k^2\vec{S} &= \vec{\nabla}\times[\vec{\nabla}^2(\vec{c}\psi)]+k^2[\vec{\nabla}\times(\vec{c}\psi)] \\ &= \vec{\nabla}\times[\vec{\nabla}^2(\vec{c}\psi)+k^2(\vec{c}\psi)] \\ &= \vec{\nabla}\times[\vec{c}(\vec{\nabla}^2\psi+k^2\psi)]\end{aligned}$$

$\vec{S}$ is the solution of the vector wave equation if ψ satisfies the scalar wave equation

$$\vec{\nabla}^2\psi+k^2\psi=0$$

Since this problem analyzes the light scattered by a spherical particle it is natural to work in spherical polar coordinates, r,θ, and ϕ. Using the separation of variables technique (i.e. set $\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$), the wave equation can be reduced to an ordinary differential equation in the radial variable. This differential equation is a spherical Bessel's equation. The details of this transformation are as follows:

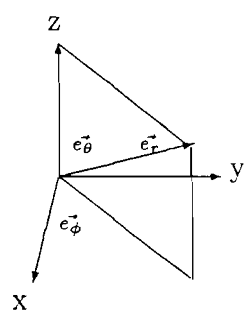

Figure 1

From reference([4, p104]), a scalar function in spherical polar coordinates satisfies the formula

$$\vec{\nabla}\psi(r,\theta,\phi)=\frac{\partial\psi}{\partial r}\vec{e_r}+\frac{1}{r}\frac{\partial\psi}{\partial\theta}\vec{e_\theta}+\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\vec{e_\phi}$$

where $\vec{e_r}, \vec{e_\theta}, \vec{e_\phi}$ are the curvilinear unit vectors in (fig-1). And for any vector $\vec{F}(r,\theta,\phi)$

$$\vec{\nabla}\cdot\vec{F}(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial(r^2F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(F_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

Thus,

$$\begin{aligned}
\vec{\nabla}^2\psi &= \vec{\nabla}\cdot(\vec{\nabla}\psi) \\
&= \vec{\nabla}\cdot(\frac{\partial\psi}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\vec{e_\phi}) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}
\end{aligned}$$

Hence, the scalar wave equation in spherical polar coordinates is

$$\begin{aligned}
&\vec{\nabla}^2\psi + k^2\psi \\
= &\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\psi}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial\psi}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0
\end{aligned}$$

Let $\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$
Then

$$\begin{aligned}
\frac{\partial\psi}{\partial r} &= \frac{dR(r)}{dr}\Theta(\theta)\Phi(\phi) \\
\frac{\partial\psi}{\partial\theta} &= R(r)\frac{d\Theta(\theta)}{d\theta}\Phi(\phi) \\
\frac{\partial\psi}{\partial\phi} &= R(r)\Theta(\theta)\frac{d\Phi(\phi)}{d\phi}
\end{aligned}$$

Substituting in the wave equation yields,

$$\begin{aligned}
&\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{dR(r)}{dr}\Theta(\theta)\Phi(\phi)) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta R(r)\Phi(\phi)\frac{d\Theta(\theta)}{d\theta}) \\
&+\frac{1}{r^2\sin^2\theta}R(r)\Theta(\theta)\frac{d^2\Phi(\phi)}{d\phi^2} + k^2R(r)\Theta(\theta)\Phi(\phi) = 0
\end{aligned}$$

Dividing by $R(r)\Theta(\theta)\Phi(\phi)$ to give

$$\begin{aligned}
&\frac{1}{r^2}\frac{1}{R(r)}\frac{d}{dr}(r^2\frac{dR(r)}{dr}) + \frac{1}{r^2\sin\theta}\frac{1}{\Theta(\theta)}\frac{d}{d\theta}(\sin\theta\frac{d\Theta(\theta)}{d\theta}) \\
&+\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2} + k^2 = 0
\end{aligned}$$

Multiply both sides by $r^2 \sin^2\theta$ to yield,

$$\frac{\sin^2\theta}{R(r)}\frac{d}{dr}(r^2\frac{dR(r)}{dr}) + \frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}(\sin\theta\frac{d\Theta(\theta)}{d\theta}) + \frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}$$
$$+k^2r^2\sin^2\theta = 0$$

and

$$\frac{\sin^2\theta}{R(r)}\frac{d}{dr}(r^2\frac{dR(r)}{dr}) + \frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}(\sin\theta\frac{d\Theta(\theta)}{d\theta}) + k^2r^2\sin^2\theta = -\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}$$

Let $-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2$. Thus,

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{2.2}$$

Then

$$\frac{\sin^2\theta}{R(r)}\frac{d}{dr}(r^2\frac{dR(r)}{dr}) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + k^2r^2\sin^2\theta = m^2$$
$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) + \frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + k^2r^2 = \frac{m^2}{\sin^2\theta}$$
$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) + r^2k^2 = -\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \frac{m^2}{\sin^2\theta}$$

For $\Theta(\theta)$:

$$-\frac{1}{\sin\theta}\frac{1}{\Theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + \frac{m^2}{\sin^2\theta} = c$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) - \frac{m^2}{\sin^2\theta}\Theta + c\Theta = 0 \tag{2.3}$$

Let c=n(n+1), n is an integer.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{d\Theta}{d\theta}) + [n(n+1) - \frac{m^2}{\sin^2\theta}]\Theta = 0$$

For $R(r)$:

$$\frac{1}{R}\frac{d}{dr}(r^2\frac{dR}{dr}) + r^2k^2 = c$$
$$\frac{d}{dr}(r^2\frac{dR}{dr}) + [r^2k^2 - n(n+1)]R = 0 \tag{2.4}$$

Let $\rho = kr$ and define $Z = R\sqrt{\rho}$, then $d\rho = kdr$, $\frac{dr}{d\rho} = \frac{1}{k}$ and

$$\begin{aligned}
\frac{dZ}{d\rho} &= \frac{d}{d\rho}(R\sqrt{\rho}) \\
&= \frac{d(R\sqrt{\rho})}{dr}\frac{dr}{d\rho} \\
&= \frac{1}{k}\frac{dR}{dr}\sqrt{kr} + \frac{R}{k}\cdot\frac{1}{2}\cdot\frac{k}{\sqrt{kr}} \\
&= \frac{\sqrt{\rho}}{k}\frac{dR}{dr} + \frac{R}{2}\frac{1}{\sqrt{\rho}}
\end{aligned}$$

Thus,

$$\frac{dR}{dr} = \frac{k}{\sqrt{\rho}}(\frac{dZ}{d\rho} - \frac{Z}{2\rho})$$

With this equation, one can show

$$\begin{aligned}
r^2\frac{dR}{dr} &= \frac{\rho^{3/2}}{k}(\frac{dZ}{d\rho} - \frac{Z}{2\rho}) \\
\frac{d}{dr}(r^2\frac{dR}{dr}) &= \frac{d\rho}{dr}\frac{d}{d\rho}[\frac{\rho^{3/2}}{k}(\frac{dZ}{d\rho} - \frac{Z}{2\rho})] \\
&= \frac{d}{d\rho}(\rho^{3/2}\frac{dZ}{d\rho} - \frac{1}{2}\rho^{1/2}Z) \\
&= \rho^{1/2}\frac{dZ}{d\rho} + \rho^{3/2}\frac{d^2Z}{d\rho^2} - \frac{1}{4}\frac{1}{\sqrt{\rho}}Z
\end{aligned}$$

Substituting in (2.4), one has

$$\begin{aligned}
&\rho^{1/2}\frac{dZ}{d\rho} + \rho^{3/2}\frac{d^2Z}{d\rho^2} - \frac{1}{4\sqrt{\rho}}Z + [\rho^2 - n(n+1)]\frac{Z}{\sqrt{\rho}} = 0 \\
&\rho\frac{dZ}{d\rho} + \rho^2\frac{d^2Z}{d\rho^2} + [\rho^2 - n(n+1) - \frac{1}{4}]Z = 0 \\
&\rho\frac{d}{d\rho}(\rho\frac{dZ}{d\rho}) + [\rho^2 - (n+\frac{1}{2})^2]Z = 0 \qquad (2.5)
\end{aligned}$$

This is a Bessel equation of order $(n+\frac{1}{2})$; its two linearly independent solutions are the Bessel functions of first and second kind $J_\nu(\rho)$ and $Y_\nu(\rho)$, where $\nu = n + 1/2$. The solutions of (2.4) are called the spherical Bessel functions. They are

$$j_n(\rho) = \sqrt{\frac{\pi}{2\rho}}J_{n+1/2}(\rho) \qquad (2.6)$$

$$y_n(\rho) = \sqrt{\frac{\pi}{2\rho}}Y_{n+1/2}(\rho) \qquad (2.7)$$

Thus, it is important to be able to accurately compute Bessel functions of fractional order. The details of one algorithm for the computation of these functions are presented in chapter 4. Note that once the solutions of (2.2), (2.3) and (2.4) have been approximated one can then approximate the solutions of the vector wave equation.

Similar to the recurrence relations (1.16),(1.17) for first kind, Bessel functions there are the following recurrence relations for spherical Bessel functions. See [8].

$$\begin{aligned} \frac{2\nu+1}{x} j_\nu(x) &= j_{\nu-1}(x) + j_{\nu+1}(x) &\quad (2.8)\\ (2\nu+1) j_\nu'(x) &= \nu j_{\nu-1} - (\nu+1) j_{\nu+1}(x) &\quad (2.9) \end{aligned}$$

From these two recurrence relations, one can show that the spherical Bessel function $j_\nu(x)$ satisfies the differential equation

$$x^2 j_\nu''(x) + 2x j_\nu'(x) + [x^2 - \nu(\nu+1)] j_\nu(x) = 0 \qquad (2.10)$$

The derivation is stated as follows:

Adding (2.8) $\times(\nu+1)$ and (2.9), yields

$$(2\nu+1) j_{\nu-1}(x) = \frac{(2\nu+1)(\nu+1)}{x} j_\nu(x) + (2\nu+1) j_\nu'(x)$$

Hence,

$$j_{\nu-1}(x) = \frac{\nu+1}{x} j_\nu(x) + j_\nu'(x) \qquad (2.11)$$

Subtracting (2.9) from (2.8)$\times\nu$, yields

$$(2\nu+1) j_{\nu+1}(x) = \frac{(2\nu+1)\nu}{x} j_\nu(x) - (2\nu+1) j_\nu'(x)$$

Hence,

$$j_{\nu+1}(x) = \frac{\nu}{x} j_\nu(x) - j_\nu'(x) \qquad (2.12)$$

From (2.11), differentiating with respect to x on both sides, yields

$$\begin{aligned} j_{\nu-1}'(x) &= \frac{\nu+1}{x} j_\nu'(x) - \frac{\nu+1}{x^2} j_\nu(x) + j_\nu''(x) \\ j_\nu''(x) &+ \frac{\nu+1}{x} j_\nu'(x) - \frac{\nu+1}{x^2} j_\nu(x) - j_{\nu-1}'(x) = 0 &\quad (2.13) \end{aligned}$$

Now, rewrite (2.12) by replacing ν by $\nu - 1$

$$j_\nu(x) = \frac{\nu-1}{x} j_{\nu-1}(x) - j_{\nu-1}'(x)$$

Using this equation and (2.11) to eliminate $j_{\nu-1}'(x)$ and $j_{\nu-1}(x)$ from (2.13), yields

$$j_\nu''(x) + \frac{\nu+1}{x} j_\nu'(x) - \frac{\nu+1}{x^2} j_\nu(x) + [j_\nu(x) - \frac{\nu-1}{x}(\frac{\nu+1}{x} j_\nu(x) + j_\nu'(x))] = 0$$

or

$$j_\nu''(x) + \frac{2}{x}j_\nu'(x) + [1 - \frac{\nu+1}{x^2} - \frac{\nu^2-1}{x^2}]j_\nu(x) = 0$$

which reduces to

$$j_\nu''(x) + \frac{2}{x}j_\nu'(x) + (1 - \frac{\nu(\nu+1)}{x^2})j_\nu(x) = 0$$

Multiplying by x^2, we finally get the differential equation (2.10).

Chapter 3

Continued Fractions

Using the recurrence formula (1.17), there is an easy way to evaluate the Bessel function of the first kind. However, the error of computation may grow very quickly, and the process of computation may become unstable. For example, if one chooses to compute $J_2(2), J_3(2), \cdots J_7(2)$, with the recurrence formula

$$J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x)$$

given the initial values $J_0(2) = 0.223890$, $J_1(2) = 0.576724$, one obtains (where error is relative to the table results from [1] within 10^{-6})

$$\begin{array}{rclll}
 & & & & \text{error} \\
J_2(2) & = & \frac{2\cdot 1}{2} J_1(2) - J_0(2) = 0.352834 & & 0 \\
J_3(2) & = & \frac{2\cdot 2}{2} J_2(2) - J_1(2) = 0.128944 & & 4 \times 10^{-6} \\
J_4(2) & = & \frac{2\cdot 3}{2} J_3(2) - J_2(2) = 0.033998 & & 2 \times 10^{-6} \\
J_5(2) & = & \frac{2\cdot 4}{2} J_4(2) - J_3(2) = 0.007048 & & 8.4 \times 10^{-6} \\
J_6(2) & = & \frac{2\cdot 5}{2} J_5(2) - J_4(2) = 0.001242 & & 39.6 \times 10^{-6} \\
J_7(2) & = & \frac{2\cdot 6}{2} J_6(2) - J_5(2) = 0.000404 & & 229.06 \times 10^{-6}
\end{array}$$

In this example, the error of $J_\nu(x)$ is always multipled by $2\nu/x$ in the next step. For our value of x, (especially, when $x \ll \nu$), the coefficient $2\nu/x$ is very large, and the process thus becomes unstable; see [2, p22] . For more discussion on the topic of the stability of the recurrence formula, the reader is referred to ([10], §5.5). Therefore, we use techniques that involve continued fraction expressions to evaluate Bessel functions. Hence this section is devoted to some background information on continued fractions.

A fraction of the form

$$f = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{3.1}$$

is called a continued fraction. In general $b_0, b_1, b_2, \cdots$ and $a_1, a_2, a_3, \cdots$ can be real or complex numbers. Continued fractions are an effective representation of computation for many functions. There are two basic types of continued fractions. First, consider the following example :

$$\begin{aligned}
\frac{75}{14} &= 5 + \frac{1}{\frac{14}{5}} \\
&= 5 + \frac{1}{2 + \frac{1}{\frac{5}{4}}} \\
&= 5 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4}}}
\end{aligned}$$

The number of terms is finite. Thus, an expression of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots \cfrac{a_{n-1}}{b_{n-1} + \frac{a_n}{b_n}}}}} \tag{3.2}$$

is called a finite continued fraction, and a_n, b_n are called the nth partial numerator and the nth partial denominator, respectively, [12, p14]. As another example of a continued fraction consider the quadratic equation:

$$x^2 - 2x - 1 = 0$$

Then

$$x = 2 + \frac{1}{x}$$

On the right-hand side of the equation, x can be replaced by $2 + \frac{1}{x}$; this gives

$$\begin{aligned}
x &= 2 + \frac{1}{2 + \frac{1}{x}} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \ddots}}}
\end{aligned}$$

The number of the terms is infinite.

It is a convenience to write the continued fraction (3.1) using the notation:

$$b_0 + \frac{a_1}{b_1 +} \ \frac{a_2}{b_2 +} \ \frac{a_3}{b_3 + \cdots} \tag{3.3}$$

Note in the second example, if

$$C_n = b_0 + \frac{a_1}{b_1 +} \ \frac{a_2}{b_2 +} \cdots \frac{a_n}{b_n}$$

then

$$
\begin{aligned}
C_3 &= 2+\cfrac{1}{2+\frac{1}{2+\frac{1}{2}}} \doteq 2.41666 \\
C_4 &= 2+\cfrac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}} \doteq 2.41379 \\
C_5 &= 2+\cfrac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}} \doteq 2.41428
\end{aligned}
$$

When n is large enough, C_n is a very good approximation to the root of the quadratic equation which is $1+\sqrt{2}=2.414213562\cdots$.

Define the n-th convergent, f_n, of a continued fraction to be

$$
f_n = \frac{P_n}{Q_n} = b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots\frac{a_n}{b_n}}} \qquad n=0,1,2,3,\cdots
$$

Then with initial values

$$
\begin{pmatrix} P_0 & P_{-1} \\ Q_0 & Q_{-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix}
$$

it is easy to find that $\frac{P_1}{Q_1}=b_0+\frac{a_1}{b_1}=\frac{b_0b_1+a_1}{b_1}$. Hence, in matrix form

$$
\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} = \begin{pmatrix} b_0b_1+a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} P_0b_1+P_{-1}a_1 \\ Q_0b_1+Q_{-1}a_1 \end{pmatrix} = \begin{pmatrix} P_0 & P_{-1} \\ Q_0 & Q_{-1} \end{pmatrix}\begin{pmatrix} b_1 \\ a_1 \end{pmatrix}
$$

and

$$
\begin{aligned}
\frac{P_2}{Q_2} &= b_0+\frac{a_1}{b_1+\frac{a_2}{b_2}} \\
&= \frac{b_2(b_0b_1+a_1)+b_0a_2}{b_1b_2+a_2} \\
&= \frac{b_2P_1+b_0a_2}{b_1b_2+a_2}
\end{aligned}
$$

Hence,

$$
\begin{pmatrix} P_2 \\ Q_2 \end{pmatrix} = \begin{pmatrix} b_2P_1+a_2P_0 \\ b_2Q_1+a_2Q_0 \end{pmatrix} = \begin{pmatrix} P_1 & P_0 \\ Q_1 & Q_0 \end{pmatrix}\begin{pmatrix} b_2 \\ a_2 \end{pmatrix}
$$

and

$$
\begin{aligned}
\frac{P_3}{Q_3} &= b_0+\cfrac{a_1}{b_1+\frac{a_2}{b_2+\frac{a_3}{b_3}}} \\
&= \frac{b_3(b_0b_1b_2+b_0a_2+a_1b_2)+a_3(b_0b_1+a_1)}{b_3(b_1b_2+a_2)+b_1a_3} \\
&= \frac{b_3P_2+a_3P_1}{b_3Q_2+a_3Q_1}
\end{aligned}
$$

Hence,

$$\begin{pmatrix} P_3 \\ Q_3 \end{pmatrix} = \begin{pmatrix} b_3P_2 + a_3P_1 \\ b_3Q_2 + a_3Q_1 \end{pmatrix} = \begin{pmatrix} P_2 & P_1 \\ Q_2 & Q_1 \end{pmatrix} \begin{pmatrix} b_3 \\ a_3 \end{pmatrix}$$

It can be shown that P_n, Q_n satisfy the following recurrence relations:

$$\begin{aligned} P_n &= b_nP_{n-1} + a_nP_{n-2} \\ Q_n &= b_nQ_{n-1} + a_nQ_{n-2} \qquad n = 1, 2, 3\cdots \end{aligned} \tag{3.4}$$

which may be expressed in matrix notation as:

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} b_n \\ a_n \end{pmatrix}$$

We now state two important definitions in the area of continued fractions [5].

Definition 1 The continued fraction f is said to converge if the sequence of nth convergents f_n tends to a limit f. The limit f is then the value of the continued fraction.

Definition 2 Let two sequences P_n and Q_n satisfy the two-term recurrence relations

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} b_n \\ a_n \end{pmatrix}$$

with the initial values

$$\begin{pmatrix} P_0 & P_{-1} \\ Q_0 & Q_{-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then the continued fraction defined by the sequences a_n, b_n has nth convergent,

$$f_n = \frac{P_n}{Q_n}$$

. These definitions may be found in [12, p14].

With this forward recurrence scheme (3.4), the continued fraction can be evaluated. However, this method also has some drawbacks. This method may generate very small or very large values of P_n and Q_n, with the possibility of overflow of floating-point representation. Also, if any Q_n tends to zero, it will be a problem to compute P_n/Q_n.

In order to avoid the overflow problem, a very good algorithm which is called the modified Lentz's method is proposed for evaluating continued fractions. This method avoids the use P_n and Q_n explicitly by instead considering the ratios

$$C_j = \frac{P_j}{P_{j-1}} \quad , \quad D_j = \frac{Q_{j-1}}{Q_j}$$

and calculating f_j by

$$f_j = f_{j-1}C_jD_j$$

From the recurrence relation (3.4), it is easy to show that the ratios satisfy the following recurrence relations

$$D_j = \frac{1}{(b_j + a_j D_{j-1})} \quad , \quad C_j = b_j + \frac{a_j}{C_{j-1}}$$

The details of this method can be found in appendix (C).

Chapter 4

Evaluation of Bessel functions of fractional order

Many algorithms have been proposed for the evaluation of Bessel functions. For integer orders, as mentioned in the previous section, there are several good methods. In this section the algorithm which is 'state of the art' for approximating the fractional order Bessel functions will be described. This algorithm and its implementation in FORTRAN appears in Numerical Recipes [10, p236-239].A copy of this code appears in Appendix E.

This algorithm is based on the idea of Steed's method [5] which was developed for evaluating Coulomb wave functions. Instead of calculating $J_\nu, J'_\nu, Y_\nu, Y'_\nu$ by series expansion, the method uses two continued fractions CF1, CF2, and one Wronskian relation of Bessel functions (their definitions will be given later), which provides a very useful routine for computing Bessel functions J_ν, J'_ν, Y_ν, and Y'_ν simultaneously.

The first continued fraction, CF1, is defined by

$$\begin{aligned} f_\nu \equiv \frac{J'_\nu(x)}{J_\nu(x)} &= \frac{\nu}{x} - \frac{J_{\nu+1}(x)}{J_\nu(x)} \\ &= \frac{\nu}{x} - \frac{1}{2(\nu+1)/x -} \; \frac{1}{2(\nu+2)/x - \cdots} \end{aligned}$$

which can be derived from the Bessel function recurrence relations of (1.17) and (1.15)

$$J_{\nu+1}(x) = \frac{2\nu}{x} - J_{\nu-1}(x)$$

and

$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x)$$

From the second equation, dividing by $J_\nu(x)$, one finds

$$f_\nu \equiv \frac{J'_\nu(x)}{J_\nu(x)} = \frac{\nu}{x} - \frac{J_{\nu+1}(x)}{J_\nu(x)} \tag{4.1}$$

From the first equation, divided by J_ν, one has

$$\begin{aligned} \frac{J_{\nu-1}(x)}{J_\nu(x)} &= \frac{2\nu}{x} - \frac{J_{\nu+1}(x)}{J_\nu(x)} \\ \frac{J_\nu(x)}{J_{\nu-1}(x)} &= \frac{1}{\frac{2\nu}{x} - \frac{J_{\nu+1}(x)}{J_\nu(x)}} \end{aligned} \tag{4.2}$$

In the denominator of (4.2), using the relation of (4.2) again with ν replaced by $\nu+1$ yields

$$\frac{J_\nu(x)}{J_{\nu-1}(x)} = \frac{1}{\frac{2\nu}{x} - \frac{1}{\frac{2(\nu+1)}{x} - \frac{J_{\nu+2}(x)}{J_{\nu+1}(x)}}}$$

In the same way in (4.1), it can be easily shown that

$$\begin{aligned} f_\nu &\equiv \frac{J'_\nu(x)}{J_\nu(x)} = \frac{\nu}{x} - \frac{J_{\nu+1}(x)}{J_\nu(x)} \\ &= \frac{\nu}{x} - \frac{1}{\frac{2(\nu+1)}{x} - \frac{J_{\nu+2}(x)}{J_{\nu+1}(x)}} \\ &= \frac{\nu}{x} - \frac{1}{\frac{2(\nu+1)}{x} - \frac{1}{\frac{2(\nu+2)}{x} - \frac{J_{\nu+3}(x)}{J_{\nu+2}(x)}}} \\ &= \frac{\nu}{x} - \frac{1}{\frac{2(\nu+1)}{x} -} \, \frac{1}{\frac{2(\nu+2)}{x} -} \, \frac{1}{\frac{2(\nu+3)}{x} -} \cdots \end{aligned} \tag{4.3}$$

The other relations needed in Steed's method are the Wronskian relation

$$W \equiv J_\nu Y'_\nu - Y_\nu J'_\nu = \frac{2}{\pi x} \tag{4.4}$$

and a complex continued fraction of the two linearly independent solutions J_ν and Y_ν of the Bessel function of order ν, which is defined by

$$p + qi = \frac{J'_\nu + iY'_\nu}{J_\nu + iY_\nu} = -\frac{1}{2x} + i + \frac{i}{x} \, \frac{(\frac{1}{2})^2 - \nu^2}{2(x+i)+} \, \frac{(\frac{3}{2})^2 - \nu^2}{2(x+2i)+} \cdots \tag{4.5}$$

and denoted by CF2. The derivations of CF2 and Wronskian relation are given in Appendix (A) and (B).

In the algorithm, both continued fractions CF1 and CF2 are evaluated by the modified Lentz's method with double precision. The modified Lentz's method is described in Appendix (C).

A special point which is called the turning point $x_{tp} = \sqrt{\nu(\nu+1)}(\approx \nu$ for large$\nu)$ plays an important role in determining the interval of convergence for the continued fractions CF1 and CF2. The explanation of the importance of the quantity $\sqrt{\nu(\nu+1)}$ is as follows.

From (2.10), the differential equation satisfied by the spherical functions $j_\nu(x)$ and $y_\nu(x)$ is

$$U''_\nu(x) + \frac{2}{x}U'_\nu(x) + (1 - \frac{\nu(\nu+1)}{x^2})U_\nu(x) = 0$$

Making a transformation defined by $V_\nu(x) = xU_\nu(x)$, the differential equation will be reduced to the form

$$V''_\nu(x) + (1 - \frac{\nu(\nu+1)}{x^2})V_\nu(x) = 0 \tag{4.6}$$

When x is sufficiently large, the solutions of (4.6) are similar to $V_\nu''(x) + V_\nu(x) = 0$ whose general solution is $V_\nu(x) = A\cos x + B\sin x$. Hence, if $x^2 \geq \nu(\nu+1)$, or $x \geq x_{tp} = \sqrt{\nu(\nu+1)} \approx \nu$, Bessel's equation has oscillatory solutions. In the nth partial denominator of (4.3) $b_n = 2(\nu+n)/x$, so the series $\sum \mid b_n \mid$ is divergent as $n \to \infty$. A necessary condition for the continued fraction with the form

$$\cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \ddots}}}$$

to be convergent is that the infinite series $\sum \mid b_n \mid$ diverges ([12, p122]). For CF1 the convergence is rapid when $x \leq x_{tp}$. In the denominator of CF1, ν is always increased by one, so the algorithm uses iteration of the continued fraction to increase ν by one until $x \leq x_{tp}$.

Since CF2 does not converge as $x \to 0$, the algorithm is divided into two situations: one in which x is not too small, and the other case is for small x. The algorithm adopts a special method for small x , which will be described in Appendix(D).

In order to calculate the four functions $J_\nu, J_\nu', Y_\nu, Y_\nu'$, the routine uses (4.3), (4.4), and (4.5) to get four relations among these functions. Three relations come from CF1 and CF2, and the fourth one is provided by the Wronskian relation. The routine first evaluates the four functions at a lower value μ of ν, then calculates them at the original value of ν.

Now, assume that x is not too small and $x \geq x_{tp}$. Arbitrarily set the initial value of J_ν

$$J_\nu = \text{arbitrary} = k[J_\nu(x)], \quad [J_\nu(x)] = \text{ the true value of } J_\nu(x)$$

Then CF1 is determined by Lentz's method, f_ν is known. Set

$$J_\nu' = f_\nu J_\nu \qquad = k[J_\nu'(x)], \quad [J_\nu'(x)] = \text{the true value of } J_\nu'(x)$$

Then by using the recurrence relations (1.14) (1.15) downwards to a value $\nu = \mu \leq x$, one will find the ratio f_μ. The recurrence relations are

$$J_{\nu-1} = \frac{\nu}{x} J_\nu + J_\nu' \tag{4.7}$$

$$J_{\nu-1}' = \frac{\nu-1}{x} J_{\nu-1} - J_\nu \tag{4.8}$$

then

$$\begin{aligned} J_{\nu-2} &= \frac{\nu-1}{x} J_{\nu-1} + J_{\nu-1}' \\ J_{\nu-2}' &= \frac{\nu-2}{x} J_{\nu-2} - J_{\nu-1} \\ &\vdots \end{aligned}$$

down to a small order μ

$$\begin{cases} J_\mu & \approx & k \cdot [J_\mu(x)] \\ J'_\mu & \approx & k \cdot [J'_\mu(x)] \end{cases} \qquad k \text{ is a real number}$$

Thus,

$$f_\mu \approx \frac{J'_\mu}{J_\mu} \approx \frac{k[J'_\mu(x)]}{k[J_\mu(x)]} \approx \frac{[J'_\mu(x)]}{[J_\mu(x)]}$$

CF2 is evaluated at $\nu = \mu$ by the modified Lentz's method. Since CF2 is defined by $p + qi$, p and q are known. Also f_μ and the Wronskian relation $W = \frac{2}{\pi x}$, provide another two relations to solve for J_μ, J'_μ, Y_μ, and Y'_μ. In terms of the known quantities f_μ, p, q, W, it will be shown that

$$\begin{aligned} J_\mu &= \pm \{W/[q + \gamma(p - f_\mu)]\}^{1/2} \\ J'_\mu &= f_\mu J_\mu \\ Y_\mu &= \gamma J_\mu \\ Y'_\mu &= Y_\mu(p + q/\gamma) \end{aligned}$$

where γ is defined by

$$\gamma = \frac{p - f_\mu}{q}$$

The procedure is stated as follows.

Since

$$\begin{aligned} p + qi &= \frac{J'_\mu + iY'_\mu}{J_\mu + iY_\mu} \\ &= \frac{(J'_\mu + iY'_\mu)(J_\mu - iY_\mu)}{J_\mu^2 + Y_\mu^2} \\ &= \frac{J_\mu J'_\mu + Y_\mu Y'_\mu + i(J_\mu Y'_\mu - J'_\mu Y_\mu)}{J_\mu^2 + Y_\mu^2} \end{aligned}$$

Thus,

$$p = \frac{J'_\mu J_\mu + Y'_\mu Y_\mu}{J_\mu^2 + Y_\mu^2}$$

and using the definition of the Wronskian (4.4)

$$q = \frac{W}{J_\mu^2 + Y_\mu^2}$$

hence

$$p - f_\mu = \frac{J'_\mu J_\mu + Y'_\mu Y_\mu}{J_\mu^2 + Y_\mu^2} - \frac{J'_\mu}{J_\mu}$$

$$
\begin{aligned}
&= \frac{J'_\mu J_\mu^2 + Y'_\mu Y_\mu J_\mu - J'_\mu (J_\mu^2 + Y_\mu^2)}{J_\mu (J_\mu^2 + Y_\mu^2)} \\
&= \frac{Y_\mu (Y'_\mu J_\mu - J'_\mu Y_\mu)}{J_\mu (J_\mu^2 + Y_\mu^2)} \\
&= \frac{Y_\mu}{J_\mu} \cdot \frac{W}{(J_\mu^2 + Y_\mu^2)} \\
&= \frac{Y_\mu}{J_\mu} \cdot q
\end{aligned}
$$

Hence,

$$\gamma = \frac{p - f_\mu}{q} = \frac{Y_\mu}{J_\mu}$$

and

$$Y_\mu = \gamma J_\mu .$$

Since

$$
\begin{aligned}
p &= \frac{J_\mu J'_\mu + Y'_\mu Y_\mu}{J_\mu^2 + Y_\mu^2} \\
q &= \frac{J_\mu Y'_\mu - Y_\mu J'_\mu}{J_\mu^2 + Y_\mu^2},
\end{aligned}
$$

multiplying p by Y_μ, q by J_μ, and adding, one gets

$$
\begin{aligned}
p \cdot Y_\mu + q \cdot J_\mu &= \frac{(J_\mu J'_\mu + Y'_\mu Y_\mu) Y_\mu + (J_\mu Y'_\mu - Y_\mu J'_\mu) J_\mu}{J_\mu^2 + Y_\mu^2} \\
&= \frac{(Y_\mu^2 + J_\mu^2) Y'_\mu}{J_\mu^2 + Y_\mu^2} = Y'_\mu .
\end{aligned}
$$

So

$$
\begin{aligned}
Y'_\mu &= p Y_\mu + q J_\mu \\
&= Y_\mu \left(p + q \frac{J_\mu}{Y_\mu}\right)
\end{aligned}
$$

since $\gamma = \frac{Y_\mu}{J_\mu}$, $\qquad = Y_\mu (p + q/\gamma).$

Using the definition of f_ν, we now have the following three relationships,

$$
\begin{aligned}
J'_\mu &= f_\mu J_\mu && (4.9) \\
Y_\mu &= \gamma J_\mu && (4.10) \\
Y'_\mu &= Y_\mu (p + q/\gamma) && (4.11)
\end{aligned}
$$

Substituting these results into the Wronskian relation

$$\begin{aligned} W &= J_\mu Y_\mu' - Y_\mu J_\mu' \\ &= J_\mu(Y_\mu(p+q/\gamma)) - (\gamma J_\mu)(f_\mu J_\mu) \\ &= J_\mu(\gamma J_\mu)(p+q/\gamma) - \gamma f_\mu J_\mu^2 \\ &= J_\mu^2(\gamma p + q - \gamma f_\mu) \\ &= J_\mu^2(q + \gamma(p - f_\mu)) \end{aligned}$$

we find that

$$J_\mu = \pm\left\{W/[q+\gamma(p-f_\mu)]\right\}^{1/2}. \tag{4.12}$$

The sign of J_μ is chosen to be the same as the sign of J_ν in the initial value.

After having all the relation formulas for J_μ, Y_μ, J_μ', and Y_μ', one must go back to compute the four quantities at the original values of order ν. In the FORTRAN code this is accomplished by first storing the initial values J_ν and J_ν' in $rjl1$ and $rjp1$, respectively , i.e,

$$\begin{aligned} rjl1 = J_\nu &= k[J_\nu(x)] \\ rjp1 = J_\nu' &= k[J_\nu'(x)] \end{aligned}$$

Then the original values of J_ν and J_ν' are scaled by taking the ratio of (4.12) to rjl which is the value found after using the recurrence relation, i.e, $rjl = J_\mu = k[J_\mu(x)]$. Then, let

$$fact = \frac{[J_\mu(x)]}{k[J_\mu(x)]} = \frac{1}{k}$$

The original values of $J_\nu(x)$ and $J_\nu'(x)$ are

$$\begin{aligned} fact * rjl1 &= \frac{1}{k}\cdot k[J_\nu(x)] = [J_\nu(x)] \\ fact * rjp1 &= \frac{1}{k}\cdot k[J_\nu'(x)] = [J_\nu'(x)] \end{aligned}$$

With (4.10) and (4.11) as the initial values, one may find the original values of Y_ν and Y_ν' by using the stable recurrence,

$$Y_{\mu+1} = \frac{2\mu}{x}Y_\nu - Y_{\mu-1} \tag{4.13}$$

$$Y_\mu' = \frac{\mu}{x}Y_\mu - Y_{\mu+1} \tag{4.14}$$

However, when x is close to zero, CF2 is not suitable. There is a good method provided by Temme [11] to handle this case. It is a complicated method but it deals with the problem well as $x \to 0$. It uses the series expansions for evaluating Y_ν and $Y_{\nu+1}'$, and hence one can get Y_ν' from (4.14). But the series expansions work only for $|\ \nu\ | \leq 1/2$. By using the recurrences (4.7) and (4.8), one finds the value of f_ν at

$\nu = \mu$ in this interval. J_μ can be calculated from the Wronskian relation. It is easy to see that

$$\begin{aligned} W &= J_\mu Y'_\mu - J'_\mu Y_\mu \\ &= J_\mu [Y'_\mu - \frac{J'_\mu}{J_\mu} Y_\mu] \\ &= J_\mu [Y'_\mu - f_\mu Y_\mu] \end{aligned}$$

Hence,

$$J_\mu = \frac{W}{Y'_\mu - f_\mu Y_\mu}$$

and by the definition of $f_\mu = \frac{J'_\mu}{J_\mu}$, we know $J'_\mu = f_\mu J_\mu$. Just like the method used before, the original values for the quantities at the order ν can be determined by scaling.

The only thing left to explain is Temme's series. They are

$$\begin{aligned} Y_\nu &= -\sum_{m=0}^{\infty} c_m g_m \\ Y_{\nu+1} &= -\frac{2}{x}\sum_{m=0}^{\infty} c_m h_m \qquad \text{where } c_m = \frac{(-x^2/4)^m}{m!} \end{aligned}$$

the coefficients g_m and h_m are defined in terms of quantities p_m, q_m, and f_m which will be shown in [10] (see §6.7), and they are

$$\begin{aligned} g_m &= f_m + \frac{2}{x}\sin^2(\frac{\nu\pi}{2}) q_m \\ h_m &= -m g_m + p_m \\ p_m &= \frac{p_{m-1}}{m-\nu} \\ q_m &= \frac{q_{m-1}}{m+\nu} \\ f_m &= \frac{m f_{m-1} + p_{m-1} + q_{m-1}}{m^2 - \nu^2} \end{aligned}$$

The initial values for these recurrences are

$$\begin{aligned} p_0 &= \frac{1}{\pi}(\frac{x}{2})^{-\nu}\Gamma(1+\nu) \\ q_0 &= \frac{1}{\pi}(\frac{x}{2})^{\nu}\Gamma(1-\nu) \\ f_0 &= \frac{2}{\pi}\frac{\nu\pi}{\sin\nu\pi}[\cos\mu\Gamma_1(\nu) + \frac{\sinh mu}{\mu}\ln(\frac{2}{x})\Gamma_2(\nu)] \end{aligned}$$

with

$$
\begin{aligned}
\mu &= \nu \ln(\frac{2}{x}) \\
\Gamma_1(\nu) &= \frac{1}{2\nu}[\frac{1}{\Gamma(1-\nu)} - \frac{1}{\Gamma(1+\nu)}] \\
\Gamma_2(\nu) &= \frac{1}{2}[\frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(1+\nu)}]
\end{aligned}
$$

The derivation of this representation is the subject of appendix (D).

Appendix A: The Confluent Hypergeometric Function

In this section, we discuss a special function called the confluent hypergeometric function. By making suitable choices of the parameters in the confluent hypergeometric function, many special functions can be expressed in a simple and compact form. In particular this function is useful in the derivation of the continued fraction, CF2.

The confluent hypergeometric function is defined by ([8] §9.9)

$$\Phi(\alpha,\gamma;z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m}{(\gamma)_m m!} z^m \qquad |\, z \,| < \infty,\ \gamma \neq 0, -1, -2, \cdots$$

Here z can be a complex variable, α and γ are arbitrary real or complex parameters, and $(\alpha)_m$ is an abbreviation.

$$\begin{aligned} (\alpha)_m &= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m) \qquad m = 1, 2, 3, \cdots \\ &= \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)} \\ (\alpha)_0 &= 1 \end{aligned}$$

This function, $\Phi(\alpha,\gamma;z)$, is a particular solution of the linear differential equation

$$zy'' + (\gamma - z)y' - \alpha y = 0 \tag{A.1}$$

for details see [3, p388]. Similar to Bessel's equation, there is a second linearly independent solution of (A.1), called the confluent hypergeometric function of the second kind, which is defined by

$$\begin{aligned} U(\alpha,\gamma;z) &= \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\Phi(\alpha,\gamma;z) + \\ &\quad \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma}\Phi(1+\alpha-\gamma, 2-\gamma; z) \qquad \gamma \neq 0, \pm 1, \pm 2, \cdots \end{aligned}$$

Like Bessel functions, $U(\alpha,\gamma;z)$ also has some useful recurrence formulas such as

$$\frac{d}{dz}U(\alpha,\gamma;z) = -\alpha U(1+\alpha, 1+\gamma; z) \tag{A.2}$$

$$zU(\alpha,\gamma+1;z) = U(\alpha-1,\gamma;z) + (\gamma-\alpha)U(\alpha,\gamma;z) \tag{A.3}$$

$$\begin{aligned} U(\alpha-1,\gamma;z) &= (z+2\alpha-\gamma)U(\alpha,\gamma;z) + \\ &\quad \alpha(\gamma-\alpha-1)U(\alpha+1,\gamma;z) \end{aligned} \tag{A.4}$$

Now, we start with the Hankel function and the confluent hypergeometric function to discuss the derivation for the expression of CF2.

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \qquad \text{where } \nu \text{ is not an integer}$$

There is relation between $H_\nu^{(1)}(x)$ and the confluent hypergeometric (see [8, p274]), which is

$$\begin{aligned} H_\nu^{(1)}(x) &= \frac{-2i}{\sqrt{\pi}} e^{i(x-\nu\pi)}(2x)^\nu U(\nu+\frac{1}{2}, 2\nu+1; 2xe^{-\pi i/2}) \\ &= \frac{-2i}{\sqrt{\pi}} e^{i(x-\nu\pi)}(2x)^\nu U(\nu+\frac{1}{2}, 2\nu+1; -2xi) \end{aligned}$$

Let $z_n(x) = U(\nu+\frac{1}{2}+n, 2\nu+1; -2xi)$; then $z_0(x) = U(\nu+\frac{1}{2}, 2\nu+1; -2xi)$ and $z_1(x) = U(\nu+\frac{1}{2}+1, 2\nu+1; -2xi)$. By the differential formula of $U(\alpha,\gamma;z)$ (A.2), one has

$$\begin{aligned} \frac{d}{dx} H_\nu^{(1)}(x) &= \frac{d}{dx}[-\frac{2i}{\sqrt{\pi}} e^{i(x-\nu\pi)}(2x)^\nu U(\nu+1/2, 2\nu+1; -2xi)] \\ &= -\frac{2i}{\sqrt{\pi}}[e^{i(x-\nu\pi)} i(2x)^\nu U(\nu+1/2, 2\nu+1; -2xi) \\ &\quad + e^{i(x-\nu\pi)} 2\nu(2x)^{\nu-1} U(\nu+1/2, 2\nu+1; -2xi) \\ &\quad + e^{i(x-\nu\pi)}(2x)^\nu \frac{d}{dx} U(\nu+1/2, 2\nu+1; -2xi)] \\ &= \frac{2i}{\sqrt{\pi}} e^{i(x-\nu\pi)}(2x)^\nu [(i+\frac{2\nu}{2x}) z_0(x) + \frac{d}{dx} U(\nu+1/2, \\ &\quad 2\nu+1; -2xi)] \end{aligned}$$

Then, let $z = -2xi$

$$\begin{aligned} \frac{d}{dx} U(\nu+1/2, 2\nu+1; -2xi) &= \frac{dz}{dx}\frac{d}{dz} U(\nu+1/2, 2\nu+1; -2xi) \\ &= (-2i)\frac{d}{dz} U(\nu+1/2, 2\nu+1; -2xi) \\ &= 2i(\nu+1/2) U(\nu+1+1/2, 2\nu+1+1; \\ &\quad -2xi) \end{aligned}$$

where, by (A.3) the above equation, $\frac{d}{dx}U(\nu+\frac{1}{2}, 2\nu+1; -2xi)$ becomes

$$\begin{aligned} &\frac{d}{dx} U(\nu+1+\frac{1}{2}, 2\nu+1; -2xi) \\ &= \frac{2i(\nu+\frac{1}{2})}{-2xi}(-2xi) U(\nu+\frac{1}{2}+1, (2\nu+1)+1; -2xi) \\ &= \frac{2i(\nu+\frac{1}{2})}{-2xi}[U(\nu+\frac{1}{2}, 2\nu+1; -2xi) \\ &\quad + ((2\nu+1)-(\nu+\frac{1}{2}+1)) U(\nu+\frac{1}{2}+1, 2\nu+1; -2xi)] \\ &= \frac{2i(\nu+\frac{1}{2})}{-2xi}\left\{ z_0(x) + (\nu-\frac{1}{2}) z_1(x) \right\} \end{aligned}$$

Thus,

$$\begin{aligned}\frac{d}{dx}H_\nu^{(1)}(x) &= \frac{-2i}{\sqrt{\pi}}e^{i(x-\nu\pi)}(2x)^\nu\left\{(i+\frac{2\nu}{2x})z_0(x)\right.\\&\left.+\frac{2i(\nu+\frac{1}{2})}{-2xi}z_0(x)+\frac{2i(\nu+\frac{1}{2})(\nu-\frac{1}{2})}{-2xi}z_1(x)\right\}\end{aligned}$$

and

$$\begin{aligned}\frac{\frac{d}{dx}H_\nu^{(1)}(x)}{H_\nu^{(1)}} &= (i+\frac{\nu}{x})-\frac{(\nu+\frac{1}{2})}{x}+\frac{((\frac{1}{2})^2-\nu^2)}{x}\frac{z_1(x)}{z_0(x)}\\&= -\frac{1}{2x}+i+\frac{((\frac{1}{2})^2-\nu^2)}{x}\frac{z_1(x)}{z_0(x)}\end{aligned}\tag{A.5}$$

Next, one needs to know $z_1(x)/z_0(x)$. By using a process similar to CF1 and (A.4), one finds

$$\begin{aligned}z_{n-1}(x) &= U(\nu+\frac{1}{2}+(n-1),2\nu+1;-2xi)\\&= [-2xi+2(\nu+\frac{1}{2}+n)-(2\nu+1)]U(\nu+\frac{1}{2}+n,2\nu+1;\\&\quad -2xi)+(\nu+\frac{1}{2}+n)(2\nu+1-(\nu+\frac{1}{2}+n)-1)U(\nu+\frac{1}{2}+n\\&\quad +1,2\nu+1;-2xi)\\&= (-2xi+2n)U(\nu+\frac{1}{2}+n,2\nu+1;-2xi)+(\nu+\frac{1}{2}+n)\cdot\\&\quad (\nu-\frac{1}{2}-n)U(\nu+\frac{1}{2}+n+1,2\nu+1;-2xi)\\&= b_nz_n(x)+a_{n+1}z_{n+1}(x)\end{aligned}$$

where

$$\begin{aligned}b_n &= 2(-xi+n)=\frac{2}{i}(x+ni)\\a_{n+1} &= (\nu^2-(n+\frac{1}{2})^2)\end{aligned}$$

Therefore,

$$\begin{aligned}z_{n-1}(x) &= b_nz_n(x)+a_{n+1}z_{n+1}\\\frac{z_{n-1}(x)}{z_n(x)} &= b_n+a_n\frac{z_{n+1}(x)}{z_n(x)}\\\frac{z_n(x)}{z_{n-1}(x)} &= \frac{1}{b_n+a_{n+1}\frac{z_{n+1}(x)}{z_n(x)}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b_n + \frac{a_{n+1}}{b_{n+1} + a_{n+2}\frac{z_{n+2}(x)}{z_{n+1}(x)}}} \\
&\ \vdots \\
&= \frac{1}{b_n+}\,\frac{a_{n+1}}{b_{n+1}+}\,\frac{a_{n+2}}{b_{n+2}+}\,\frac{a_{n+3}}{b_{n+3}+\cdots}
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{z_1(x)}{z_0(x)} &= \frac{1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots \\
&= \frac{1}{\frac{2}{i}(x+i) + \frac{\nu^2-(\frac{3}{2})^2}{\frac{2}{i}(x+2i) + \frac{\nu^2-(\frac{5}{2})^2}{\frac{2}{i}(x+3i)+\cdots}}} \\
&= \frac{i}{2(x+i) + \frac{(\frac{3}{2})^2-\nu^2}{2(x+2i) + \frac{(\frac{5}{2})^2-\nu^2}{2(x+3i)+\cdots}}} \qquad \text{(A.6)}
\end{aligned}
$$

From (A.5) and (A.6), one has

$$
\begin{aligned}
p + qi &= \frac{\frac{d}{dx}H_\nu^{(1)}(x)}{H_\nu^{(1)}} = \frac{J_\nu'(x) + iY_\nu'(x)}{J_\nu(x) + iY_\nu(x)} \\
&= \frac{-1}{2x} + i + \frac{(\frac{1}{2})^2 - \nu^2}{x}\,\frac{z_1(x)}{z_0(x)} \\
&= \frac{-1}{2x} + i + \frac{i}{x}\,\frac{((\frac{1}{2})^2 - \nu^2)}{2(x+i)+}\,\frac{((\frac{3}{2})^2 - \nu^2)}{2(x+2i)+}\,\frac{((\frac{5}{2})^2 - \nu^2)}{2(x+3i)+\cdots}
\end{aligned}
$$

Appendix B: Wronskian Relation

In this appendix, we will discuss the derivation of the Wronskian relation of Bessel functions. Suppose u_1, u_2 are two solutions of Bessel's equation:

$$y'' + \frac{1}{x}y' + (1 - (\frac{\nu}{x})^2)y = 0 \quad x \neq 0$$

Then

$$u_1'' + \frac{1}{x}u_1' + (1 - (\frac{\nu}{x})^2)u_1 = 0$$
$$u_2'' + \frac{1}{x}u_2' + (1 - (\frac{\nu}{x})^2)u_2 = 0$$

Multipling the first equation by $-u_2$, the second by u_1, and adding yields

$$\begin{aligned} u_2''u_1 + \frac{1}{x}u_2'u_1 - u_1''u_2 - \frac{1}{x}u_1'u_2 &= 0 \\ (u_2''u_1 - u_1''u_2) + \frac{1}{x}(u_2'u_1 - u_1'u_2) &= 0 \qquad \text{(B.1)} \end{aligned}$$

Let

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1u_2' - u_2u_1'$$

Then (B.1) becomes

$$\begin{vmatrix} u_1 & u_2 \\ u_1'' & u_2'' \end{vmatrix} + \frac{1}{x}\begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = 0$$

That is

$$\frac{dW}{dx} + \frac{1}{x}W = 0$$
$$W = C\,exp[-\int^x \frac{1}{t}dt]$$

This is known as Abel's identity

$$C = W\,exp[\int^x \frac{1}{t}dt]$$

where C is a constant independent of x. If one chooses $u_1 = J_\nu(x)$ and $u_2 = J_{-\nu}(x)$, where ν is not an integer, then

$$C = xW\{J_\nu(x), J_{-\nu}(x)\}$$

With the series expressions of $J_\nu(x)$ and $J_{-\nu}(x)$

$$
\begin{aligned}
J_\nu(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(\nu+m+1)}(\frac{x}{2})^{2m+\nu} \\
J_{-\nu}(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(-\nu+m+1)}(\frac{x}{2})^{2m-\nu}
\end{aligned}
$$

it is clear that

$$
\begin{aligned}
J_\nu(x) &= \frac{1}{\Gamma(1+\nu)}(\frac{x}{2})^\nu + \frac{-1}{\Gamma(2-\nu)}(\frac{x}{2})^{2+\nu} + \cdots \\
J_{-\nu}(x) &= \frac{(1)}{\Gamma(1-\nu)}(\frac{x}{2})^{-\nu} + \frac{-1}{\Gamma(2-\nu)}(\frac{x}{2})^{2-\nu} + \cdots \\
J'_\nu(x) &= \frac{\nu/2}{\Gamma(1+\nu)}(\frac{x}{2})^{\nu-1} + \frac{-(1+\nu/2)}{\Gamma(2+\nu)}(\frac{x}{2})^{1+\nu} + \cdots \\
J'_{-\nu}(x) &= \frac{-\nu/2}{\Gamma(1-\nu)}(\frac{x}{2})^{-\nu-1} + \frac{-1+\nu/2}{\Gamma(2-\nu)}(\frac{x}{2})^{1-\nu} \cdots
\end{aligned}
$$

Then one may find a value with a simple form for C:

$$
\begin{aligned}
& xW\{J_\nu(x), J_{-\nu}(x)\} \\
= \; & x(J_\nu(x)J'_{-\nu}(x) - J_{-\nu}(x)J'_\nu(x)) \\
= \; & \frac{-2\nu}{\Gamma(\nu+1)\Gamma(-\nu+1)}[1+O(x)^2]
\end{aligned}
$$

Since C is a parameter that depends on the particular Bessel functions u_1 and u_2, and C is independent of x, it may be identified at any convenient point such as $x = 0$, then

$$
\begin{aligned}
C &= \lim_{x\to 0^+} xW\{J_\nu(x), J_{-\nu}(x)\} \\
&= \frac{-2\nu}{\Gamma(1+\nu)\Gamma(1-\nu)}
\end{aligned}
$$

Next, from the Gamma function formulas (see[3]):

$$
\begin{aligned}
\Gamma(x+1) &= x\Gamma(x) \\
\Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin \pi x}
\end{aligned}
$$

it is easy to see that

$$
\Gamma(1+\nu)\Gamma(1-\nu) = \nu\Gamma(\nu) \cdot \frac{\pi}{\Gamma(\nu)\sin \pi\nu} = \frac{\nu\pi}{\sin \pi\nu}
$$

Thus,

$$C = \frac{-2\nu}{\Gamma(1+\nu)\Gamma(1-\nu)} = \frac{-2\nu\sin\pi\nu}{\nu\pi} = \frac{-2\sin\nu\pi}{\pi}$$

which implies

$$W\left\{J_\nu(x), J_{-\nu}(x)\right\} = -\frac{2\sin\pi\nu}{\pi x}$$

$J_\nu(x)$ and $Y_\nu(x)$ is another pair of solutions of Bessel's equation, and

$$\begin{aligned} Y_\nu(x) &= \frac{J_\nu(x)\cos\nu\pi - J_{-\nu}(x)}{\sin\nu\pi} \\ Y'_\nu(x) &= \frac{J'_\nu(x)\cos\nu\pi - J'_{-\nu}(x)}{\sin\nu\pi} \end{aligned}$$

Hence,

$$\begin{aligned} W\left\{J_\nu(x), Y_\nu(x)\right\} &= J_\nu Y'_\nu - Y_\nu J'_\nu \\ &= \frac{J_\nu(x)J'_\nu(x)\cos\nu\pi - J_\nu(x)J'_{-\nu}(x)}{\sin\nu\pi} \\ &\quad - \frac{J'_\nu(x)J_\nu(x)\cos\nu\pi - J'_\nu(x)J_{-\nu}(x)}{\sin\nu\pi} \\ &= \frac{J'_\nu(x)J_{-\nu}(x) - J_\nu(x)J'_{-\nu}(x)}{\sin\nu\pi} \\ &= \frac{-W\left\{J_\nu(x), J_{-\nu}(x)\right\}}{\sin\nu\pi} \\ &= \frac{-2\sin\nu\pi}{\pi x\sin\nu\pi} \\ &= \frac{2}{\pi x} \end{aligned}$$

So one has

$$W \equiv J_\nu Y'_\nu - Y_\nu J'_\nu = \frac{2}{\pi x}$$

Appendix C: The Modified Lentz's Method

There are several general methods for computing continued fractions, but the best method seems to be the Modified Lentz's method. It uses two ratios defined by

$$C_j = \frac{P_j}{P_{j-1}}, \quad \text{and} \quad D_j = \frac{Q_{j-1}}{Q_j}$$

to calculate the continued fraction f_j:

$$\begin{aligned} f_j &= \frac{P_j}{Q_j} \\ &= \frac{P_j}{P_{j-1}} \frac{P_{j-1}}{Q_{j-1}} \frac{Q_{j-1}}{Q_j} \\ &= f_{j-1} C_j D_j \end{aligned}$$

From (3.4), one knows that P_j and Q_j satisfy the following recurrence relations

$$\begin{aligned} P_j &= P_{j-1} b_j + P_{j-2} a_j \\ Q_j &= Q_{j-1} b_j + Q_{j-2} a_j \qquad\qquad j = 1, 2, 3, \cdots \end{aligned}$$

Dividing by P_{j-1} and Q_{j-1} respectively, one finds

$$\begin{aligned} \frac{P_j}{P_{j-1}} &= b_j + \frac{P_{j-2}}{P_{j-1}} a_j \\ \frac{Q_j}{Q_{j-1}} &= b_j + \frac{Q_{j-2}}{Q_{j-1}} a_j \end{aligned}$$

ıce,

$$C_j = b_j + a_j / C_{j-1} \quad \text{and} \quad D_j = \frac{1}{b_j + D_{j-1} a_j}$$

ever, there is still a problem for these two equations; the denominator of D_j or uantity C_j might approach zero. To avoid division by zero, one can shift them very small amount, e.g., 10^{-50} if necessary. This yields a very accurate result $_{+1}$. The details of this algorithm can be found in [10].

Appendix D: Temme's Series

In Chapter 4, the analysis based on the use of CF2 is not appropriate when x is small. For this case, Temme [11] provides a very good method for evaluating the four quantities J_ν, J_ν', Y_ν, and Y_ν', which uses series expansions to handle the singularity as $x \to 0$. The goal of this appendix is to explain Temme's series.

From the series expansion for J_ν and the definition of Y_ν (see section 1.2), one may find that

$$\begin{aligned}
Y_\nu(x) &= \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \\
&= \frac{(1-2\sin^2\frac{\nu\pi}{2})J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi} \\
&= \frac{1}{\sin\nu\pi}[(J_\nu(x) - J_{-\nu}(x)) - 2\sin^2\frac{\nu\pi}{2}J_\nu(x)] \\
&= \frac{-\nu}{\sin\nu\pi}[\frac{1}{\nu}(J_{-\nu}(x) - J_\nu(x)) + \frac{2}{\nu}\sin^2\frac{\nu\pi}{2}J_\nu(x)] \\
&= \frac{-\nu}{\sin\nu\pi}\left\{\frac{1}{\nu}\left[\sum_{m=0}^{\infty}\frac{(\frac{x}{2})^{-\nu}(-\frac{x^2}{4})^m}{m!\Gamma(m-\nu+1)} - \sum_{m=0}^{\infty}\frac{(\frac{x}{2})^{\nu}(-\frac{x^2}{4})^m}{m!\Gamma(m+\nu+1)}\right]\right. \\
&\qquad \left. + \frac{2}{\nu}\sin^2\frac{\nu\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{x}{2})^{\nu}(-\frac{x^2}{4})^m}{m!\Gamma(m-\nu+1)}\right\} \\
&= -\sum_{m=0}^{\infty}\frac{(-\frac{x^2}{4})^m}{m!}\left\{\frac{1}{\nu}\left[\frac{\nu}{\sin\nu\pi}\frac{(\frac{x}{2})^{-\nu}}{\Gamma(m-\nu+1)} - \frac{\nu}{\sin\nu\pi}\frac{(\frac{x}{2})^{\nu}}{\Gamma(m+\nu+1)}\right]\right. \\
&\qquad \left. + \frac{2}{\nu}\sin^2(\frac{\nu\pi}{2})\cdot\frac{\nu}{\sin\nu\pi}\frac{(\frac{x}{2})^{\nu}}{\Gamma(m+\nu+1)}\right\}
\end{aligned}$$

t

$$\begin{aligned}
C_m &= \frac{(-\frac{x^2}{4})^m}{m!} &\text{(D.1)}\\
p_m &= \frac{\nu}{\sin\nu\pi}\cdot\frac{(\frac{x}{2})^{-\nu}}{\Gamma(m-\nu+1)} &\text{(D.2)}\\
q_m &= \frac{\nu}{\sin\nu\pi}\cdot\frac{(\frac{x}{2})^{\nu}}{\Gamma(m+\nu+1)} &\text{(D.3)}\\
f_m &= (p_m - q_m)/\nu &\text{(D.4)}\\
g_m &= f_m + \frac{2}{\nu}\sin^2(\frac{\nu\pi}{2})q_m \\
h_m &= -mg_m + p_m
\end{aligned}$$

Then

$$Y_\nu(x) = -\sum_{m=0}^{\infty}C_m(f_m + \frac{2}{\nu}\sin^2(\frac{\nu\pi}{2})q_m) = -\sum_{m=0}^{\infty}C_m g_m$$

One may have an expression for $Y_{\nu+1}(x)$ with a similar process which replaces ν by $\nu+1$ in the series expansion of $Y_\nu(x)$ and uses the recurrence formula (1.17) such that

$$
\begin{aligned}
Y_{\nu+1}(x) &= \frac{1}{\sin(\nu+1)\pi}[\cos(\nu+1)\pi J_{\nu+1}(x) - J_{-(\nu+1)}(x)] \\
&= \frac{1}{\sin(\nu+1)\pi}[-\cos\nu\pi J_{\nu+1} + (\frac{2\nu}{x}J_{-\nu} - J_{-\nu+1})]
\end{aligned}
$$

And substituting the series expansion for the Bessel functions will yield

$$
Y_{\nu+1}(x) = -\frac{2}{x}\sum_{m=0}^{\infty} C_m h_m
$$

In the formulas (D.2), (D.3), it is easy to find the recurrence relations:

$$
p_m = p_{m-1}/(m-\nu), \qquad q_m = q_{m-1}/(m+\nu)
$$

Then (D.4) becomes

$$
f_m = (mf_{m-1} + p_{m-1} + q_{m-1})/(m^2 - \nu^2)
$$

From the relations of gamma function it is known that

$$
\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}
$$

and

$$
\Gamma(1+x) = x\Gamma(x).
$$

then

$$
\Gamma(1-x) = \frac{\pi}{\sin\pi x} \times \frac{x}{\Gamma(1+x)}
$$

Since

$$
p_m = \frac{\nu}{\sin\nu\pi}\frac{(\frac{x}{2})^{-\nu}}{\Gamma(m-\nu+1)}
$$

one finds

$$
\begin{aligned}
p_0 &= \frac{\nu}{\sin\nu\pi}\frac{(\frac{x}{2})^{-\nu}}{\Gamma(-\nu+1)} \\
&= \frac{1}{\pi}(\frac{x}{2})^{-\nu}\Gamma(1+\nu)
\end{aligned}
$$

Similarly, since

$$
q_m = \frac{\nu}{\sin\nu\pi}\frac{(\frac{x}{2})^{\nu}}{\Gamma(m+\nu+1)}
$$

then

$$
\begin{aligned}
q_0 &= \frac{\nu}{\sin \nu\pi} \frac{(\frac{x}{2})^{\nu}}{\Gamma(\nu+1)} \\
&= \frac{1}{\pi}(\frac{x}{2})^{\nu}\Gamma(1-\nu)
\end{aligned}
$$

Next, one needs to know the expression for f_0. Since

$$
\begin{aligned}
f_m &= (p_m - q_m)/\nu \\
&= \frac{\nu}{\sin \nu\pi}[\frac{1}{\nu}(\frac{(\frac{x}{2})^{-\nu}}{\Gamma(m-\nu+1)} - \frac{(\frac{x}{2})^{\nu}}{\Gamma(m+\nu+1)}]
\end{aligned}
$$

then

$$
\begin{aligned}
f_0 &= \frac{\nu}{\sin \nu\pi}\left[\frac{1}{\nu}\frac{(\frac{x}{2})^{-\nu}}{\Gamma(1-\nu)} - \frac{1}{\nu}\frac{(\frac{x}{2})^{\nu}}{\Gamma(1+\nu)}\right] \\
&= \frac{\nu}{\sin \nu\pi}\left[\frac{1}{\nu}\frac{(\frac{2}{x})^{\nu}}{\Gamma(1-\nu)} - \frac{1}{\nu}\frac{(\frac{2}{x})^{-\nu}}{\Gamma(1+\nu)}\right] \\
&= \frac{\nu}{\sin \nu\pi}\left[\frac{1}{\nu\Gamma(1-\nu)}\left(\frac{(\frac{2}{x})^{\nu}+(\frac{2}{x})^{-\nu}}{2} + \frac{(\frac{2}{x})^{\nu}-(\frac{2}{x})^{-\nu}}{2}\right)\right. \\
&\quad \left. - \frac{1}{\nu\Gamma(1+\nu)}\left(\frac{(\frac{2}{x})^{\nu}+(\frac{2}{x})^{-\nu}}{2} - \frac{(\frac{2}{x})^{\nu}-(\frac{2}{x})^{-\nu}}{2}\right)\right] \\
&= \frac{1}{\pi}\frac{\pi\nu}{\sin \nu\pi}\left\{\frac{1}{\nu}\left[(\frac{(\frac{2}{x})^{\nu}+(\frac{2}{x})^{-\nu}}{2})(\frac{1}{\Gamma(1-\nu)} - \frac{1}{\Gamma(1+\nu)})\right]\right. \\
&\quad \left. + \frac{1}{\nu}\left[\left(\frac{(\frac{2}{x})^{\nu}-(\frac{2}{x})^{-\nu}}{2}\right)\left(\frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(1+\nu)}\right)\right]\right\} \\
&= \frac{2}{\pi}\frac{(\nu\pi)}{\sin \nu\pi}\left\{\frac{1}{2\nu}(\cosh\mu)\left(\frac{1}{\Gamma(1-\nu)} - \frac{1}{\Gamma(1+\nu)}\right)\right. \\
&\quad \left. + \frac{1}{2\nu}(\sinh\mu)\left(\frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(1+\nu)}\right)\right\} \\
&= \frac{2}{\pi}\frac{(\nu\pi)}{\sin \nu\pi}\left\{(\cosh\mu)\left[\frac{1}{2\nu}\left(\frac{1}{\Gamma(1-\nu)} - \frac{1}{\Gamma(1+\nu)}\right)\right]\right. \\
&\quad \left. + \frac{\sinh\mu}{\nu}\left[\frac{1}{2}\left(\frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(1+\nu)}\right)\right]\right\}
\end{aligned}
$$

wh $= \nu\ln(\frac{2}{x})$ and hence

$$
\begin{aligned}
\cosh\mu &= \frac{1}{2}(e^{\mu}+e^{-\mu}) = \frac{1}{2}((\frac{2}{x})^{\nu} + (\frac{2}{x})^{-\nu}) \\
\sinh\mu &= \frac{1}{2}(e^{\mu}-e^{-\mu}) = \frac{1}{2}((\frac{2}{x})^{\nu} - (\frac{2}{x})^{-\nu})
\end{aligned}
$$

Let

$$\begin{aligned}\Gamma_1(\nu) &= \frac{1}{2\nu}\left[\frac{1}{\Gamma(1-\nu)} - \frac{1}{\Gamma(1+\nu)}\right]\\ \Gamma_2(\nu) &= \frac{1}{2}\left[\frac{1}{\Gamma(1-\nu)} + \frac{1}{\Gamma(1+\nu)}\right]\end{aligned}$$

Hence,

$$f_0 = \frac{2}{\pi}\frac{(\nu\pi)}{\sin\nu\pi}\left\{(\cosh\mu)\Gamma_1(\nu) + \frac{\sinh\mu}{\mu}\ln\left(\frac{2}{x}\right)\Gamma_2(\nu)\right\}$$

The advantage of writing f_0 in such complicated form is that as $\nu \to 0, f_0$ can be controlled by evaluating $\nu\pi/\sin(\nu\pi), \sinh\mu/\mu$, and Γ_1 (see [10], §6.7).

Appendix (E)

FORTRAN routines found in Numerical Recipes for the evaluation of Bessel functions of fractional order

```
      SUBROUTINE bessjy(x,xnu,rj,ry,rjp,ryp)
      INTEGER MAXIT
      REAL rj,rjp,ry,ryp,x,xnu,XMIN
      DOUBLE PRECISION EPS,FPMIN,PI
      PARAMETER (EPS=1.e-10,FPMIN=1.e-30,MAXIT=10000,XMIN=2.,
     *     PI=3.141592653589793d0)
C USES beschb
```

Returns the Bessel functions $\mathtt{rj} = J_\nu$, $\mathtt{ry} = Y_\nu$ and their derivatives $\mathtt{rjp} = J'_\nu$, $\mathtt{ryp} = Y'_\nu$, for positive x and for $\mathtt{xnu} = \nu \geq 0$. The relative accuracy is within one or two significant digits of EPS, except near a zero of one of the functions, where EPS controls its absolute accuracy. FPMIN is a number close to the machine's smallest floating-point number. All internal arithmetic is in double precision. To convert the entire routine to double precision, change the REAL declaration above and decrease EPS to 10^{-16}. Also convert the subroutine beschb.

```
      INTEGER i,isign,l,nl
      DOUBLE PRECISION a,b,br,bi,c,cr,ci,d,del,del1,den,di,dlr,dli,
     *     dr,e,f,fact,fact2,fact3,ff,gam,gam1,gam2,gammi,gampl,h,
     *     p,pimu,pimu2,q,r,rjl,rjl1,rjmu,rjp1,rjpl,rjtemp,ry1,
     *     rymu,rymup,rytemp,sum,sum1,temp,w,x2,xi,xi2,xmu,xmu2
      if(x.le.0..or.xnu.lt.0.) pause 'bad arguments in bessjy'
      if(x.lt.XMIN)then
        nl=int(xnu+.5d0)
      else
        nl=max(0,int(xnu-x+1.5d0))
      endif
```

nl is the number of downward recurrences of the J's and upward recurrences of Y's. xmu lies between $-1/2$ and $1/2$ for $x <$ XMIN, while it is chosen so that x is greater than the turning point for $x \geq$ XMIN.

```
      xmu=xnu-nl
      xmu2=xmu*xmu
      xi=1.d0/x
      xi2=2.d0*xi
      w=xi2/PI
      isign=1
      h=xnu*xi
      if(h.lt.FPMIN)h=FPMIN
      b=xi2*xnu
      d=0.d0
      c=h
      do 11 i=1,MAXIT
        b=b+xi2
        d=b-d
        if(abs(d).lt.FPMIN)d=FPMIN
        c=b-1.d0/c
        if(abs(c).lt.FPMIN)c=FPMIN
        d=1.d0/d
        del=c*d
        h=del*h
        if(d.lt.0.d0)isign=-isign
```

w=xi2/PI: The Wronskian.

isign=1: Evaluate CF1 by modified Lentz's method (§5.2).

h=xnu*xi: isign keeps track of sign changes in the denominator.

```
        if(abs(del-1.d0).lt.EPS)goto 1
    enddo 11
    pause 'x too large in bessjy; try asymptotic expansion'
1   continue
    rjl=isign*FPMIN                 Initialize $J_\nu$ and $J'_\nu$ for downward recurrence.
    rjpl=h*rjl
    rjl1=rjl                        Store values for later rescaling.
    rjp1=rjpl
    fact=xnu*xi
    do 12 l=nl,1,-1
        rjtemp=fact*rjl+rjpl
        fact=fact-xi
        rjpl=fact*rjtemp-rjl
        rjl=rjtemp
    enddo 12
    if(rjl.eq.0.d0)rjl=EPS
    f=rjpl/rjl                      Now have unnormalized $J_\mu$ and $J'_\mu$.
    if(x.lt.XMIN) then              Use series.
        x2=.5d0*x
        pimu=PI*xmu
        if(abs(pimu).lt.EPS)then
            fact=1.d0
        else
            fact=pimu/sin(pimu)
        endif
        d=-log(x2)
        e=xmu*d
        if(abs(e).lt.EPS)then
            fact2=1.d0
        else
            fact2=sinh(e)/e
        endif
        call beschb(xmu,gam1,gam2,gampl,gammi)      Chebyshev evaluation of $\Gamma_1$ and $\Gamma_2$.
        ff=2.d0/PI*fact*(gam1*cosh(e)+gam2*fact2*d) $f_0$.
        e=exp(e)
        p=e/(gampl*PI)                              $p_0$.
        q=1.d0/(e*PI*gammi)                         $q_0$.
        pimu2=0.5d0*pimu
        if(abs(pimu2).lt.EPS)then
            fact3=1.d0
        else
            fact3=sin(pimu2)/pimu2
        endif
        -=PI*pimu2*fact3*fact3
         =1.d0
         =-x2*x2
         um=ff+r*q
         m1=p
d         1,MAXIT
          i*ff+p+q)/(i*i-xmu2)
           d/i
           (i-xmu)
           i+xmu)
            *(ff+r*q)
            um+del
            *p-i*del
             um1+del1
             (del).lt.(1.d0+abs(sum))*EPS)goto 2
endd
```

```
      pause 'bessy series failed to converge'
2     continue
      rymu=-sum
      ry1=-sum1*xi2
      rymup=xmu*xi*rymu-ry1
      rjmu=w/(rymup-f*rymu)                        Equation (6.7.13).
    else                                           Evaluate CF2 by modified Lentz's method
      a=.25d0-xmu2                                     (§5.2).
      p=-.5d0*xi
      q=1.d0
      br=2.d0*x
      bi=2.d0
      fact=a*xi/(p*p+q*q)
      cr=br+q*fact
      ci=bi+p*fact
      den=br*br+bi*bi
      dr=br/den
      di=-bi/den
      dlr=cr*dr-ci*di
      dli=cr*di+ci*dr
      temp=p*dlr-q*dli
      q=p*dli+q*dlr
      p=temp
      do 14 i=2,MAXIT
        a=a+2*(i-1)
        bi=bi+2.d0
        dr=a*dr+br
        di=a*di+bi
        if(abs(dr)+abs(di).lt.FPMIN)dr=FPMIN
        fact=a/(cr*cr+ci*ci)
        cr=br+cr*fact
        ci=bi-ci*fact
        if(abs(cr)+abs(ci).lt.FPMIN)cr=FPMIN
        den=dr*dr+di*di
        dr=dr/den
        di=-di/den
        dlr=cr*dr-ci*di
        dli=cr*di+ci*dr
        temp=p*dlr-q*dli
        q=p*dli+q*dlr
        p=temp
        if(abs(dlr-1.d0)+abs(dli).lt.EPS)goto 3
      enddo 14
      pause 'cf2 failed in bessjy'
        tinue
        =(p-f)/q                          Equations (6.7.6) - (6.7.10).
        l=sqrt(w/((p-f)*gam+q))
        l=sign(rjmu,rjl)
         =rjmu*gam
         p=rymu*(p+q/gam)
         xmu*xi*rymu-rymup
    en
    fa     l/rjl
           jl1*fact                             Scale original J_ν and J'_ν.
           jp1*fact
           i=1,nl                               Upward recurrence of Y_ν.
           ytemp=(xmu+i)*xi2*ry1-rymu
           mu=ry1
            1=rytemp
            5
```

```
      ry=rymu
      ryp=xnu*xi*rymu-ry1
      return
      END
```

```
      SUBROUTINE beschb(x,gam1,gam2,gampl,gammi)
      INTEGER NUSE1,NUSE2
      DOUBLE PRECISION gam1,gam2,gammi,gampl,x
      PARAMETER (NUSE1=5,NUSE2=5)
C     USES chebev
```

Evaluates Γ_1 and Γ_2 by Chebyshev expansion for $|x| \leq 1/2$. Also returns $1/\Gamma(1+x)$ and $1/\Gamma(1-x)$. If converting to double precision, set NUSE1 = 7, NUSE2 = 8.

```
      REAL xx,c1(7),c2(8),chebev
      SAVE c1,c2
      DATA c1/-1.142022680371172d0,6.516511267076d-3,
     *     3.08709017308d-4,-3.470626964d-6,6.943764d-9,
     *     3.6780d-11,-1.36d-13/
      DATA c2/1.843740587300906d0,-.076852840844786d0,
     *     1.271927136655d-3,-4.971736704d-6,-3.3126120d-8,
     *     2.42310d-10,-1.70d-13,-1.d-15/
      xx=8.d0*x*x-1.d0
      gam1=chebev(-1.,1.,c1,NUSE1,xx)
      gam2=chebev(-1.,1.,c2,NUSE2,xx)
      gampl=gam2-x*gam1
      gammi=gam2+x*gam1
      return
      END
```

Multiply x by 2 to make range be -1 to 1, and then apply transformation for evaluating even Chebyshev series.

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Chung-Yen Shih.

Signature of Author

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Date

An Investigation of a Stable Numerical Algorithm for the Evaluation of Fractional Order Bessel Functions

Title of Thesis

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