Title: Elements of Linear Representations of Finite Groups

This paper is a survey of elementary concepts of the theory of representation of finite groups; wherein abstract groups are realized as groups of linear transformations or matrices. Basic definitions and examples of the above are given, as well as a notion of an equivalence relation for representations. The regular representation is presented through the concept of a group algebra. Other properties, such as sub-representations and irreducible representations, lead to an important result about the reducibility of representations of subgroups known as Clifford's Theorem. The character of a representation is then defined as the trace of the linear transformation or matrix which represents each element of the group. The relationship between characters and representations is developed including: (1) orthogonality relations from Schur's Lemma; and (2) the fact that the number of irreducible representations of a group is equal to its number of conjugacy classes. Finally, the concepts of induced representations and characters are explored, culminating in the Frobenius reciprocity formula.
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Chapter 1
Basic Definitions and Examples

Insights into the structure of abstract groups can be gained by their realization as familiar groups. In particular, we will consider homomorphisms of finite groups into groups of linear transformations and groups of matrices.

1.1 Representation by Automorphism

In the following, G will be a finite multiplicative group, and V will be a vector space of finite dimension over F, a subfield of \( \mathbb{C} \), the complex numbers. Let GL(V) denote the set of all invertible linear transformations of V onto itself. GL(V) acquires a group structure if multiplication is defined as composition: \((f \circ g)(v) = f(g(v))\) for all \( f, g \in GL(V), \ v \in V \).

**1.1 Definition:** The map \( \rho: G \to GL(V) \) is a linear representation of G over F provided \( \rho \) is a group homomorphism from G into GL(V).

So for all \( g \in G, \ \rho(g) \in GL(V) \). Of course, the basic properties of group homomorphisms hold for \( \rho \):

1. \( \rho(gg') = \rho(g) \circ \rho(g') \) for all \( g, g' \in G \);
2. \( \rho(1) = \text{Id}_V \) (The group identity 1 maps to the identity transformation on V.)
3. \( p(g^{-1}) = (p(g))^{-1} \); and
4. \( p(g^m) = (p(g))^m \) for any integer \( m \).

In the sequel, representation will mean linear representation.

Strictly speaking, a representation of a group \( G \) over a field \( F \) is given by a pair \( (V, \rho) \) and we say that \( V \) is a representation space of \( G \). However, when clear from the context, we will simply refer to the representation \( (V, \rho) \) as \( \rho \) or \( V \). Further, since \( GL(V) \) is often called the group of automorphisms of \( V \), we will often refer to \( \rho \) as a automorphism representation of \( G \).

1.2 Definition: Let \( (V, \rho) \) be a representation of \( G \), with \( \dim V = n \). Then \( (V, \rho) \) has degree \( n \).

Examples:

1. Let \( G = S_n \), the symmetric group of degree \( n \), and let \( V = \mathbb{F}^n \). As a basis for \( V \), take \( B = \{ e_1, e_2, \ldots, e_n \} \), where \( e_i = (0, 0, \ldots, 1, \ldots, 0) \). For any \( \sigma \in S_n \), define a \( i \)th position mapping \( \rho(\sigma) \) on \( V \) by:

   1. \( \rho(\sigma)(e_i) = e_{\sigma(i)}, \ i = 1, \ldots, n; \)
   2. \( \rho(\sigma)(e_i + e_j) = e_{\sigma(i)} + e_{\sigma(j)}, \ i, j = 1, \ldots, n; \)
   3. For any \( \alpha \in \mathbb{F} \), \( \rho(\sigma)(\alpha e_i) = \alpha e_{\sigma(i)}. \)

In other words, \( \rho(\sigma) \) is a linear transformation on \( V \).

Furthermore, for any \( \sigma \in S_n \), \( (\rho(\sigma^{-1}) \circ (\rho(\sigma))(e_i) = \rho(\sigma^{-1})(\rho(\sigma)(e_i)) = \rho(\sigma^{-1})(e_{\sigma(i)}) = e_{\sigma^{-1}(\sigma(i))} = e_i. \)
Similarly, \((\rho(\sigma) \circ \rho(\sigma^{-1}))(e_i) = e_i\), so \(\rho(\sigma)\) is invertible and we have \(\rho: \text{Sn} \to \text{GL}(V)\). For any \(\sigma, \tau \in G = \text{Sn}\), \(\rho(\sigma \tau)(e_i) = e_{(\sigma \tau)(i)} = e_{\sigma(\tau(i))} = \rho(\sigma)(e_{\tau(i)}) = \rho(\sigma)(\rho(\tau)(e_i)) = (\rho(\sigma) \circ \rho(\tau))(e_i)\). Hence, \(\rho\) is a group homomorphism and a representation of \(\text{Sn}\) over \(F\). This representation is called the canonical representation of \(\text{Sn}\).

It is worthy to note here that while the order of \(\text{Sn}\) is \(n!\), the degree of the representation is \(n\). The question of whether a representation of degree equal to the order of the group can always be found will be addressed later.

2. The trivial representation of degree \(n\): Let \(G\) be a finite group and \(V\) a vector space of dimension \(n\). Define: \(\rho: G \to \text{GL}(V)\) by \(\rho(g) = \text{Id}_V\) for all \(g \in G\). Clearly \(\rho\) is a representation of \(G\). Both of these examples will surface again in other settings.

1.2 Matrix Representation

It is necessary to recall from linear algebra that there is an \(nxn\) matrix associated with a given linear transformation of an \(n\)-dimensional vector space and a given basis. If \(V\) is such a space over \(F\) with basis \(B = \{V_1, \ldots, V_n\}\), and \(T \in \text{GL}(V)\), then \(T(V_j) = \sum_{i=1}^{n} a_{ij} V_i\), \(1 \leq j \leq n\), \(a_{ij} \in F\).

The matrix \([T]_B = (a_{ij})\), where the \(a_{ij}\) are taken to be the \(j^{th}\) column of \([T]_B\), is called the matrix of \(T\) relative to the basis \(B\). Clearly, \([T]_B\) is uniquely determined by \(T\) and \(B\).
This relationship provides a second major type of representation.

Denote by $GL(n, F)$ the set of all $n \times n$ nonsingular matrices with entries from the field $F$. Under the operation of usual matrix multiplication, $GL(n, F)$ is a group. This group is called the general linear group of degree $n$ over $F$.

1.3 Definition: The mapping $\theta : G \to GL(n, F)$ is a matrix representation of $G$ over $F$ of degree $n$ provided $\theta$ is a group homomorphism.

So for $x, y \in G$, $\theta(x)$ and $\theta(y)$ are $n \times n$ matrices such that $\theta(xy) = \theta(x) \theta(y)$.

Before giving examples, we shall make concrete the connection between automorphism and matrix representations suggested by remarks preceding the definition.

1.4 Theorem: Let $\rho : G \to GL(V)$ be an automorphism representation of $G$. Let $B = \{V_1, \ldots, V_n\}$ be a basis for $V$ and let $[\rho(g)]_B$ be the matrix of $\rho(g)$ relative to $B$ for any $g \in G$. Then the mapping $\theta : G \to GL(n, F)$ given by $\theta(g) = [\rho(g)]_B$ is a matrix representation of $G$.

Proof: For any $g, g' \in G$, let $[\rho(g)]_B = (a_{ij})$ and $[\rho(g')]_B = (b_{ij})$ where $a_{ij}, b_{ij} \in F$. Then we have $\rho(g)(V_j) = \sum_{i=1}^n a_{ij} V_i$ and $\rho(g')(V_j) = \sum_{i=1}^n b_{ij} V_i$ for any $V_j \in B$. Now, the $k, j$ entry of $[\rho(g)]_B [\rho(g')]_B = (a_{ij})(b_{ij})$ is given by $\sum_{i=1}^n a_{ki} b_{ij}$. 

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On the other hand, to look at the entries in $[\rho(gg')]_B$, we compute:

$$\rho(gg')(V_j) = (\rho(g) \circ \rho(g'))(V_j) = \rho(g)(\rho(g')(V_j))$$

$$= \rho(g) \left( \sum_{i=1}^{n} b_{ij}v_i \right) = \sum_{i=1}^{n} b_{ij} \rho(g)(v_i) = \sum_{i=1}^{n} b_{ij} \left( \sum_{k=1}^{n} a_{ki}v_k \right)$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ij} a_{ki}v_k = \sum_{k=1}^{n} \left( \sum_{i=1}^{n} a_{ki} b_{ij} \right) v_k$$

So the $k,j$ entry of $[\rho(gg')]_B$ is $\sum_{k=1}^{n} a_{ki} b_{ij}$. But this is exactly the $k,j$ entry of $(a_{ij})(b_{ij}) = [\rho(g)]_B [\rho(g')]_B$ and $\theta$ is a matrix representation.\(\square\)

Suppose now that a matrix representation $\theta : G \to \text{GL}(n,F)$ is given. Can a corresponding automorphism representation be found? If we let $V = F^n$ and define $\rho : G \to \text{GL}(F^n)$ by $\rho(g) : F^n \to F^n$ such that $\rho(g)(v) = \theta(g)v$ (*) for all $g \in G$, $v \in F^n$, $\rho$ is a representation of $G$. Note that the left-hand side of *, $\rho(g)(v)$, is a linear transformation acting on $v \in F^n$; while the right-hand side of * is the product of an $n \times n$ matrix with an $n \times 1$ matrix.

To see that $\rho$ is, in fact, a group homomorphism, let $g, g' \in G$ and $v \in F^n$ and compute as follows:

$$\rho(gg')(v) = \theta(gg')v = \theta(g)\theta(g')v = \theta(g)(\theta(g')v) = \theta(g)(\rho(g')(v)) = \rho(g)(\rho(g')(v)) = (\rho(g) \circ \rho(g'))(v).$$

Note that, for any $g \in G$, the matrix of $\rho(g)$ relative to the
standard basis $B = \{e_1, e_2, \ldots, e_n\}$ for $F^n$ is $\theta(g)$. That is, $[\rho(g)]_B = \theta(g)$.

**Examples:**

3. Let $G = S_n$ as in example 1. Define $\theta: S_n \to GL(n, F)$ by $\theta(\sigma) = (a_{ij})$ where $a_{\sigma(i)j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ for all $\sigma \in S_n$.

Notice that $\theta(\sigma)$ has a 1 in the $\sigma(j)$, $j$ positions, for $j = 1, \ldots, n$. That is, the $j^{th}$ column contains a 1 in the $\sigma(j)^{th}$ row. To see that $\theta$ is indeed a matrix representation, take the automorphism representation $\rho$ from Example 1 and express it in matrix form relative to $B$, the standard basis of $F^n$.

Then $[\rho(g)]_B = \theta(g)$ for all $g \in G$ and, by Theorem 1.4, $\theta$ is a matrix representation. For instance, consider $\sigma = (1 \ 3)$ and $\tau = (1\ 2\ 3\ 4)$ as elements of $G = S_4$. (Note: sometimes $\sigma$ and $\tau$ are written as $(1\ 2\ 3\ 4)$ and $(1\ 2\ 3\ 4)$, respectively.)

In $\theta(\sigma)$ we have $j = 1 \Rightarrow \sigma(j) = 3$; $j = 2 \Rightarrow \sigma(j) = 2$; $j = 3 \Rightarrow \sigma(j) = 1$; and $j = 4 \Rightarrow \sigma(j) = 4$. So $\theta(\sigma) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

In $\theta(\tau)$, $j = 1 \Rightarrow \tau(j) = 2$; $j = 2 \Rightarrow \tau(j) = 3$; $j = 3 \Rightarrow \tau(j) = 4$; $j = 4 \Rightarrow \tau(j) = 1$. Hence, $\theta(\tau) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

Because the matrices in Example 3 are $n \times n$, the representation is said to be of degree $n$. Another representation of $S_n$, this one of degree 1, is given below.
4. Let $G = S_n$ and $F = \mathbb{C}$. Define $\theta : S_n \to GL(1, \mathbb{C})$ by:

$$\theta(\sigma) = \begin{cases} 
1 & \text{if } \sigma \text{ is an even permutation} \\
-1 & \text{if } \sigma \text{ is an odd permutation.}
\end{cases}$$

(Note that $GL(1, \mathbb{C})$ is just $\mathbb{C}^*$, the multiplicative group of non-zero complex numbers.) For any $\sigma, \tau \in S_n$:

$$\theta(\sigma \circ \tau) = \begin{cases} 
1 & \text{if } \sigma \circ \tau \text{ even} \\
-1 & \text{if } \sigma \circ \tau \text{ odd}
\end{cases}$$

$$= \begin{cases} 
1 & \text{if both } \sigma \text{ and } \tau \text{ even or both } \sigma \text{ and } \tau \text{ odd} \\
-1 & \text{if } \sigma \text{ even and } \tau \text{ odd or } \sigma \text{ odd and } \tau \text{ even}
\end{cases}$$

So $\theta$ is a matrix representation of degree 1; in fact, it is known as the alternating representation.

5. The set $G$ of rotations about the origin in the real plane is a group under composition of rotations. Note that if $g_\alpha, g_\beta \in G$ are rotations through angles $\alpha$ and $\beta$, respectively, then $g_\alpha g_\beta = g_\beta g_\alpha \in G$ is just the rotation through $\alpha + \beta$. Let $\mathbb{R}$ be the real numbers.

For $(x, y) \in \mathbb{R}^2$, $g_\alpha(x, y) = (x', y')$ where

$$x' = x \cos \alpha - y \sin \alpha$$

$$y' = x \sin \alpha + y \cos \alpha$$

Define $\theta : G \to GL(2, \mathbb{R})$ by

$$\theta(g_\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$ 

Then $\theta$ is a matrix representation of $G$ of degree 2 since:

$$\theta(g_\alpha) \theta(g_\beta) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix}.$$
(Applying trig identities)
\[
\begin{bmatrix}
\cos(\alpha + \beta) & -\sin(\alpha + \beta) \\
\sin(\alpha + \beta) & \cos(\alpha + \beta)
\end{bmatrix} = \theta(g_\alpha g_\beta).
\]

Now let \( G \) be the group of rotations about the origin in three dimensional space. Suppose that, as the result of a rotation, the \( x-, y-, \) and \( z- \) axes are transformed into the \( x'-, y'-, \) and \( z'- \) axes.

The rotation can be considered as the product (composition) of three rotations: (1) \( g_{\alpha_1} \rightarrow \) through angle \( \alpha_1 \) about the \( z \)-axis carrying the \( x \)-axis to line \( L; \) \( L \) is the line of intersection of the \( x-y \) plane and the \( x'-y' \) plane.

(2) \( g_{\alpha_2} \rightarrow \) through angle \( \alpha_2 \) about \( L \) carrying the \( z \)-axis to the \( z' \)-axis; and

(3) \( g_{\alpha_3} \rightarrow \) through angle \( \alpha_3 \) about the \( z' \)-axis.

\( \theta : G \rightarrow (\text{GL}_3, \mathcal{R}) \) is as follows:

\[
\theta(g_{\alpha_1}) = \begin{bmatrix}
\cos \alpha_1 & -\sin \alpha_1 & 0 \\
\sin \alpha_1 & \cos \alpha_1 & 0 \\
0 & 0 & 1
\end{bmatrix} ;
\]
\[ \theta(ga_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_2 & -\sin \alpha_2 \\ 0 & \sin \alpha_2 & \cos \alpha_2 \end{bmatrix} \]; and

\[ \theta(ga_3) = \begin{bmatrix} \cos \alpha_3 & -\sin \alpha_3 & 0 \\ \sin \alpha_3 & \cos \alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

Thus, the entire rotation is represented by:

\[
\theta(ga_1 ga_2 ga_3) = \theta(ga_1) \theta(ga_2) \theta(ga_3) = \\
\begin{bmatrix}
\cos \alpha_1 \cos \alpha_3 & -\cos \alpha_1 \sin \alpha_3 & \sin \alpha_1 \sin \alpha_2 \\
\sin \alpha_1 \cos \alpha_2 \sin \alpha_3 & \sin \alpha_1 \cos \alpha_2 \cos \alpha_3 & \sin \alpha_1 \sin \alpha_2 \\
\cos \alpha_1 \cos \alpha_2 \sin \alpha_3 & \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 & \cos \alpha_1 \sin \alpha_2 \\
\sin \alpha_2 \sin \alpha_3 & \sin \alpha_2 \cos \alpha_3 & \cos \alpha_2
\end{bmatrix}.
\]

Example 6 is presented as an important concrete realization of group theoretic results for the physical sciences.

1.3 Representation Modules

Representations can be viewed in a somewhat more general context via the representation module defined below.

**Definition:** Let \( G \) be a group and let \( V \) be a vector space over \( F \). \( V \) is called a *left \( G \)-module* if a multiplication is defined on \( V \) such that:

- for all \( g, g' \in G; \, v, v' \in V, \, a \in F, \)
  
  \( (1) \quad gv \in V \)

  \( (2) \quad g(v + v') = gv + gv' \)
(3) \( g(av) = a(gv) \)
(4) \( (g g') v = g(g'v) \)
(5) \( 1v = v \); where \( 1 \) is the identity of \( G \).

A right \( G \)-module can be defined in a similar way. For our purposes, \( G \)-module will mean left \( G \)-module.

The connection between representation modules and the representation of \( G \) by automorphism is given by:

1.6 Theorem: The mapping \( \rho : G \to GL(V) \) is an automorphism representation of \( G \) if and only if \( V \) is a \( G \)-module.

Proof: (=\( \to \)) If \( \rho \) is an automorphism representation, then \( \rho(g) \) is a linear transformation on \( V \), for all \( g \in G \).

Define the multiplication by \( gv = \rho(g)(v) \) \( \forall \ g \in G, \ v \in V \).

It is completely routine to establish from the conditions of the definition that \( V \) is a \( G \)-module. Parts (2) and (4) are shown here:

(2) \( g(v + v') = \rho(g)(v + v') = \rho(g)(v) + \rho(g)(v') = gv + g v' \);

and

(4) \( (g g')v = (\rho(g) \circ \rho(g'))(v) = \rho(g)(\rho(g')(v)) = \rho(g)(g'v) = g(g'v) \).

(\( \leq \)) Let \( G \) be a group and \( V \) a \( G \)-module. Define \( \rho(g) : V \to V \) by \( \rho(g)(v) = gv \). That \( \rho(g) \) is a linear transformation is clear from (2) and (3) in the definition of a \( G \)-module. Also, \( \rho(g) \) is invertible \( \forall g \in G \) since \( (\rho(g))^{-1} = \rho(g^{-1}) \).

Hence, \( \rho(g) \in GL(V) \) for all \( g \in G \).
Finally, for all \( v \in V, g, g' \in G \): \( \rho(gg')(v) = (gg')v = g(g'v) = \rho(g)(\rho(g')(v)) = (\rho(g) \circ \rho(g'))(v) \). Hence, \( \rho \) is a group homomorphism from \( G \) into \( GL(V) \) and, therefore, a representation.

1.4 Equivalence

The relationships between representation by automorphisms, matrices, and modules have now been outlined. But what can be said about, say, two automorphism representations of the same group \( G \)? Consider this motivation for a definition: Let \((V,\rho)\) be a representation of \( G \) over \( F \), and let \( V' \) be a vector space isomorphic to \( V \). Now define a map \( \rho' : G \rightarrow GL(V') \) by \( \rho'(g) : V' \rightarrow V' \) where \( \rho'(g) = \alpha \circ \rho(g) \circ \alpha^{-1} \) where \( \alpha \) is a vector space isomorphism from \( V \) to \( V' \).

Then, for all \( g, g' \in G \): \( \rho'(gg') = \alpha \circ \rho(gg') \circ \alpha^{-1} = \alpha \circ \rho(g) \circ \rho(g') \circ \alpha^{-1} = \alpha \circ \rho(g) \circ (\alpha^{-1} \circ \alpha) \circ \rho(g') \circ \alpha^{-1} = (\alpha \circ \rho(g) \circ \alpha^{-1}) \circ (\alpha \circ \rho(g') \circ \alpha^{-1}) = \rho'(g) \circ \rho'(g'). \)

Hence \((V',\rho')\) is also a representation of \( G \).

1.7 Definition: Two representations of \( G \) over \( F', (V,\rho) \) and \((V',\rho')\), are equivalent provided there exists a vector space isomorphism \( \alpha : V \rightarrow V' \) such that \( \alpha \circ \rho(g) = \rho(g') \circ \alpha \) \( \forall g \in G \). In this case, we write \( \rho \equiv \rho' \). A similar definition applies to matrix representations.
1.8 Definition: Two matrix representations, \( \theta \) and \( \Phi \), of \( G \) of degree \( n \) over \( F \) are equivalent \((\theta \sim \Phi)\) provided there exists a fixed \( P \in \text{GL}(n,F) \) such \( \theta(g) = P \Phi(g) P^{-1} \) for all \( g \in G \).

Suppose that \( \theta, \Phi, \) and \( \Psi \) are matrix representations of \( G \) of degree \( n \) over \( F \); and that \( \theta \sim \Phi \) and \( \Phi \sim \Psi \). Then we have \( P \in \text{GL}(n,F) \) such that \( \theta(g) = P \Phi(g) P^{-1} \), and \( Q \in \text{GL}(n,F) \) such that \( \Phi(g) = Q \Psi(g) Q^{-1} \), for all \( g \in G \). Combining:

\[
\theta(g) = P (Q \Psi(g) Q^{-1}) P^{-1} = (PQ) \Psi(g) (PQ)^{-1}.
\]

\( \therefore \theta \sim \Psi \), and we have transitivity in equivalence of matrix representations.

It is similarly easy to establish, for both matrix and automorphism representations, the other criteria necessary to show:

1.9 Theorem: Equivalence of representations is an equivalence relation on the set of all representations of \( G \) of degree \( n \) over \( F \).

Two observations are in order concerning equivalence:

1. Recall that for a representation \((V,\rho)\), we were able to compute a corresponding matrix representation \( \theta \). But \( \theta \) was dependent upon the choice of basis for \( V \). Another choice of basis would result in another matrix representation \( \theta' \).

However, from linear algebra, the relationship between the
matrices of a linear transformation relative to two distinct bases is precisely that of the definition of $\theta = \theta'$.

2. Similarly, given a matrix representation $\theta$, every choice of a vector space of dimension $n$ yields an equivalent automorphism representation of degree $n$. 
Chapter 2

The Group Algebra of G

2.1 Definition of the Group Algebra

Representation of a group can be given a more algebraic (and abstract) formulation by introducing the group algebra of G.

First, it is necessary to recall the definition of an algebra over a field.

2.1 Definition: A vector space V over a field F is called an algebra over F if a multiplication is defined on V such that, for all u, v, w ∈ V, α ∈ F:

1. \(u(v+w) = uv + uw\);
2. \((u+v)w = uw + vw\); and
3. \(\alpha(uv) = (\alpha u)v = u(\alpha v)\).

Now let G be a finite group and let F be a field.

2.2 Definition: The group algebra \(FG\) is the set of all formal sums:

\[
FG = \left\{ \sum_{g \in G} a_g g \mid a_g \in F \right\}
\]

with the operations:

1. \[
\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g)g;
\]
2. \[
\alpha \left( \sum_{g \in G} a_g g \right) = \sum_{g \in G} \alpha a_g g \text{ for all } \alpha \in F; \text{ and}
\]

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Theorem: FG as defined above is an algebra over $F$.

Proof: It is routine to verify that FG is a vector space over $F$ under the addition and scalar multiplication in (1) and (2) of the definition (2.2).

To see that the multiplication in (3) of definition 2.2 makes FG an algebra over $F$, we use definition 2.1: For $a_g$, $b_h$, $c_h \in F$,

(1) $\sum_{g \in G} a_g g \left( \sum_{h \in G} b_h h + \sum_{h \in G} c_h h \right) =
\sum_{g \in G} a_g g \left( \sum_{h \in G} [b_h + c_h] h \right) =
\sum_{g \in G} a_g (b_h + c_h) gh = \sum_{g, h \in G} [a_gb_h + a_gc_h] gh =
\sum_{g, h \in G} [a_gb_h] gh + \sum_{g, h \in G} [a_gc_h] gh =
\left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} b_h h \right) + \left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} c_h h \right) .

(2) The proof is similar to the proof of (1).

(3) For any $\alpha \in F$, $\alpha \left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} b_h h \right) =
\alpha \sum_{g, h \in G} [a_gb_h] gh = \sum_{g, h \in G} [\alpha a_g] [b_h] gh =
\left( \sum_{g, h \in G} [\alpha a_g] g \right) \left( \sum_{h \in G} b_h h \right) = \left( \alpha \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} b_h h \right) .

Also, $\sum_{g, h \in G} [a_gb_h] gh = \left( \sum_{g \in G} a_g g \right) \left( \sum_{h \in G} [\alpha b_h] h \right) =$
\[
\left( \sum_{g \in G} a_g \cdot g \right) \left( \alpha \sum_{h \in G} b_h \cdot h \right), \text{ and (3) is established.}
\]

Hence, \( FG \) is an algebra over \( F \). \( \square \)

Note that, with the multiplication so defined, \( FG \) is also a ring and has as unity the one-term sum \( 1_{FG} = 1_F 1_G \).

In addition, as a vector space, \( FG \) has a most interesting basis. Every element of \( FG \) is, by definition, simply a linear combination of the elements of \( G \) over \( F \). So the set of elements \( \{1_g\}_{g \in G} \) forms a basis for \( FG \) over \( F \). As a consequence, \( \dim(FG) = |G| \), where \( |G| \) is the order of \( G \).

2.2 Representation of \( FG \)

To continue to move toward more algebraic concepts, such as representation of the group algebra \( FG \), we must extend some fundamental notions to answer the question: What, precisely, is meant by a representation of an algebra (vis-à-vis, a group)?

If \( V \) is a vector space over \( F \), denote by \( \text{Hom}(V,V) \) the set of all linear transformation of \( V \) into itself. For comparison, notice that \( GL(V) \subset \text{Hom}(V,V) \). For all \( T_1, T_2, \in \text{Hom}(V,V), v \in V, \alpha \in F \), we define addition, scalar multiplication, and multiplication as:

1. \( (T_1 + T_2)(v) = T_1(v) + T_2(v) \);
2. \( (\alpha T_1)(v) = \alpha(T_1(v)) \); and
3. \( (T_1 T_2)(v) = (T_1 \circ T_2)(v) = T_1(T_2(v)) \), i.e.,

multiplication is the composition of linear transformations.
With the operations so defined, it is easy to check that 
\( \text{Hom}(V,V) \) is an algebra over \( F \).

2.4 Definition: Let \( A \) be an algebra with unity over \( F \) 
and \( V \) a vector space over \( F \). Then the mapping 
\( \Phi : A \rightarrow \text{Hom}(V,V) \) is a representation of \( A \) over \( F \) provided:

For all \( a, b \in A, \alpha \in F \),

1. \( \Phi(a + b) = \Phi(a) + \Phi(b) \);
2. \( \Phi(\alpha a) = \alpha \Phi(a) \);
3. \( \Phi(ab) = \Phi(a) \circ \Phi(b) \); and
4. \( \Phi(1_A) = \text{Id}_V \), where \( \text{Id}_V \) is the identity transformation on 
\( V \) and \( 1_A \) is the multiplicative identity of \( A \). So \( \Phi \) is just 
an algebra homomorphism.

We wish to define such a representation for the algebra 
\( FG \). This can be accomplished by a natural and unique 
extension of the representation of the group \( G \).

Let \( (V,\rho) \) be a representation of \( G \). Define 
\( \rho^* : FG \rightarrow \text{Hom}(V,V) \) by:

\[
\rho^* \left( \sum_{g \in G} \alpha_g g \right) = \sum_{g \in G} \alpha_g \rho(g) \text{ for all } \sum_{g \in G} \alpha_g g \in FG.
\]

Since \( \rho(g) \) is a linear transformation for each \( g \in G \),
the image of \( \sum_{g \in G} \alpha_g g \) under \( \rho^* \) is itself a linear 
transformation which is the sum of scalar multiples of linear transformations.
2.5 Theorem: The mapping $\rho^*$ is an algebra homomorphism and, thus, a representation of $FG$.

Proof: We refer to conditions (1) - (4) in definition 2.4. For all $\sum a_g g, \sum b_g g \in FG, \alpha \in F$, we have:

1. $\rho^* \left( \sum a_g g + \sum b_g g \right) = \rho^* \left( \sum (a_g + b_g) g \right) =

\sum [a_g + b_g] \rho(g) = \sum (a_g \rho(g) + b_g \rho(g)) =

\sum a_g \rho(g) + \sum b_g \rho(g) = \rho^* \left( \sum a_g g \right) + \rho^* \left( \sum b_g g \right).

2. $\rho^* (\alpha \sum a_g g), = \rho^* \left( \sum [\alpha a_g] g \right) = \sum [\alpha a_g] \rho(g) =

\alpha \sum a_g \rho(g) = \alpha \rho^* \left( \sum a_g g \right). (Since \alpha does not depend on $g \in G$.)

3. $\rho^* \left( \sum a_g g \right) \left( \sum b_h h \right) =

\rho^* \left( \sum (a_g b_h) gh \right) = \sum (a_g b_h) \rho(gh)) = \sum (a_g b_h) \rho(gh).

Let us pause for a moment to see where we are. The sum in (*) calls for the composition of all linear transformations $\rho(g) \circ \rho(h)$ such that the product of the corresponding scalars, $a_g b_h$, is not zero. That is, $a_g \neq 0$ and $b_h \neq 0$.  

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Now since \( p(g), p(h) \in \text{Hom}(V,V) \) (*) is equal to

\[
\sum_{g \in G} a_g p(g) \circ b_h p(h) (**). \text{ But we can achieve the same composition of all combinations of } p(g) \text{ and } p(h) \text{ with } a_g \neq 0 \text{ and } b_h \neq 0 \text{ by writing (**) as}
\]

\[
\sum_{g \in G} a_g p(g) \circ \sum_{h \in G} b_h p(h).
\]

And \( \sum_{g \in G} a_g p(g) \circ \sum_{h \in G} b_h p(h) = p^* \left( \sum_{g \in G} a_g g \right) \circ p^* \left( \sum_{h \in G} b_h h \right) \).

(4) Recall that the unit element of \( FG \) is \( l_{FG} = l_F l_G \). So \( p^*(1_{FG}) = p^*(1_F 1_G) = 1_F p(1_G) = 1_F \text{Id}_V = \text{Id}_V \),

since \( p \) is a group homomorphism from \( G \) to \( GL(V) \).

Therefore, \( p^* \) is a representation of \( FG \).

A couple of remarks about this result are in order.

1. The representation \( p^* \) of \( FG \) is said to be the representation corresponding to the representation \( p \) of \( G \).

2. Recall from definition 1.5 and theorem 1.6 that if \( p : G \to GL(V) \) is a representation, then \( V \) is called a \( G \)-module. Now, if for any \( \gamma = \sum_{g \in FG} a_g g \in FG \), \( v \in V \), we define \( p^*(\gamma) = \gamma v \), we have that \( V \) is an \( FG \) - module.

Suppose now that \( p^* : FG \to \text{Hom}(V,V) \) is any representation of \( FG \). Recall that the elements of \( G \) form a basis for \( FG \) as a vector space; hence, \( G \subset FG \) (Under the identification \( g \leftrightarrow l_F g \)). Our aim now is to define
\( \rho: G \to \text{GL}(V) \) so that \( \rho \) will be a representation of \( G \).

Consider the restriction of \( \rho^* \) to \( G \) and define \( \rho = \rho^*|_G \).

Since \( \text{GL}(V) \subset \text{Hom}(V,V) \), we must first be convinced that \( \rho(g) \in \text{GL}(V) \) for all \( g \in G \). Let \( 1 \) be the identity of \( G \).

Then \( \rho(1) = \rho^*(1) = \text{Id}_V \). On the other hand \( \rho(1) = \rho(g g^{-1}) = \rho^*(g g^{-1}) = \rho^*(g) \circ \rho^*(g^{-1}) = \rho(g) \circ \rho(g^{-1}) = \text{Id}_V \), for any \( g \in G \). Similarly, \( \rho(g^{-1}) \circ \rho(g) = \text{Id}_V \), so \( \rho(g)^{-1} = \rho(g^{-1}) \) and \( \rho(g) \in \text{GL}(V) \).

To see that \( \rho \) is a group homomorphism, let \( g, g' \in G \) and compute: \( \rho(g g') = \rho^*(g g') = \rho^*(g) \circ \rho^*(g') = \rho(g) \circ \rho(g') \).

Hence, we have shown:

**2.6 Theorem:** The mapping \( \rho = \rho^*|_G : G \to \text{GL}(V) \) is a representation of \( G \) with representation space \( V \) over \( F \).

So to every representation \( \rho^* \) of \( FG \), there corresponds a representation \( \rho \) of \( G \). We summarize the discussion concerning theorems 2.5 and 2.6 by observing:

**2.7 Theorem:** There is a one-to-one correspondence between the representations of a finite group \( G \) over \( F \) with representation space \( V \) and the group algebra \( FG \) over \( F \) with representation space \( V \).

### 2.3 Regular Representation of a Group

We shall now pursue a question posed in section 1.1:

Does every finite group of order \( n \) have a representation of degree \( n \)?
Let $G$ be a group with $|G| = n$ ($|G|$ is the order of $G$). Assign to the elements of $G$ some fixed ordering: $1 = g_1, g_2, \ldots, g_n$. Consider the group algebra $FG$ as a vector space of dimension $n$. In fact, we will use $V = FG$ as our representation space. The elements of $G$ form a basis for $FG = V$ and any $v \in V$ can be uniquely expressed as:

$$\sum_{i=1}^{n} a_i g_i$$

where $a_i \in F$, and $\{g_1, \ldots, g_n\}$ is the basis of $V$.

Now, for each $x \in G$, define $\rho(x) : FG \to FG$ by $\rho(x)(g_i) = xg_i$ on the basis elements and extend $\rho(x)$ linearly:

$$\rho(x) \left( \sum_{i=1}^{n} a_i g_i \right) = \sum_{i=1}^{n} a_i \rho(x)(g_i) = \sum_{i=1}^{n} a_i (xg_i).$$

**2.8 Theorem:** The mapping $\rho$ as defined above is a representation of $G$ over $F$ with representation space $FG$.

**Proof:** To see that $\rho(x)$ is invertible for each $x \in G$, note that for any $\sum_{i=1}^{n} a_i g_i \in FG$, $(\rho(x) \circ \rho(x^{-1})) \left( \sum_{i=1}^{n} a_i g_i \right) = \rho(x) \left( \rho(x^{-1}) \left( \sum_{i=1}^{n} a_i g_i \right) \right) = \rho(x) \left( \sum_{i=1}^{n} a_i (x^{-1}g_i) \right) = \sum_{i=1}^{n} a_i (xx^{-1}g_i) = \sum_{i=1}^{n} a_i g_i.

Similarly, $(\rho(x^{-1}) \circ \rho(x)) (\sum_{i=1}^{n} a_i g_i) = \sum_{i=1}^{n} a_i g_i$. So each $\rho(x)$ has an inverse; namely, $\rho(x^{-1})$. Then we have $\rho(x) \in GL(FG)$ for all $x \in G$. 21
Finally, we establish that $\rho : G \to \text{GL}(F G)$ is a group homomorphism. Let $x, y \in G$. Then for each $i = 1, 2, \ldots, n$,

$$
\rho(xy)(g_i) = xyg_i = x(yg_i) = \rho(x)(yg_i) = \rho(x)\rho(y)(g_i) = (\rho(x) \circ \rho(y))(g_i).
$$

Hence $\rho(xy) = \rho(x) \circ \rho(y)$. Therefore, $\rho$ is a representation of $G$ over $F$ with representation space $FG$. □

2.9 Definition: The representation $\rho : G \to \text{GL}(F G)$ above is called the (left) **regular representation** of $G$ over $F$.

Remarks concerning the regular representation:

1. We can similarly define the right regular representation by defining $\rho(x)(g_i) = g_ix$, for all $x \in G$, $i=1, \ldots, n$.

2. Since the degree of the regular representation equals $\dim(F G) = n$, we have an affirmative answer to the question at the outset of this section.

The basis $\{1=g_i, g_2, \ldots, g_n\}$ of $FG$ is nothing but the set of images of $1 \in FG$ under the $\rho(g_i)$, $i=1, \ldots, n$. That is, $\rho(g_1)(1) = 1 = g_1, \rho(g_2)(1) = g_2, \ldots, \rho(g_n)(1) = g_n$. We will show that any representation, with the property that the images of one vector from the representation space form a basis of that space, is equivalent to the regular representation $(FG, \rho)$.

Let $(W, \rho')$ be a representation of $G$ over $F$ such that there exists a $w \in W$ and $(\rho'(g)(w) | g \in G)$ form a basis of $W$. 

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To show that \((FG, \rho) = (W, \rho')\), we have need of a vector space isomorphism from \(FG\) to \(W\). Define \(\Psi : FG \to W\) by

\[
\Psi(g_i) = \rho'(g_i)(w),
\]
and extend \(\Psi\) linearly so that

\[
\Psi\left( \sum_{i=1}^{n} a_i g_i \right) = \sum_{i=1}^{n} a_i \rho'(g_i)(w).
\]

Now let \(\sum_{i=1}^{n} a_i g_i, \sum_{i=1}^{n} b_i g_i \in FG\) and suppose

\[
\Psi\left( \sum_{i=1}^{n} a_i g_i \right) = \sum_{i=1}^{n} b_i \rho'(g_i)(w).
\]

We know that the \(\rho'(g_i)(w), \) for \(i = 1, \ldots, n\) form a basis for the vector space \(W\). But since each element of \(W\), such as

\[
\sum_{i=1}^{n} a_i \rho'(g_i)(w) = \sum_{i=1}^{n} b_i \rho'(g_i)(w),
\]

can be uniquely written as a linear combination of basis elements, we have that \(a_i = b_i\) for \(i = 1, \ldots, n\). Hence, \(\sum_{i=1}^{n} a_i g_i = \sum_{i=1}^{n} b_i g_i\) and \(\Psi\) is one-to-one.

It is a basic property of linear transformations that, since \(\Psi\) is one-to-one and \(\dim(\text{dim}(FG)) = \dim W\), then \(\Psi\) is an isomorphism from \(FG\) to \(W\).

To establish \((FG, \rho) = (W, \rho')\), we must now show that, for any \(x \in G\), \(\Psi \circ \rho(x) = \rho'(x) \circ \Psi\). That is, we show that this diagram commutes:
If \( g_i \) is any basis element of \( FG \), \((\Psi \circ \rho(x))(g_i) = \Psi'((\rho(x)(g_i)) = \Psi(xg_i) = \rho'(xg_i)(w) = (\rho'(x) \circ \rho'(g_i))(w) = \rho'(x)(\rho'(g_i)(w)) = \rho'(x)(\Psi(g_i)) = (\rho'(x) \circ \Psi)(g_i) \). Hence;

\((FG, \rho) = (W, \rho')\).

So far, we have examined the regular representation only in terms of representation by automorphism. As in Chapter 1, for any choice of basis of \( FG \), we have a corresponding matrix representation of \( G \) over \( F \). If we stick with our basis \( G = \{g_1, \ldots, g_n\} \) of \( FG \), then \( \Phi : G \rightarrow GL(n, F) \) will be defined by \( \Phi(x) = [\rho(x)]_G \) for any \( x \in G \), where \([\rho(x)]_G \) is the matrix of the linear transformation \( \rho(x) \) relative to \( G \). Recall that the jth column of \([\rho(x)]_G \) is just the coefficients of \( \rho(x)(g_j) \). But since \( \rho(x)(g_j) = xg_j \), another basis vector of \( FG \), the jth column will consist of zeros except for a 1 in the i\(^{th}\) row where i is such that \( g_i = xg_j \). The matrix \([\rho(x)]_G \) can be expressed as \([\rho(x)]_G = (\delta_{g_i, xg_j})\), where \( \delta_{g_i, xg_j} \) is Kronecker delta. This matrix is sometimes called a permutation matrix.
Examples:

1. Let $G = \{1, x, x^2\}$ and let $F = \mathbb{R}$, the real numbers. If we denote the elements of $G$ by $g_1 = 1$, $g_2 = x$, $g_3 = x^2$, then a typical element of $FG = \mathbb{R}G$ has the form: $a_1 1 + a_2 x + a_3 x^2 = a_1 g_1 + a_2 g_2 + a_3 g_3$, where $a_1, a_2, a_3 \in \mathbb{R}$. (Note that $\mathbb{R}G$ is isomorphic to $\mathbb{R}^3$ as a vector space.)

If $(\mathbb{R}G, \rho)$ is the regular representation of $G$ via automorphism, then the matrix representing $g_k \in G$, $k = 1, 2, 3$, is given by $[\rho(g_k)]_G = (\delta_{g_1}, g_k g_j)$. For instance, $[\rho(g_3)]_G = (\delta_{g_1}, g_3 g_j)$.

We compute the entries of $[\rho(g_3)]_G$ by:

$\rho(g_3)(1) = 0 g_1 + 0 g_2 + 1 g_3$
$\rho(g_3)(x) = 1 g_1 + 0 g_2 + 0 g_3$
$\rho(g_3)(x^2) = 0 g_1 + 1 g_2 + 0 g_3$. 

so $[\rho(g_3)]_G = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

by similar computations;

$[\rho(g_2)]_G = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$; and

$[\rho(g_1)]_G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Let $G = C_4$, the cyclic group of order 4.

So $G = \{x, x^2, x^3, x^4 = 1\}$. As in example 1, let us denote the elements by $g_1 = x^4 = 1$; $g_2 = x$, $g_3 = x^2$, $g_4 = x^3$.

Also, let $(\mathbb{R}G, \rho)$ be the regular representation by automorphism, and let $\Phi(g_k) = [\rho(g_k)]_G$ be the matrix
representing $g_k$, $k = 1, 2, 3, 4$. So we have, again, $\Phi(g_k) = (\delta_{g_k}, g_k g_3)$.

But since we know that $x = g_2$ is the generator of $G = C_4$, we need only compute $\Phi(g_2)$. The computations are essentially the same as in example 1, save that $\Phi(g_k)$ will be a 4x4 matrix. The computations yield:

$$\Phi(x) = \Phi(g_2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$  Also,

$$\Phi(x^2) = \Phi(g_3) = \Phi(x)\Phi(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};$$

$$\Phi(x^3) = \Phi(g_4) = \Phi(x)\Phi(x^2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix};$$ and

$$\Phi(1) = \Phi(x^4) = \Phi(g_1) = \Phi(x)\Phi(x^3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

After examining examples 1 and 2, it is not difficult to obtain the regular matrix representation of $C_n$ for any integer $n \geq 3$.

3 Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ and assign the ordering $g_1 = (1), g_2 = (12), g_3 = (13), g_4 = (23), g_5 = (123), g_6 = (132)$. Let $(\mathcal{R}_G, \rho)$ be the regular representation by automorphism. We wish to specify the regular matrix representation $\Phi : G \rightarrow \text{GL}(6, \mathcal{R})$.

Since $\Phi$ is a group homomorphism, it is sufficient to give the matrices corresponding to a set of generators of
$G = S_3$, such as $\{(12), (123)\} = \{g_2, g_5\}$. We display the computations for the first three columns of $\Phi((12))$:

$\rho(g_2)((1)) = (12)(1) = (12) = Og_1 + 1g_2 + Og_3 + \ldots + Og_6$; 
$\rho(g_2)((12)) = (12)(12) = (1) = 1g_1 + Og_2 + \ldots + Og_6$; and 
$\rho(g_2)((13)) = (12)(13) = (132) = 0g_1 + \ldots + 0g_5 + 1g_6$.

The remaining columns of $\Phi((12))$ and those of $\Phi((123))$ are computed in a similar manner, yielding:

$$\Phi((12)) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}; \text{ and}$$

$$\Phi((123)) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}.$$ 

We now return to the general development.

Another method that can be used to obtain the regular representation works through permutations. Let $S_n$ be the symmetric group of degree $n$ and let $V$ be any vector space of dimension $n$ with basis $\{v_1, v_2, \ldots, v_n\}$.

For every $\sigma \in S_n$, define a mapping $\eta(\sigma) : V \to V$ by $\eta(\sigma)(v_i) = v_{\sigma(i)}$ for $i = 1, \ldots, n$. By comparing to example $1$ in section 1.1, two things are clear:

1. For all $\sigma \in S_n$, $\eta(\sigma) \in GL(V)$, so $\eta : S_n \to GL(V)$; and
2. $\eta$ is a group homomorphism.
Hence, \( \eta \) is a representation of \( S_n \).

Now let \( G = \{g_1, \ldots, g_n\} \) be a group of order \( n \). A classic result of group theory due to Cayley assures us that \( G \) is isomorphic to a subgroup of \( S_n \). For every \( g \in G \), define \( \pi : G \to S_n \) by \( \pi(g) = (g_1, \ldots, g_n) \).

(It is easy to check that, for \( g, g' \in G \), \( \pi(gg') = \pi(g)\pi(g') \); so \( \pi \) is a group homomorphism. The mapping \( \pi \) is called the left-permutation representation of \( G \).

Our goal here is to find a linear representation of \( G \), i.e., we need a homomorphism from \( G \) to \( GL(V) \). Define \( \tilde{\eta} : G \to S_n \to GL(V) \) to be the composition: \( \tilde{\eta} = \eta \circ \pi \). It is routine to check that \( \tilde{\eta} \) is a representation of \( G \) over \( F \). In fact, we have:

**2.10 Theorem:** Let \( (FG, \rho) \) be the regular representation of \( G \) over \( F \), and let \( (V, \tilde{\eta}) \) be as defined above. Then \( \rho = \tilde{\eta} \).

**Proof:** We require a vector space isomorphism \( \alpha : FG \to V \) such that, for all \( g \in G \), the following diagram commutes:
Recall that the elements of $G$ form a basis for $FG$ and let \( \{v_1, \ldots, v_n\} \) be a basis of $V$. Define $\alpha : FG \to V$ by

$$
\alpha \left( \sum_{i=1}^{n} a_i g_i \right) = \sum_{i=1}^{n} a_i v_i.
$$

(In terms of basis elements, $\alpha(g_i) = v_i$.) Then for any $\sum_{i=1}^{n} a_i g_i, \sum_{i=1}^{n} b_i g_i \in FG$, $\gamma$,

$$
\beta \in \mathbb{F}; \quad \alpha \left( \gamma \sum_{i=1}^{n} a_i g_i + \beta \sum_{i=1}^{n} b_i g_i \right) = \alpha \left( \sum_{i=1}^{n} (\gamma a_i + \beta b_i) g_i \right) = 
$$

$$
\sum_{i=1}^{n} (\gamma a_i + \beta b_i) \alpha(g_i) = \sum_{i=1}^{n} (\gamma a_i \beta b_i) v_i = 
$$

$$
\gamma \alpha \left( \sum_{i=1}^{n} a_i g_i \right) + \beta \alpha \left( \sum_{i=1}^{n} b_i g_i \right); \text{ so } \alpha \text{ is a linear transformation from } FG \text{ to } V.
$$

To see that $\alpha$ is 1-1, and hence an isomorphism, let $\alpha \left( \sum_{i=1}^{n} a_i g_i \right) = 

\alpha \left( \sum_{i=1}^{n} b_i g_i \right)$. Then $\sum_{i=1}^{n} a_i V_i = \sum_{i=1}^{n} b_i V_i$, and since any vector can be uniquely written in $V$ as a linear combination
of the $v_i, i=1, \ldots, n$, $a_i = b_i$ for $i = 1, \ldots, n$. So $\sum_{i=1}^{n} a_i g_i = \sum_{i=1}^{n} b_i g_i$.

To see that $\alpha \circ \rho(g) = \bar{\eta}(g) \circ \alpha$ for any $g \in G$, let $g_i \in FG$ and compute as follows: $(\alpha \circ \rho(g))(g_i) = \alpha(\rho(g)(g_i))$

$= \alpha(gg_i) = \alpha(g_j) = V_j$, where we have let $gg_i = g_j$ for some $j = 1, \ldots, n$.

On the other hand, $(\bar{\eta}(g) \circ \alpha))(g_i) = \bar{\eta}(g)(\alpha(g_i)) = \bar{\eta}(g)(V_i) = \eta(\pi(g))(V_i) = V_{\pi(g)}(i)$. The only hurdle remaining is the fact that two different common notations have been used here for permutations acting on groups. The "i" in the subscript $\pi(g)(i)$ represents $g_i \in G$. So $\pi(g)(i) \equiv \pi(g)(g_i) = gg_i = g_j \equiv j$, where, as above, we let $gg_i = g_j$. Hence, $V_{\pi(g)}(i) = V_j$, and the diagram commutes as required. $\square$
Given a representation \((V, \rho)\) of \(G\) over \(F\), some natural questions can arise: Are there nontrivial subspaces of \(V\) that can serve, under the action of \(\rho\), as representation spaces of \(G\)? How are representations on such subspaces of \(V\) related to each other and to \((V, \rho)\)? How is the representation of a subgroup of \(G\) related to the representation of \(G\)? Much of the material in Chapters 3 and 4 concerns these questions.

### 3.1 Subrepresentations

**3.1 Definition:** Let \((V, \rho)\) be a representation of \(G\) over \(F\). A vector subspace \(W\) of \(V\) is invariant under the action of \(G\) provided \(\rho(g)(w) \in W\) for all \(g \in G\) and \(w \in W\).

Equivalently, \(W\) is said to be **stable** under \(G\) or under \(\rho\).

**3.2 Definition:** For \(W\) invariant under \(\rho\), let \(\rho^W\) be the restriction of \(\rho\) to \(W\), i.e., \(\rho^W = \rho|_W : G \to GL(W)\).

In light of these definitions we have:

**3.3 Theorem:** If \(W\) is invariant under \(\rho\), then \(\rho^W : G \to GL(W)\) is a representation of \(G\) over \(F\).
Proof: It is clear that $\rho^W(g)$ is a linear transformation on $W$ for all $g \in G$. All that needs to be shown, then, is that $\rho^W$ is a group homomorphism.

Let $g, g' \in G$ and $w \in W$. Then: $\rho^W(gg')(w) = \rho(g)(\rho(g')(w))$, since $w \in V$ and $(V, \rho)$ is a representation of $G$. But $\rho(g)(\rho(g')(w)) = (\rho^W(g) \circ \rho^W(g'))(w)$, since $V$ is invariant under $\rho$ and $w \in W$. Therefore $\rho^W$ is a representation of $G$ over $F$. 

It will be common in the remainder of this paper to denote a representation by the representation space. For instance, in the discussion above, we have that $V$ and $W$ are representations of $G$ over $F$. In fact, we say that $W$ is a subrepresentation of $V$, or $W$ is a $G$-subspace of $V$.

From linear algebra, if $W$ is a subspace of $V$, then the quotient space $V/W$ is also a vector space. Given a representation $(V, \rho)$ of $G$, we can define $(V/W, \tilde{\rho})$ the quotient representation of $G$ induced by $\rho$.

3.4 Theorem: Let $(V, \rho)$ be a representation of $G$ over $F$, and let $W$ be a subrepresentation of $V$. For all $g \in G$, define $\tilde{\rho}(g) : V/W \to V/W$ by $\tilde{\rho}(g)(v + W) = \rho(g)(v) + W$ for each $v + W \in V/W$. Then:

(1) $\tilde{\rho}(g)$ is a linear transformation of $V/W$;

(2) $\tilde{\rho}(g)$ is one-to-one (hence, $\tilde{\rho}(g)$ is invertible and onto, i.e., $\tilde{\rho}(g) \in \text{GL}(V/W)$); and

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(3) \((V/W, \tilde{\rho})\) is a representation of \(G\) over \(F\).

**Proof:**

(1) That \(\tilde{\rho}(g) : V/W \rightarrow V/W\) is a linear transformation is a direct consequence of the vector space properties of \(V/W\), and the fact that \(\rho(g) : V \rightarrow V\) is a linear transformation.

Let \(\alpha, \beta \in F\) and \(v_1, v_2 \in V\), \(g \in G\). Then:
\[
\tilde{\rho}(g) (\alpha (v_1 + W) + \beta (v_2 + W)) = \tilde{\rho}(g) ((\alpha v_1 + \beta v_2) + W) = \\
(\alpha \rho(g)(v_1) + \beta \rho(g)(v_2)) + W = (\alpha \rho(g)(v_1) + W) + (\beta \rho(g)(v_2) + W) = \\
\alpha\tilde{\rho}(g)(v_1 + W) + \beta\tilde{\rho}(g)(v_2 + W).
\]
So \(\tilde{\rho}(g)\) is a linear transformation on \(V/W\).

(2) Let \(v_1, v_2 \in V\), \(g \in G\), and suppose that \(\rho(g)(v_1 + W) = \rho(g)(v_2 + W)\). Then
\[
\rho(g)(v_1) + W = \rho(g)(v_2) + W; \text{ which means}
\]
\[
\rho(g)(v_1) - \rho(g)(v_2) \in W. \text{ So } \rho(g)(v_1 - v_2) \in W.
\]
But this means that \(\rho(g^{-1})(\rho(g)(v_1 - v_2)) \in W\) since \(W\) is a subrepresentation of \(G\). Now, \(\rho(g^{-1})(\rho(g)(v_1 - v_2)) = v_1 - v_2\), so we have \(v_1 - v_2 \in W\), which implies that
\[
v_1 + W = v_2 + W. \text{ Hence } \tilde{\rho}(g) \text{ is one-to-one}
\]
and \(\tilde{\rho}(g) \in GL(V/W)\).

(3) Let \(g, g' \in G\) and \(v + W \in V/W\).

Then:
\[
\tilde{\rho}(gg')(v + W) = \rho(gg')(v) + W = \\
(\rho(g) \circ \rho(g'))(v) + W = \rho(g)(\rho(g')(v)) + W = \\
\tilde{\rho}(g)(\rho(g')(v) + W) = \tilde{\rho}(g)(\tilde{\rho}(g')(v + W)) =
\]

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(\tilde{\rho}(g) \circ \tilde{\rho}(g'))(v + W). Therefore (V/W, \tilde{\rho}) is a representation of G over F. \quad \square

If G is a group with \(|G| \geq 2\) (\(|G| \equiv \text{order of } G\)), the regular representation \((FG, \rho)\) always has a nontrivial subrepresentation \((W, \rho^W)\). Let W be the one-dimensional subspace of FG with basis element \(x = \sum g\).

Then \(W = \{ax | a \in F\}\). To see that W is, indeed, invariant under \(\rho\), let \(g \in G, w \in W\) and compute: \(\rho^W(g)(w) = \rho(g)(w) = \rho(g)(ax)\), for some \(a \in F\). But, under the regular representation of G, \(\rho(g)(ax) = a(gx)\). However, \(gx = x\), so \(a(gx) = ax = w \in W\). The subrepresentation W is called the unit representation. It will be of importance later when we, in some sense, characterize all of the subrepresentations of the regular representation.

3.2 Direct Sums

Suppose V is a vector space with subspaces W and W'. Recall from linear algebra that V is the direct sum of W and W' provided each \(v \in V\) can be uniquely written as \(v = w + w'\), where \(w \in W\) and \(w' \in W'\). Equivalently, \(W \cap W' = \{0\}\) and \(\dim V = \dim W + \dim W'\). We write \(V = W \oplus W'\) and call W' a complement of W in V. Also, the map \(p : V \to W\) given by \(p(v) = p(w + w') = w\) is called the projection of V onto W.

In terms of representations, we may ask the following: If \(V\) is a representation of G and W is a subrepresentation
of $V$, does there exist a subrepresentation $W'$ of $V$ such that $V = W \oplus W'$? Since $W$ is stable under $G$, it is called an invariant complement of $W$ in $V$.

We shall answer this question in a theorem whose proof will require:

**3.5 Lemma:** There is a one-to-one correspondence between the projections of $V$ onto $W$ and the complements of $W$ in $V$.

**Proof:** Given $V = W \oplus W'$, $W'$ is a complement of $W$ in $V$. Define $p : V \to W$ as above, i.e., for $v \in V$, $p(v) = p(w + w') = w$ where $w \in W, w' \in W'$. The image of $p$ is $W$ and $p(w) = w$ for all $w \in W$. So for each complement $W'$ of $W$, we have a projection $p$ of $V$ onto $W$.

Conversely, suppose $p : V \to W$ is a projection. Then $\ker p = \{v \in V | p(v) = 0\}$. We claim that $V = W \oplus \ker p$, i.e., that $\ker p$ is the desired complement of $W$ in $V$.

It is obvious that $W \oplus \ker p \subset V$. So let $v \in V$ and choose a basis $(v_1, \ldots, v_n)$ for $V$ such that $(v_1, \ldots, v_r)$ ($r \leq n$) is a basis for $W$. Then: $v = a_1v_1 + \ldots + a_rv_r + a_{r+1}v_{r+1} + \ldots + a_nv_n; a_i \in F, i = 1, \ldots, n$.

Let $a_1v_1 + \ldots + a_rv_r = w \in W$ and $a_{r+1}v_{r+1} + \ldots + a_nv_n = x \in W$. Now we have $v = w + x$ with $w \in W$ and $x \in \ker p$ since $p(x) = 0$. So $v \in W \oplus \ker p$ and $V \subset W \oplus \ker p$.

Hence, $V = W \oplus \ker p$ and $\ker p$ is the complement of $W$ corresponding to the projection $p$. □
3.6 Theorem: Let \( (V, \rho) \) be a representation of \( G \) and let \( W \) be a \( G \)-subspace of \( V \). Then there exists an invariant complement \( W' \) of \( W \) in \( V \).

**Proof:** From linear algebra, a subspace \( W \) of \( V \) has a complement in \( V \). Let \( W^0 \) be any complement of \( W \) in \( V \), and by the lemma let \( p \) be the corresponding projection of \( V \) onto \( W \). Define a map \( p' \) to be the average of the conjugates of \( p \) by the elements of \( G \):

\[
p' = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho^{-1}(g).
\]

We will show:

1. \( p' \) is a projection of \( V \) onto \( W \); and
2. The complement of \( W \) corresponding to \( p' \) is the desired invariant complement \( W' \).

**Proof of 1:**

Since \( p \) maps \( V \) into \( W \) and \( \rho(g)(w) \in W \) for every \( g \in G \), \( w \in W \), then \( p' \) maps \( V \) into \( W \).

Now let \( w \in W \). Then \( \rho^{-1}(g)(w) \in W \) since \( W \) is a \( G \)-subspace. So \( p \circ \rho^{-1}(g)(w) = \rho^{-1}(g)(w) \). Thus we have, for each \( g \in G \),

\[
[p(g) \circ p \circ \rho^{-1}(g)](w) = [p(g) \circ \rho^{-1}(g)](w) = w.
\]

Hence, \( p'(w) = w \) and \( p' \) is onto. Therefore \( p' \) is a projection of \( V \) onto \( W \).
Proof of (2):
Again using lemma 3.5, let $W'$ be the complement of $W$ corresponding to $p'$. From the proof of the lemma, $W' = \ker p'$. To show that $W'$ is stable under $G$, we need one more intermediate result, namely, that $p'$ commutes with $\rho(g)$ for every $g \in G$. Let $g' \in G$. Then:

$$\rho(g') \circ p' \circ \rho^{-1}(g') = \frac{1}{|G|} \sum_{g \in G} \rho(g') \circ (\rho(g) \circ p \circ \rho^{-1}(g)) \circ \rho^{-1}(g') =$$

$$\frac{1}{|G|} \sum_{g \in G} \rho(g'g) \circ p \circ \rho^{-1}(g'g).$$

But the last sum can only permute the terms of $\frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho^{-1}(g) = p'$. Hence, $\rho(g') \circ p' = p' \circ \rho(g')$. Now for $w' \in W'$, $g' \in G$, $p'(w') = 0$ since $p'$ is a projection onto $W$. So $p' \circ \rho(g')(w') = \rho(g') \circ p'(w') = 0$. But this means that $\rho(g')(w') \in W'$; hence $W'$ is a $G$-subspace of $V$, and the theorem is proved.$\square$

A couple of comments concerning theorem 3.6 and direct sums in general:
1. If $\text{char } F \mid |G|$, then, for all $a \in F$, $v \in V$, $p'(av) = 0$. We shall henceforth assume $\text{char } F \nmid |G|$.
2. Suppose that $(V, \rho)$ is a representation of $G$ and $V$ can be decomposed into a direct sum of any finite number of $G$-subspaces. That is, $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, where the $W_i$, $i = 1, \ldots, k$, are invariant under $\rho$. For all $g \in G$,
\( \rho_i(g) \in \text{GL}(W_i) \), where \( \rho_i(g) \) is the restriction of \( \rho(g) \) to \( W_i \). It is easy to check that \( \rho_i : G \to \text{GL}(W_i) \) is a representation of \( G \). Then we say that \( \rho \) is the direct sum of the \( \rho_i \), \( i = 1, \ldots, k \), and write \( \rho = \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_k \).

### 3.3 Matrices of Subrepresentations

Sections 3.1 and 3.2 examined subrepresentations, and the quotient representation, and direct sums in terms of representation by automorphism. It is both interesting and useful for later developments to see how matrix representations are affected.

Let \((V, \rho)\) be a representation of \( G \) and let \( W \) be a nontrivial \( G \)-subspace of \( V \) such that \( \dim V = n \) and \( \dim W = r \) with \( 0 < r < n \). Then a basis \( B = \{v_1, v_2, \ldots, v_n\} \) for \( V \) can be found such that \( B' = \{v_1, \ldots, v_r\} \) is a basis for \( W \). So for any \( j = 1, 2, \ldots, r \), and any \( g \in G \), \( \rho(g)(v_j) = \rho^W(g)(v_j) = \sum_{i=1}^{r} a_{ij}v_i \). The \( a_{ij} \) are elements of \( F \) that will constitute the \( j \)th column of \([\rho^W(g)]_{B'}\). At this point we have, for the matrix of \( \rho(g) \) relative to the basis \( B \):

\[
[\rho(g)]_B = \begin{bmatrix}
 0 & & & \vdots \\
 & & & \\
& & & \\
& & & \\
& & & \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
 0 & & & \vdots \\
 & & & \ddots \\
 & & & \\
& & & \\
& & & \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1r} \\
a_{21} & a_{22} & \cdots & a_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nr}
\end{bmatrix}
\]

\[
\begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1r} \\
a_{21} & a_{22} & \cdots & a_{2r} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nr}
\end{bmatrix}
\]
What happens to $\rho(g)(v_j)$ for $r + 1 \leq j \leq n$? We are no longer restricted to $W$, so we can write $\rho(g)(v_j) = \sum_{i=1}^{n} a_{ij}v_i$; $r + 1 \leq j \leq n$. Hence $\rho(g)(v_j) = a_{1j}v_1 + \ldots + a_{rj}v_r + a_{r+1j}v_{r+1} + \ldots + a_{nj}v_n$.

Now we must note that the set $B'' = \{v_{r+1} + W, v_{r+2} + W, \ldots, v_n + W\}$ is a basis for the quotient space $V/W$. Hence, for $r + 1 \leq j \leq n$, $\bar{\rho}(g)(v_j) + W = \rho(g)(v_j) + W$. So the $j$th column of $[\bar{\rho}(g)]_{B''}$ will consist of $a_{r+1j}, a_{r+2j}, \ldots, a_{nj}$. We can now write:

$$[\rho(g)]_B = \begin{bmatrix}
0 & \rho^W(g) |_{B'} & \ldots & \rho^W(g) |_{B''} \\
\end{bmatrix}.$$ 

Suppose now that $V = W \oplus W'$ where $W$ and $W'$ are subrepresentations of $G$. Then a basis $B = \{v_1, \ldots, v_n\}$ for $V$ can be chosen so that $B' = \{v_1, \ldots, v_r\}$ and $B'' = \{v_{r+1}, \ldots, v_n\}$ are bases for $W$ and $W'$ respectively.

This leaves $[\rho^W(g)]_{B'}$ as above. But now, since $W'$ is invariant under $\rho$, for $r + 1 \leq j \leq n$, $\rho(g)(v_j) = \rho^W(g)(v_j) = \sum_{i=r+1}^{n} a_{ij}v_i$. So the matrix of $\rho(g)$ relative to the basis $B$
becomes:

$$[P(g)]_B = \begin{bmatrix} \rho^{W_1}(g)_{B'} & 0 \\ 0 & \rho^{W_k}(g)_{B''} \end{bmatrix}$$

We can extend to \( V = W_1 \oplus \ldots \oplus W_k \), where each \( W_i \), \( i=1, \ldots, k \) is a \( G \)-subspace. A basis \( B \) can be found for \( V \) so that \([P(g)]_B\) consists of a submatrix corresponding to the restriction of \( P(g) \) to each \( W_i \) along the diagonal, and zeros elsewhere.
4.1 Irreducible Representations

We have seen that if a representation $V$ has a nontrivial $G$-subspace $W$, then there exists a complement $W'$ of $W$ in $V$ which is also invariant under $G$. Hence, $V$ can be written as the direct sum of two nontrivial representations: $V = W \oplus W'$. $V$ is then said to be a reducible representation.

On the other hand:

4.1 Definition: Let $(V, \rho)$ be a representation of $G$. $V$ (or $\rho$) is said to be irreducible provided $V \neq \{0\}$ and the only $G$-subspaces of $V$ are $\{0\}$ and $V$. Otherwise, $V$ is reducible.

A few observations are immediate:

1. If $V$ is irreducible, then it cannot be written as the direct sum of two nontrivial $G$-subspaces;
2. Any representation of degree 1 is irreducible; and
3. If $V = \{0\}$, then it is reducible. We say that $V$ is the direct sum of the empty family of irreducible representations.

If $V$ is a representation, we should like to know if it can be "broken down" into irreducible representations.

4.2 Theorem: (Maschke) Every representation is a direct sum of irreducible representations.
Proof: Suppose V is a representation of G. The proof is by induction on dim V. But let us look at the first few cases before trying the induction:

(i) If dim V = 0, then V = {0} and the theorem holds by observation ℘ above;

(ii) If dim V = 1 then V is irreducible. So the only sum we may write is V = V ⊕ {0}. But by part (i), the theorem holds for {0}; hence, it is true for V ⊕ {0}. In fact, the theorem is valid for irreducible representations of any dimension.

(iii) Let V be reducible and dim V = 2. Then V can be written as the direct sum V = W₁ + W₂, where W₁ and W₂ are nontrivial G-subspaces. Hence, dim W₁ = dim W₂ = 1 and the theorem holds.

Now suppose V is reducible with dim V = n and n > 2. Our induction hypothesis is that the theorem holds for all reducible representations of dimension less than n. V, being reducible, can be written as V = V' ⊕ V'' where dim V' < n and dim V'' < n. Hence, V' and V'' can be written as direct sums of irreducible representations; and therefore, so can V. ♦

Since theorem 4.2 depends on theorem 3.6 we are operating under the assumption that char F ∤ |G|.

Recall from theorem 1.4 that if (V,ρ) is a representation of G by automorphism and B is a basis for V,
then \( \theta(g) = [\rho(g)]_B \), for all \( g \in G \), is a matrix representation of \( G \). In particular, if \((V,\rho)\) is irreducible, then we will say that \( \theta \) is an **irreducible matrix representation** of \( G \). We then get an interesting corollary to theorem 4.2:

**4.3 Corollary:** Let \( \theta : G \to GL(n,F) \) be a matrix representation of \( G \). Then there exists a fixed matrix \( P \in GL(n,F) \) such that for all \( g \in G \),

\[
P\theta(g)P^{-1} = \begin{bmatrix}
\theta_1(g) & & & 0 \\
& \theta_2(g) & & \\
& & \ddots & \\
0 & \cdots & \cdots & \theta_r(g)
\end{bmatrix}
\]

where the \( \theta_i(g) \) are irreducible matrix representations of \( G \).

**Proof:** As in section 1.2, let \( V = F^n \) and define \( \rho:G \to GL(F^n) \) by \( \rho(g): F^n \to F^n \) where \( \rho(g)(v) = \theta(g)v \) for all \( v \in V \). Then \((V,\rho) = (F^n,\rho)\) is a representation of \( G \) by automorphism.

By theorem 4.2, \( V = V_1 \oplus V_2 \oplus \ldots \oplus V_r \), where the \( V_i \), \( i = 1, \ldots, r \), are irreducible. Denote the restriction of \( \rho \) to \( V_i \) by \( \rho^{V_i} \).

Now, from section 3.3, a basis \( B \) can be chosen for \( V \) such that for \( g \in G \),
where $B_i \subseteq B$ is a basis for $V_i$ with $\bigcup_{i=1}^r B_i = B$ and $B_i \cap B_j = \emptyset$ for $i \neq j$.

Note that each $[\rho^V_i(g)]_{B_i}$ is an irreducible matrix representation of $G$. Let $[\rho^V_i(g)]_{B_i} = \theta_i(g)$ for $i=1, \ldots, r$.

Finally, from section 1.2 again, if $E$ is the standard basis for $F^n$, then $[\rho(g)]_E = \theta(g)$ for all $g \in G$. And by observation 1 following theorem 1.9, $[\rho(g)]_E = \theta(g) \sim [\rho(g)]_B$.

Hence, by the definition of equivalence (1.8), there exists a matrix $P \in GL(n, F)$ such that $P\theta(g) P^{-1} = [\rho(g)]_B$. \square

4.2 Clifford's Theorem

Clifford's theorem is a result about the reducibility of the representation of $H$, where $H$ is a subgroup of the group $G$. However, this development does not rely on theorem 3.6; in fact, we need make no assumptions about the relationship of char $F$ to $|H|$ or $|G|$.

Some preliminary definitions and lemmas are needed.

4.4 Definition: Let $(V, \rho)$ be a representation of $G$ over $F$ and let $H$ be any subgroup of $G$. The representation of $H$ by restricting $\rho$ to $H$ is denoted by $\text{Res}_H^G \rho: H \rightarrow GL(V_H)$.

The representation space $V_H$ is the same as $V$ as a vector space, but we define only the action of $H$ on $V_H$. $V_H$, then, is an $H$ - module.
4.5 Lemma: Let $H$ be a normal subgroup of $G$ ($H \Delta G$), and let $\sigma: H \rightarrow GL(W)$ be a representation of $H$. For any $g \in G$, the map $\sigma^g$ on $H$ given by $\sigma^g(h) = \sigma(ghg^{-1})$ is a representation of $H$.

**Proof:** First note that since $H \Delta G$, $ghg^{-1} \in H$ for any $g \in G$, $h \in H$. Thus, $\sigma^g(h) \in GL(W)$.

Now let $h, h' \in H$. Then we simply check:

$$\sigma^g(hh') = \sigma(ghh'g^{-1}) = \sigma(gh[gh^{-1}]h'g^{-1}) = \sigma(gh^{-1}) \circ \sigma(h') = \sigma^g(h) \circ \sigma^g(h').$$

So $\sigma^g$ is a group homomorphism from $H$ to $GL(W)$ and, hence, a representation of $H$. \( \Box \)

4.6 Definition: The representation of $\sigma^g$ (or any equivalent to it) is called a **conjugate** of $\sigma$.

4.7 Lemma: Let $\sigma: H \rightarrow GL(W)$ be a representation of $H(\Delta G)$. Then:

1. If $U$ is an invariant subspace of $W$ under $\sigma^g$ for $g \in G$, then $U$ is invariant under $\sigma$; and

2. If $\sigma$ is irreducible, then $\sigma^g$ is irreducible for any $g \in G$.

**Proof:** 1. Let $U$ be an invariant subspace of $W$ under $\sigma^g$ for $g \in G$. Then for all $u \in U$, $h \in H$, $\sigma^g(h)(u) \in U$.

Now, for any $u \in U$, $h \in H$, $g \in G$, $\sigma(h)(u) = \sigma(gh'g^{-1})(u)$, where $h' \in H$ such that $gh'g^{-1} = h$. We know that such an $h'$ exists since $H \Delta G \Rightarrow ghg^{-1} = H$. 45
But \( \sigma(g'h'g^{-1})(u) = \sigma(g)(h')(u) \in U \) since \( U \) is invariant under \( \sigma^g \). Therefore \( \sigma(h)(u) \in U \) and \( U \) is invariant under \( \sigma \).

2. Suppose \( \sigma \) is irreducible. The \( \sigma^g \) must also be irreducible. For, from (1), any invariant subspace of \( W \) under \( \sigma^g \) must be invariant under \( \sigma \). But the only invariant subspaces under \( \sigma \) are \( \{0\} \) and \( W \).

4.8 Theorem: (Clifford)

Let \( H \Delta G \), let \( (V, \rho) \) be an irreducible representation of \( G \), and let \( W \) be an irreducible \( H \)-submodule of \( V_H \) via 

\[
\text{Res}_H^G : \sigma : H \to \text{GL}(W). \]

Then \( V_H \) is a direct sum of irreducible \( H \)-submodules, each of which is conjugate to \( (W, \sigma) \).

**Proof:** First we show that for each \( g \in G \), \( \rho(g)W \) (the image of \( W \) under \( \rho(g) \)) is an \( H \)-submodule of \( V_H \). To see that \( \rho(g)W \) is invariant, let \( h \in H \), and \( y \in W \), Then:

\[
(\text{Res}_H^G)(h)(\rho(g)(y)) = \rho(h)(\rho(g)(y)) = \rho(hg)(y) = \rho(g^{-1}hg)(y) = \rho(g)(\rho(g^{-1}h)(y)); \]

where \( h' = g^{-1}hg \) and \( h' \in H \) since \( H \Delta G \). But \( \rho(g)(\rho(h')(y)) = \rho(g)((\text{Res}_H^G)(h')(y)) \in \rho(g)W \); since \( y \in W \), an invariant subspace of \( V_H \) under \( \text{Res}_H^G \). Let \( \sigma_g' : H \to \text{GL}(\rho(g)W) \) be the representation of \( H \) with representation space \( \rho(g)W \). The relationship \( (\text{Res}_H^G)(h)(\rho(g)(y)) = \rho(g)((\text{Res}_H^G)(h')(y)) \), where \( h' \) is conjugate to \( h \), will be needed again, call it equation 4.9.

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Next we show that $\sigma_g' = \sigma^g$. For $g \in G$, define a map $\alpha : W \to \rho(g)W$ by $\alpha(y) = \rho(g)(y)$ for all $y \in W$. Since $\rho(g)W$ is just the image of $W$ under $\rho(g)$, $\alpha$ is clearly a vector space isomorphism. To establish the desired equivalence, we must show that, for all $h \in H$, $\alpha \circ \sigma^g(h) = \sigma_g'(h) \circ \alpha$.

For any $y \in W$, $(\alpha \circ \sigma^g(h))(y) = \alpha(\sigma(ghg^{-1})(y)) = \rho(g)(\sigma(ghg^{-1})(y)) = \rho(g)[(\text{Res}_H^G)(ghg^{-1})(y)] = (\text{Res}_H^G)(h)(\rho(g)(y))$ by equation 4.9. Now, since $\rho(g)(y) \in \rho(g)W$, $(\text{Res}_H^G)(h)(\rho(g)(y)) = (\sigma_g'(h) \circ \alpha)(y)$. So we have $(\rho(g)W, \sigma_g') = (W, \sigma^g)$.

Now, by hypothesis, $(W, \sigma)$ is irreducible, and since $(\rho(g)W, \sigma_g') = (W, \sigma^g)$, $(\rho(g)W, \sigma_g')$ is conjugate to $(W, \sigma)$. Hence, applying lemma 4.7, $(\rho(g)W, \sigma_g')$ is irreducible.

All that remains is to show that $V_H = V = \bigoplus_{g \in G} \rho(g)W$. Since $(V, \rho)$ is an irreducible representation of $G$, it suffices to show that $\bigoplus_{g \in G} \rho(g)W$ is invariant under the action of $G$.

Let $|G| = n$ and fix some ordering of the elements of $G : g_1, g_2, \ldots, g_n$. Let $w_i \in \rho(g_i)W$. Then for any $g_j \in G$, $w \in \bigoplus_{g_i \in G} \rho(g_i)W$: 47
\[ \rho(g_j)(w) = \rho(g_j)(w_1 + w_2 + \ldots + w_n) = \rho(g_j)(w_1) + \rho(g_j)(w_2) + \ldots + \rho(g_j)(w_n) = \rho(g_j)(\rho(g_1)(w^{(1)})) + \rho(g_j)(\rho(g_2)(w^{(2)})) + \ldots + \rho(g_j)(\rho(g_n)(w^{(n)})) = \]
\[ \bigoplus_{g_i \in G} \rho(g_i)W_i \] where \( w^{(i)} \in W \) such that \( \rho(g_i)(w^{(i)}) = w_i \).

So \( \bigoplus_{g \in G} \rho(g)W = V_H \), where each \( (\rho(g)W, \sigma_g) \) is irreducible and conjugate to \( (W, \sigma) \). \( \square \)

Note from the hypotheses of Clifford's theorem that we were given one irreducible \( H \)-submodule. In that case, we can now know all of the irreducible \( H \)-submodules of \( V_H \).

### 4.3 Schur's Lemma

This section is more a foreshadowing of Chapter 5 than a continuation of material in Chapter 4. Still, getting some of the very technical computations done here will facilitate a smoother flow of results about characters in Chapter 5.

We first record some definitions and properties (without proof) of the trace of a matrix or linear transformation.

**4.10 Definition:** If \( A = (a_{ij}) \) is any \( nxn \) matrix over a field \( F \), then we denote the trace of \( A \) by \( tr(A) \) and \( \text{tr}(A) = \sum_{i=1}^{n} a_{ii} \).

**4.11 Theorem:** For \( nxn \) matrices \( A, B \) over \( F \) and \( \lambda \in F \):

1. \( tr(\lambda A) = \lambda tr(A) \);
2. \( tr(A + B) = tr(A) + tr(B) \); and
3. \( tr(AB) = tr(BA) \).

A direct consequence of 3 above is

**4.12 Corollary:** If \( A \in GL(n, F) \) (i.e. \( A \) is invertible), then \( tr(ACA^{-1}) = tr(C) \) for an \( nxn \) matrix \( C \) over \( F \).

**4.13 Definition:** If \( V \) is a finite dimensional vector space with basis \( B \) and \( T \) is any linear transformation on \( V \), then we define the **trace of \( T \)** by \( tr(T) = tr[T]_B \).

From definition 4.13 and corollary 4.12, we have that \( tr(T) \) does not depend on the choice of basis for \( V \).

For the proof of Schur's lemma, it is also necessary to recall that \( \lambda \in F \) is an **eigenvalue** of \( T \) if there exists a \( v \in V \) with \( v \neq 0 \) and \( T(v) = \lambda v \).

We shall need the following properties of eigenvalues:

**4.14 Theorem:** If \( T \) is a linear transformation on \( V \) over \( F \), then \( tr(T) \) is equal to the sum of the eigenvalues of \( T \) counted with their multiplicities.

**4.15 Theorem:** If \( \lambda \) is an eigenvalue of an invertible linear transformation \( T \), then \( \lambda^{-1} \) is an eigenvalue of \( T^{-1} \).

**4.16 Lemma:** (Schur) Let \( \rho^1 : G \to GL(V_1) \) and \( \rho^2 : G \to GL(V_2) \) be irreducible representations of \( G \) over \( F = \mathbb{C} \). Let \( f : V_1 \to V_2 \) be a linear transformation such that \( \rho^2(s) \circ f = f \circ \rho^1(s) \) for all \( s \in G \). Then:

1. If \( \rho^1 \) and \( \rho^2 \) are not equivalent, then \( f = 0 \); and
2. If \( V_1 = V_2 \) and \( \rho^1 = \rho^2 \), then \( f \) is a scalar multiple of the identity map on \( V_1 \).
Proof: We prove the contrapositive of (1). Suppose \( f \neq 0 \). Let \( W_1 = \ker f = \{ x \in V_1 \mid f(x) = 0 \} \). If \( x \in W_1 \), then \( f(\rho^1(s)(x)) = \rho^2(s)(f(x)) = \rho^2(s)(0) = 0 \). So for \( x \in W_1 \), \( \rho^1(s)(x) \in W_1 \) and \( W_1 \) is invariant under \( \rho^1 \).

Since \( V_1 \) is irreducible, either \( W_1 = \{0\} \) or \( W_1 = V_1 \). But if \( W_1 = V_1 \), then \( f = 0 \), contrary to our assumption. So \( W_1 = \{0\} \), which means that \( f \) is one-to-one and an isomorphism of \( V_1 \) into \( V_2 \).

Now let \( W_2 = \text{Im} f = \{ f(x) \mid x \in V_1 \} \). For \( y \in W_2 \), there is an \( x \in V_1 \) such that \( f(x) = y \). So, for \( s \in G \),

\[
(\rho^2(s) \circ f)(x) = \rho^2(s)(y) = (f \circ \rho^1(s))(x) = f(\rho^1(s)(x)) \in W_2.
\]

Hence, \( W_2 \) is invariant under \( \rho^2 \), so \( W_2 = V_2 \) or \( W_2 = \{0\} \). But if \( W_2 = \{0\} \), then \( f = 0 \). It must be then, that \( W_2 = V_2 \). Now \( f \) is an isomorphism of \( V_1 \) onto \( V_2 \), whence \( (V_1, \rho^1) = (V_2, \rho^2) \).

For (2), suppose that \( V_1 = V_2 \) and \( \rho^1 = \rho^2 \). Let \( \lambda \) be an eigenvalue of \( f \). Define \( f' : V_1 \to V_2 (= V_1) \) by \( f' = f - \lambda \) (\( = f - \lambda \text{Id}_V \)). Now since \( \lambda \) is an eigenvalue of \( f \), there is a \( v \in V_1 \) with \( v \neq 0 \) and \( f(v) = \lambda v \). Then \( f'(v) = f(v) - \lambda v = \lambda v - \lambda v = 0 \). So \( v \in \ker f' \) and \( \ker f' \neq \{0\} \).

But we also have, for any \( s \in G \) and any \( v' \in V_1 = V_2 \),

\[
(\rho^2(s) \circ f')(v') = \rho^2(s)(f'(v')) = \rho^2(s)((f - \lambda)(v')) = \rho^2(s)(f(v') - \lambda v') = (\ast) \rho^2(s)(f(v')) - \rho^2(s)(\lambda v').
\]

On the other hand, \( (f' \circ \rho^1(s))(v') = (f - \lambda)(\rho^1(s)(v')) = f(\rho^1(s)(v')) - \lambda \rho^1(s)(v') = (\ast\ast) f(\rho^1(s)(v')) - \rho^1(s)(\lambda v'). \)
Comparing (*) and (**), $\rho^2(s)(f(v')) = f(\rho^1(s)(v'))$ by hypothesis. And, since $\rho^1 = \rho^2$, we have $\rho^2(s)(\lambda v') = \rho^1(s)(\lambda v')$. Therefore $\rho^2(s) \circ f' = f' \circ \rho^1(s)$.

Now we may argue as in the proof of 1 that the ker $f'$ is invariant under $\rho^1$, and since ker $f \neq \{0\}$, then ker $f' = V_1$. Hence, $f' = 0$ and so, by definition of $f'$, $f = \lambda$, a scalar multiple of the identity. 2

We shall prove three corollaries to Schur’s lemma. Two of them involve quite technical matrix computations, but will be most useful in the discussion of characters.

**4.17 Corollary:** Let $(V_1, \rho^1)$ and $(V_2, \rho^2)$ be irreducible representations of $G$ over $F = \mathbb{C}$. Let $h \neq 0$ be a linear transformation of $V_1$ into $V_2$ and define:

$$h' = \frac{1}{|G|} \sum_{t \in G} \rho^2(t)^{-1} \circ h \circ \rho^1(t).$$

Then:

1. If $\rho^1$ and $\rho^2$ are not equivalent, then $h' = 0$; and
2. If $V_1 = V_2$ and $\rho^1 = \rho^2$, then $h' = \frac{1}{n} \text{tr}(h)$, where $n = \dim V_1$.

**Proof:**

1. Clearly, $h'$ is a linear transformation from $V_1$ into $V_2$.

We need that $\rho^2(s) \circ h' = h' \circ \rho^1(s)$ for all $s \in G$. Note that

$$\rho^2(s)^{-1} \circ h' \circ \rho^1(s) =$$

$$\rho^2(s)^{-1} \circ \left( \frac{1}{|G|} \sum_{t \in G} \rho^2(t)^{-1} \circ h \circ \rho^1(t) \right) \circ \rho^1(s) =$$
\[
\frac{1}{|G|} \sum_{t \in G} \rho^2(s)^{-1} \circ \rho^2(t)^{-1} \circ h \circ \rho^1(t) \circ \rho^1(s) = \\
\frac{1}{|G|} \sum_{t \in G} \rho^2(ts)^{-1} \circ h \circ \rho^1(ts).
\]

Now, as \( t \) runs over the elements of \( G \), so does the product \( ts \). So \( \sum_{t \in G} \rho^2(ts)^{-1} \circ h \circ \rho^1(ts) = h' \). Hence, \( h' \circ \rho^1(s) = \rho^2(s) \circ h' \). So we can apply part \( \text{i} \) of Schur's lemma with \( f = h' \). Therefore \( h' = 0 \).

For \( \text{ii} \), let \( h' = \lambda \), a scalar multiple of the identity. Now we apply some properties of the trace of a linear transformation: Since \( \rho^1 = \rho^2 \),
\[
\text{tr} \ h' = \text{tr} \left( \frac{1}{|G|} \sum_{t \in G} \rho^1(t)^{-1} \circ h \circ \rho^1(t) \right) = \\
\frac{1}{|G|} \sum_{t \in G} \text{tr}(\rho^1(t)^{-1} \circ h \circ \rho^1(t)) = \frac{1}{|G|} \sum_{t \in G} \text{tr}(h) = \text{tr}(h).
\]

On the other hand, since \( \lambda \) is a scalar multiple of the identity, \( \text{tr}(h') = \text{tr}(\lambda) = n\lambda \), where \( n = \text{dim} \ V_1 \). Comparing the two expressions for \( \text{tr}(h') \), we have \( \lambda = \frac{1}{n} \text{tr}(h) \). \( \square \)

Now let \( B_1 \) and \( B_2 \) be bases of \( V_1 \) and \( V_2 \), respectively. The particular bases chosen have no bearing on the results that follow. We choose them simply because the matrix representations corresponding to \( \rho^1 \) and \( \rho^2 \) must be computed relative to some basis. For \( t \in G \), let \( [\rho^1(t)]_{B_1} = (a_{ij}(t)) \) and \( [\rho^2(t)]_{B_2} = (a_{k1}(t)) \).
The matrices of the linear transformations $h$ and $h'$ from $V_1$ into $V_2$ depend on both $B_1$ and $B_2$. So let $[h]_{B_1,B_2} = (x_{ki})$ and $[h']_{B_1,B_2} = (x'_{ki})$.

Then the $k, i$ entry of $(x'_{ki})$ is, by definition of $h'$:

\[(4.18) \quad x'_{ki} = \frac{1}{|G|} \sum_{t,j,l} a_{kl}(t^{-1}) x_{lj} a_{ji}(t). \]

The composition of maps is now expressed as matrix multiplication. The products $a_{kl}(t^{-1}) a_{ji}(t)$ can be considered as coefficients in a linear form with respect to $x_{lj}$. 

We can now state the second corollary to Schur's lemma.

**4.19 Corollary:** Let $\rho^1, \rho^2, h, \text{ and } h'$ be as in corollary 4.17. Suppose $\rho^1$ and $\rho^2$ are not equivalent. Then

\[\frac{1}{|G|} \sum_{t \in G} a_{kl}(t^{-1}) a_{ji}(t) = 0 \text{ for all } i, j, k, l.\]

**Proof:** Under these hypotheses, corollary 4.15 gives $h' = 0$. So $x'_{ki} = 0$ for all $k$ and $i$. This means that each term on the right side of equation 4.18 is zero. Since $h \neq 0$, we must have the products $a_{kl}(t^{-1}) a_{ji}(t) = 0$ for all $j$ and $l$; hence

\[\frac{1}{|G|} \sum_{t \in G} a_{kl}(t^{-1}) a_{ji}(t) = 0 \text{ for all } i, j, k, \text{ and } l.\]

And finally:

**4.20 Corollary:** Let $\rho^1, \rho^2, h, \text{ and } h'$ be as in corollary 4.17. Suppose $V_1 = V_2$ and $\rho^1 = \rho^2$. Let $n = \dim V_1$. Then:

\[\frac{1}{|G|} \sum_{t,j,l} a_{kl}(t^{-1}) a_{ji}(t) = \frac{1}{n} \delta_{ki} \delta_{lj} = \begin{cases} \frac{1}{n} & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}.\]
Proof: By corollary 4.17, $h' = \lambda$. So $x'_{ki} = \lambda \delta_{ki}$. But

$$\lambda = \frac{1}{n} \text{tr}(h) = \frac{1}{n} \sum_{j=1}^{n} x_{jj} = \frac{1}{n} \sum_{i,j} \delta_{ki} \delta_{lj}.$$ Then: $x'_{ki} = \delta_{ki} \lambda = \delta_{ki} \left( \frac{1}{n} \sum_{i,j} \delta_{lj} x_{1j} \right) = (*) \frac{1}{n} \sum_{i,j} \delta_{ki} \delta_{lj} x_{1j}$; another linear form with respect to $x_{1j}$.

For any $i,j$, let us now equate the coefficients from (*)

and equation 4.18

$$\frac{1}{|G|} \sum_{t \in G} a_{kj}(t^{-1})a_{ji}(t) = \frac{1}{n} \delta_{ki} \delta_{lj} = \begin{cases} \frac{1}{n} & \text{if } k = i \text{ and } l = j \quad \square \\ 0 & \text{otherwise} \end{cases}$$

It will soon be evident in Chapter 5 that these results will yield much information about representations.
Chapter 5
Character Theory

5.1 Definition and Basic Properties

In the hypotheses of Schur's lemma (4.16) we let $F = \mathbb{C}$, the field of complex numbers. We shall continue with that assumption. The following definition will result in much information about group representations and their irreducible constituents.

5.1 Definition: Let $\rho: G \rightarrow \text{GL}(V)$ be a representation of $G$ over $F$. Then the character of (or afforded by) $\rho$ is the mapping $\chi: G \rightarrow F$ given by $\chi(s) = \text{tr}(\rho(s))$, for all $s \in G$.

We can already begin to see the importance of characters in:

5.2 Theorem: Let $(V_1, \rho_1)$ and $(V_2, \rho_2)$ be equivalent representations of $G$ with characters $\chi_1$ and $\chi_2$, respectively. Then $\chi_1(s) = \chi_2(s)$ for all $s \in G$, i.e., equivalent representations have the same characters.

Proof: Since $\rho_1 = \rho_2$, there exists a vector space isomorphism $\alpha : V_1 \rightarrow V_2$ such that $\alpha \circ \rho_1(s) = \rho_2(s) \circ \alpha$, for all $s \in G$. That is,

$$\rho_1(s) = \alpha^{-1} \circ \rho_2(s) \circ \alpha.$$ 

So, $\text{tr}(\rho_1(s)) = \text{tr}(\alpha^{-1} \circ \rho_2(s) \circ \alpha)$. But from the properties of trace (corollary 4.12 and definition 4.13),

$$\text{tr}(\alpha^{-1} \circ \rho_2(s) \circ \alpha) = \text{tr}(\rho_2(s)).$$

Hence $\chi_1(s) = \chi_2(s)$ for all
(The converse of this theorem is true and its proof will be given later.)

Let us also get our first insight into the behavior of characters of subrepresentations:

5.3 Theorem: Let \((V, \rho)\) be a representation of \(G\) with \(G\)-subspaces \(W_1\) and \(W_2\) such that \(V = W_1 \oplus W_2\). Let \(\chi, \chi_1\) and \(\chi_2\) be the characters of \(V, W_1,\) and \(W_2,\) respectively. Then \(\chi = \chi_1 + \chi_2\).

Proof: As in section 3.3, appropriate bases \(B, B_1, B_2\) for \(V, W_1, W_2\) (resp.), can be found so that:

\[
[\rho(g)]_B = \begin{bmatrix}
[\rho^{W_1}(g)]_{B_1} & 0 \\
0 & [\rho^{W_2}(g)]_{B_2}
\end{bmatrix}
\text{ for all } g \in G.
\]

Now simply note that the trace of the left-hand side is \(\chi(g)\), while the trace of the right-hand side is \(\chi_1(g) + \chi_2(g)\). □

Also as in section 3.3, this result may be extended to any finite number of \(G\)-subspaces of \(V\).

We close this section with a lemma that will be called on from time to time:

5.4 Lemma: Let \(\chi\) be the character of representation \((V, \rho)\) of degree \(n\). Then:
1. $\chi(1_G) = n$ where $1_G$ is the identity of $G$;
2. $\chi(s^{-1}) = \overline{\chi(s)}$ for all $s \in G$, where $\overline{\chi(s)}$ denotes the complex conjugate of $\chi(s)$; and
3. $\chi(tst^{-1}) = \chi(s)$ for all $s, t \in G$.

**Proof:**

1. Since $\rho(1) = \text{Id}_V$, we have $\chi(1) = \text{tr} (\text{Id}_V) = n$.
2. For all $s \in G$, $\rho(s)$ is an element of the group $\text{GL}(V)$. If the order of $s \in G$ is $m$, then the same is true of $\rho(s)$:

   $$[\rho(s)]^m = \rho(s^m) = \text{Id}_V.$$  

   If $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ are eigenvalues of $\rho(s)$ with eigenvectors $v_1, \ldots, v_n$, then

   $$(\rho(s))^m(v_i) = \lambda_i^m v_i = v_i,$$  

   for $i = 1, \ldots, n$.

   Hence, $\lambda_i^m = 1$, or $\lambda_i$ is an $m$th root of unity. But this means that $|\lambda_i| = 1$ and $\lambda_i^{-1} = \overline{\lambda_i}$. Then:

   $$\chi(s) = \text{tr}(\rho(s)) = (\sum_{i=1}^{n} \lambda_i) = \sum_{i=1}^{n} \overline{\lambda_i} = \sum_{i=1}^{n} \lambda_i^{-1} = \text{tr}(\rho(s)^{-1})$$

   $$= \text{tr}(\rho(s^{-1})) = \chi(s^{-1}).$$

   We have used theorems 4.14 and 4.15 in the above string of equalities.

3. We have that $\chi(tst^{-1}) = \text{tr}(\rho(tst^{-1})) = \text{tr}(\rho(t) \circ \rho(s) \circ \rho(t)^{-1}) = \text{tr}(\rho(s)) = \chi(s)$ from the properties of the trace function. $\Box$

**5.2 The Space of Class Functions**

Any function $f : G \rightarrow \mathbb{C}$ such that $f(tst^{-1}) = f(s)$, for $s, t \in G$, is called a class function. The name is due to the
fact that \( f \) is constant on the elements of a given conjugacy class of \( G \). The character \( \chi \) of any representation of \( G \) is, therefore, a class function. We seek to impose a vector space structure on \( \text{CG} = \{ f \mid f \text{ is a class function on } G \} \), which contains the characters of \( G \) as a subset.

Let \( f, h \in \text{CG} \) and \( s \in G \). Define \( f + h \) by \( (f + h)(s) = f(s) + h(s) \). Then if \( t, s \in G \), \( (f + h)(tst^{-1}) = f(tst^{-1}) + h(tst^{-1}) = f(s) + h(s) = (f + h)(s) \). Hence \( f + h \in \text{CG} \).

Also, for \( \alpha \in \mathbb{F} = \mathbb{C} \), \( f \in \text{CG} \), \( s \in G \), define \( \alpha f \) by \( (\alpha f)(s) = \alpha(f(s)) \). Then, for \( t, s \in G \), \( (\alpha f)(tst^{-1}) = \alpha(f(tst^{-1}) = \alpha(f(s)) = (\alpha f)(s) \). So \( \alpha f \in \text{CG} \).

We will omit the routine verification of the vector space axioms for \( \text{CG} \). However, there is much more to say about this space.

Define \( (\ , \ ) : \text{CG} \times \text{CG} \to \mathbb{C} \) by: \( (f,h) = \frac{1}{|G|} \sum_{t \in G} f(t) \overline{h(t)} \) where \( f, h \in \text{CG} \).

5.5 Theorem: For any \( f, h, k \in \text{CG} \), \( \alpha, \beta \in \mathbb{F} \), we have:

1. \( (f, h) = (h, f) \);
2. \( (f, f) \geq 0 \) with equality iff \( f = 0 \); and
3. \( (\alpha f + \beta h, k) = \alpha(f, k) + \beta(h, k) \).

That is, \( (\ , \ ) \) is an inner product on \( \text{CG} \).

Proof:

Let \( f, h, k \in \text{CG} \), \( \alpha, \beta \in \mathbb{F} \).

1. \( (h, f) = \frac{1}{|G|} \sum_{t \in G} h(t) \overline{f(t)} = \)
\[ \frac{1}{|G|} \sum_{t \in G} h(t) f(t) = \frac{1}{|G|} \sum_{t \in G} h(t) f(t) = (f, h). \]

2. \((f, f) = \frac{1}{|G|} \sum_{t \in G} f(t)^2 \). Since \( f : G \to \mathbb{C} \), \( f(t) = a + bi \) and \( f(t) = a - bi \) for \( t \in G \) and some \( a, b \in \mathbb{R} \).

Hence, each term of the sum \( \sum_{t \in G} f(t)^2 \) has the form \( a^2 + b^2 \); so \((f, f) \geq 0\). Further, the sum is zero iff each term is zero iff \( f(t) = 0 \) for each \( t \in G \) iff \( f(t) = 0 \) for all \( t \in G \).

3. \((\alpha f + \beta h, k) = \frac{1}{|G|} \sum_{t \in G} (\alpha f(t) + \beta h(t)) k(t) = \frac{1}{|G|} \sum_{t \in G} (\alpha f(t) k(t) + \beta h(t) k(t)) = \alpha \frac{1}{|G|} \sum_{t \in G} f(t) k(t) + \beta \frac{1}{|G|} \sum_{t \in G} h(t) k(t) = \alpha (f, k) + \beta (h, k) \).

Any inner product as defined above also has the following property:

5.6 Corollary: If \( f, h, k \in \text{CF}_G \) and \( \alpha, \beta \in \mathbb{F} \), then \((f, \alpha h + \beta k) = \alpha (f, h) + \beta (f, k) \).

An important observation is necessary before making the connection between the inner product above and characters. The vector space \( \text{CF}_G \) is actually a subspace of
A = \{all functions from G into \mathcal{E}\}. For instance, from corollaries 4.19 and 4.20, \(a_{k1}(t^{-1})\) and \(a_{j1}(t)\) are elements of A.

Furthermore, theorem 5.5 never uses the fact that \(f\), \(h\), and \(k\) are class functions. The inner product as defined here holds for all elements of A.

Now let \((V,\rho)\) be an irreducible representation of \(G\) of degree \(n\) with character \(\chi\). In this case, we call \(\chi\) an irreducible character. Let \(\rho(t)\) be given in matrix form by \(\left( a_{ij}(t) \right)\) for \(t \in G\). Then \(\chi(t) = \sum_{i=1}^{n} a_{ii}(t)\).

Consider the inner product \((\chi, \chi) = \frac{1}{|G|} \sum_{t \in G} \chi(t) \overline{\chi(t)}\).

By lemma 5.4 \(\overline{\chi(t)} = \chi(t^{-1})\), so \((\chi, \chi) = \frac{1}{|G|} \sum_{t \in G} \chi(t) \chi(t^{-1})\). Now using the definitions of \(\chi\) and \(\left( , \right)\), along with theorem 5.5 and corollary 5.6:

\[
\begin{align*}
(\chi, \chi) &= \frac{1}{|G|} \sum_{t \in G} \left( \sum_{i=1}^{n} a_{ii}(t) \right) \left( \sum_{j=1}^{n} a_{jj}(t^{-1}) \right) \\
&= \sum_{i=1}^{n} a_{ii}(t) \sum_{j=1}^{n} a_{jj}(t^{-1}) \\
&= \sum_{i,j=1}^{n} \left( a_{ii}(t), a_{jj}(t^{-1}) \right) = \sum_{i,j=1}^{n} \left( \frac{1}{|G|} \sum_{t \in G} a_{ii}(t) a_{jj}(t^{-1}) \right).
\end{align*}
\]
We may now apply corollary 4.20 since we have $\rho = \rho^1 = \rho^2$ and $V = V_1 = V_2$. Hence, $\frac{1}{|G|} \sum_{t \in G} a_{ii}(t)a_{jj}(t^{-1}) = \frac{1}{n} \delta_{ij}$.

Finally, then:

$$\langle \chi, \chi \rangle = \sum_{i,j} \frac{1}{n} \delta_{ij} = \frac{n}{n} = 1$$

So we have proven:

**5.7 Theorem:** If $\chi$ is the character of an irreducible representation of $G$, then $\langle \chi, \chi \rangle = 1$.

Since equivalent representations have the same characters (theorem 5.2), we immediately have:

**5.8 Corollary:** If $\chi$ and $\Psi$ are irreducible characters of equivalent representations of $G$, then $\langle \chi, \Psi \rangle = \langle \Psi, \chi \rangle = 1$.

On the other hand suppose $\chi$ and $\Psi$ are irreducible characters of two representation $\rho^1$ and $\rho^2$ of $G$ that are not equivalent. For $t \in G$, let $\rho^1(t)$ and $\rho^2(t)$ be given in matrix form by $(a_{ij}(t))$ and $(a_{kl}(t))$, respectively. Now we may argue as in the proof of theorem 5.7 and arrive at:

$$\langle \chi, \Psi \rangle = \sum_{i,1} \left( \frac{1}{|G|} \sum_{t \in G} a_{ii}(t)a_{11}(t^{-1}) \right).$$

But corollary 4.19 applies and we have

$$\frac{1}{|G|} \sum_{t \in G} a_{ii}(t)a_{11}(t^{-1}) = 0 \text{ for all } i, 1.$$

Hence, $\langle \chi, \Psi \rangle = 0$ and we have shown:
5.9 Theorem: If $\chi$ and $\Psi$ are irreducible characters of two nonequivalent representations of $G$, then $(\chi,\Psi) = 0$.

Theorems 5.7 and 5.9 give us orthogonality relations for the characters in $\text{CFG}$. From theorem 5.7, we have that the norm of an irreducible character $\chi$, $||\chi||$, is $||\chi|| = \sqrt{(\chi,\chi)} = 1$. And in theorem 5.8, we say that $\chi$ and $\Psi$ are orthogonal since $(\chi,\Psi) = 0$.

5.3 Applications of the Orthogonality Relations

Let $V$ be a representation of $G$. We know from Maschke's theorem (4.2) that $V$ can be written as $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, where $W_i$, $i = 1, \ldots, k$, is an irreducible representation of $G$. If we fix one of the irreducibles and call it $W$, we can then ask: how many of the $W_i$ are equivalent to $W$? This is answered in:

5.10 Theorem: Let $V$, $W$, and the $W_i$, $i = 1, \ldots, k$, be as above. Let $V$ have character $\Phi$ and $W$ have character $\chi$. Then the number of $W_i$ equivalent to $W$ is $(\Phi,\chi)$.

Proof: Let $\chi_i$ be the character of $W_i$, $i = 1, \ldots, k$. By theorem 5.3, $\Phi = \chi_1 + \chi_2 + \ldots + \chi_k$. So by the linearity of the inner product (theorem 5.5 3), $(\Phi,\chi) = (\chi_1,\chi) + (\chi_2,\chi) + \ldots + (\chi_k,\chi)$. But each term of the sum is 0 (if $W_i$ is not equivalent to $W$) or 1 (if $W_i$ is equivalent to $W$).

Hence $(\Phi,\chi)$ equals the number of $W_i$ equivalent to $W$. $\square$

Now since $(\Phi,\chi)$ does not depend on the decomposition of $V$, we immediately have:
5.11 Corollary: The number of $W_i$ equivalent to $W$ is independent of the chosen decomposition of $V$.

We call $(\Phi, \chi_i)$ the multiplicity of $\chi_i$ in $\Phi$ or, more intuitively, the number of times that $W_i$ is contained in $V$. In fact, if we let $\chi_1, \ldots, \chi_h$ be the distinct irreducible characters corresponding to $W_1, \ldots, W_h$ (we are possibly changing some subscript from above), and let $m_i = (\Phi, \chi_i), i = 1, \ldots, h$, then we can write: $V = m_1W_1 \oplus \cdots \oplus m_hW_h$.

Each $m_iW_i$ represents the direct sum of the irreducible representations equivalent to $W_i$. Furthermore, we can also write:

$$\Phi = m_1\chi_1 + \ldots + m_h\chi_h;$$ which yields:

5.12 Corollary: Keeping the notation above, $(\Phi, \Phi) = \sum_{i=1}^{h} m_i^2$.

Proof:

$(\Phi, \Phi) = (m_1\chi_1 + m_2\chi_2 + \ldots + m_h\chi_h, m_1\chi_1 + \ldots + m_h\chi_h);$ of which the only non-zero terms are:

$$m_1\overline{m_1}(\chi_1, \chi_1) + m_2\overline{m_2}(\chi_2, \chi_2) + \ldots + m_h\overline{m_h}(\chi_h, \chi_h) =$$

$$m_1\overline{m_1} + m_2\overline{m_2} + \ldots + m_h\overline{m_h} = \sum_{i=1}^{n} m_i^2;$$ since $m_i$ is a positive integer for each $i = 1, \ldots, h.$ \[ \square \]

The harvest of results from theorem 5.10 is rich indeed. Another is the converse of theorem 5.2:
5.13 Theorem: If two representations of $G$ have the same character, then they are equivalent.

Proof: Let $V_1$ and $V_2$ be representations of $G$ with character $\Phi$. Let $U_1, \ldots, U_r$ be the set of all nonequivalent irreducible representations contained in $V_1$ or $V_2$ (i.e., equivalent representation contained in $V_1$ and $V_2$ are denoted by a single $U_i$, $i = 1, \ldots, r$). Then $V_1 = \oplus_{i=1}^{r} a_i U_i$ and $V_2 = \oplus_{i=1}^{r} b_i U_i$; $a_i, b_i$ integers.

So $\Phi = \sum_{i=1}^{r} a_i \chi_i = \sum_{i=1}^{r} b_i \chi_i$, where $\chi_i$ is the character of $U_i$. But by theorem 5.10, $(\Phi, \chi_i) = a_i = b_i$ for each $i=1, \ldots, r$; whence $\oplus_{i=1}^{r} a_i U_i = \oplus_{i=1}^{r} b_i U_i$. Thus $V_1 = V_2$. \( \square \)

Also, a direct consequence of corollary 5.12 is a criterion for irreducibility:

5.14 Theorem: Let $\Phi$ be the character of a representation $V$ of $G$. Then $(\Phi, \Phi) = 1$ iff $V$ is irreducible.

Proof: Let $V = m_1 W_1 \oplus \cdots \oplus m_h W_h$ where $W_i$ is irreducible and $m_i$ is the number of times that $W_i$ is contained in $V$. Then $(\Phi, \Phi) = \sum_{i=1}^{h} m_i^2 = 1$ iff $m_j = 1$ for some $1 \leq j \leq h$ (and all other terms are 0) iff $V = W_j$. \( \square \)

We can glean more useful results from the character of the regular representation of $G$ as described in section 2.3.
Denote the regular representation by \((FG, \rho_R)\) and recall that its degree is \(|G|\). Using the elements of \(G\) as a basis for \(FG\), we wrote the matrices of \(\rho_R\) as:

\[
[p(s)]_G = (\delta g_i, s g_j) \text{ for } s \in G.
\]

Now denote the character of \(\rho_R\) by \(\chi_R\) and we have:

\[
\chi_R(s) = \text{tr}([p(s)]_G) \text{ for } s \in G.
\]

But \((\delta g_i, s g_j)\) is a permutation matrix. The \(j^{th}\) column will have a 1 in the \(j^{th}\) row (i.e. on the diagonal) iff \(g_j = s g_j\) iff \(s = 1_G\), the identity of \(G\).

**5.15 Theorem:** With the above notation,

\[
\chi_R(1_G) = |G| \text{ and } \chi_R(s) = 0 \text{ if } s \neq 1_G.
\]

**Proof:** By the preceding comments,

\[
\text{tr}([p(s)]_G) = \begin{cases} 
\text{degree of } \rho_R = |G| \text{ if } s = 1_G \\
0 \text{ if } s \neq 1_G
\end{cases}.
\]

In chapter 6, we will describe all of the irreducible representations of some groups by displaying their characters in an organized table. It is desirable, then, to have some place to look for irreducible representations, and how many of each to expect.

**5.16 Theorem:** Every irreducible representation \(W_i\) of \(G\) is contained in the regular representation of \(G\) with multiplicity equal to its degree \(n_i\).

**Proof:** Let \(\chi_R\) be the character of the regular representation of \(G\) and let \(\chi_i\) be the character of \(W_i\).
By theorem 5.10, the number of times that $V_i$ is contained in the regular representation is:

$$(\chi_R, \chi_1) = \frac{1}{|G|} \sum_{t \in G} \chi_R(t)\chi_1(t^{-1}) = \frac{1}{|G|} |G| \chi_1(l_G) = \chi_1(l_G) = n_i,$$

since $t = l_G$ gives the only non-zero term of the sum. \(\square\)

We should note that saying any irreducible representation $V_i$ is contained in the regular representation is intended in the broader sense that $V_i$ is equivalent to an irreducible representation contained in the regular representation. There are yet other results related to theorem 5.16 which will aid us in determining the characters of the irreducible representations. Let $\{W_1, ..., W_r\}$ be all of the nonequivalent irreducible representations of $G$ and let $n_i$ and $\chi_i$ be the degree and character of $W_i$, respectively.

5.17 Corollary: With the hypotheses above:

1. \(\sum_{i=1}^{r} n_i^2 = |G|\); and
2. \(\sum_{i=1}^{r} n_i \chi_i(s) = 0\), where $s \in G$ but $s \neq l_G$.

Proof: 1 If $\chi_R$ is the character of the regular representation of $G$, then $\chi_R(s) = \sum_{i=1}^{r} n_i \chi_i(s)$ for all $s \in G$.

In particular, if $s = l_G$: $\chi_R(l_G) = \sum_{i=1}^{r} n_i \chi_i(l_G) = \sum_{i=1}^{r} n_i n_i$

$= \sum_{i=1}^{r} n_i^2$; where $\chi_i(l_G) = n_i$ from lemma 5.4 1.
On the other hand, theorem 5.15 gives $\chi_R(1_G) = |G|$.

Let $s \in G$ but $s \neq 1_G$. Then $\chi_R(s) = 0$ (theorem 5.15) and, hence, $\sum_{i=1}^{r} n_i \chi_i(s) = 0$. \hfill \Box
Chapter 6

Applications of Characters

6.1 Conjugacy Classes

and

Irreducible Representations

In section 5.2, we introduced the vector space \( C_{FG} \), the space of class functions on \( G \). That is, \( C_{FG} = \{ f | f(t s t^{-1}) = f(s) \text{ for all } s, t \in G \} \). Included in \( C_{FG} \) are the characters of any representation of \( G \). In particular, \( \{ \chi_1, \chi_2, \ldots, \chi_h \} \), the complete set of characters of nonequivalent irreducible representations is a subset of \( C_{FG} \). We intend to show that this subset is, in fact, an orthonormal basis of \( C_{FG} \). First, a somewhat technical lemma is needed:

**6.1 Lemma:** Let \( f \in C_{FG} \) and let \( (V, \rho) \) be a representation of \( G \). Define a linear transformation \( \rho(f) : V \to V \) by \( \rho(f) = \sum_{t \in G} f(t) \rho(t) \). If \( V \) is irreducible of degree \( n \) and character \( \chi \), then \( \rho(f) = \lambda, a \text{ scalar multiple of the identity and:} \)

\[
\lambda = \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f, \chi).
\]

**Proof:** We will apply part 2 of Schur's lemma (4.16), but we must have \( \rho(f) \circ \rho(s) = \rho(s) \circ \rho(f) \) for \( s \in G \). To that end, we compute:
\[ \rho(s)^{-1} \circ \rho(f) \circ \rho(s) = \rho(s)^{-1} \circ \sum_{t \in G} f(t) \rho(t) \circ \rho(s) = \]

\[ \sum_{t \in G} f(t) (\rho(s)^{-1} \circ \rho(t) \circ \rho(s)), \] since \( f(t) \) is just a scalar and \( \rho(s) \) and \( \rho(s)^{-1} \) do not depend on \( t \in G \). Now, since \( \rho \) is a group homomorphism,

\[ \sum_{t \in G} f(t) \rho(s^{-1}ts). \] Now substitute \( u = s^{-1}ts \) noting: (i) then \( t = sus^{-1} \); and (ii) as \( t \) runs over the elements of \( G \), so does \( u \). So we have:

\[ \rho^{-1}(s) \circ \rho(t) \circ \rho(s) = \sum_{u \in G} f(sus^{-1}) \rho(u) = \]

\[ \sum_{u \in G} f(u) \rho(u) = \rho(f), \] since \( f \in \text{CFG} \). Hence, \( \rho(f) \circ \rho(s) = \rho(s) \circ \rho(f) \), and we may apply part 2 of 4.16.

Then \( \rho(f) \) is a scalar \( \lambda \) and \( \text{tr}(\rho(f)) = \text{tr}(\lambda) = n\lambda \). By theorem 4.11 2, \( \text{tr}(\rho(f)) = \text{tr}( \sum_{t \in G} f(t) \rho(t) ) = \)

\[ \sum_{t \in G} f(t) \text{tr}(\rho(t)). \] But \( \text{tr}(\rho(t)) = \chi(t) \). So, setting the two expressions for \( \text{tr}(\rho(f)) \) equal to each other:

\[ n\lambda = \sum_{t \in G} f(t) \text{tr}(\rho(t)) = \sum_{t \in G} f(t) \chi(t) \Rightarrow \lambda = \]

\[ \frac{1}{n} \sum_{t \in G} f(t) \chi(t). \] Finally, by definition of \( ( , ) \),

\[ \frac{1}{n} \sum_{t \in G} f(t) \chi(t) = \frac{|G|}{n} (f,\chi). \]

Now we can prove:

**6.2 Theorem:** The set \( \{ \chi_1, \chi_2, \ldots, \chi_n \} \) forms an orthonormal basis of \( \text{CFG} \).
**Proof:** From the orthogonality relations, we already know that \( \{\chi_1, \ldots, \chi_h\} \) is an orthonormal set, i.e.,

\[
(\chi_i, \chi_j) = \begin{cases} 
1 & \text{if } i=j \\
0 & \text{otherwise}
\end{cases}
\]

Then

\[
(\overline{\chi_i}, \overline{\chi_j}) = \frac{1}{|G|} \sum_{t \in G} \overline{\chi_i(t)} \overline{\chi_j(t)}
\]

\[
= (\overline{\chi_j}, \overline{\chi_i}) = \begin{cases} 
1 & \text{if } i=j \\
0 & \text{otherwise}
\end{cases}
\]

So \( B = \{\overline{\chi_1}, \overline{\chi_2}, \ldots, \overline{\chi_h}\} \) is also an orthonormal set in \( \text{CFG} \).

We will actually show \( B \) is a basis of \( \text{CFG} \) by establishing that the only element of \( \text{CFG} \) orthogonal to the \( \overline{\chi_i}, i = 1, \ldots, h \), is the zero class function.

So let \( f \in \text{CFG} \) be such that \( (f, \overline{\chi_i}) = 0, i=1, \ldots, h \). Let \( (V, \rho) \) be any representation of \( G \) and define \( \rho(f) : V \to V \) by

\[
\rho(f) = \sum_{t \in G} f(t) \rho(t).
\]

There are two cases to consider:

(i) If \( (V, \rho) \) is irreducible, then by lemma 6.1, \( \rho(f) = \frac{|G|}{n} (f, \chi) \), where \( \chi \) is the irreducible character of \( (V, \rho) \).

Then \( \rho(f) = 0 \) since \( (f, \overline{\chi}) = 0 \);

(ii) If \( (V, \rho) \) is reducible, let \( W_1, \ldots, W_k \) be the nonequivalent irreducible constituents of \( V \), with characters \( \chi_1, \ldots, \chi_k \), \( k \leq h \). Then, by lemma 6.1:

\[
\rho(f) |_{W_i} = \sum_{t \in G} f(t) \rho |_{W_i}(t) = \frac{|G|}{n} (f, \overline{\chi_i}), \quad i = 1, \ldots, k.
\]

Since \( \rho(f) = 0 \) on each \( W_i \), then \( \rho(f) = 0 \) on all of \( V \). So we have, in either case, \( \rho(f) = 0 \).
Finally, we show $\rho(f) = 0$ implies that $f = 0$. We do so in $(FG, \rho_R)$, the regular representation of $G$, since it contains every irreducible representation of $G$. Let $g_i$ be a basis element of $FG$. Then:

$$
\rho_R(f)(g_i) = \sum_{t \in G} f(t) \rho_R(t)(g_i) = \sum_{t \in G} f(t)(tg_i).
$$

On the other hand, $\rho(f) = 0$ in any representation, so $\rho_R(f)(g_i) = 0$. This can only be true if $f(t) = 0$ for all $t \in G$. Hence, $f = 0$.

So we have that $B = \{\overline{\chi_1}, \ldots, \overline{\chi_h}\}$, is an orthonormal basis for $CFG$. But $\{\chi_1, \ldots, \chi_h\}$ is a set of $h$ orthonormal functions in $CFG$, so they must also form an orthonormal basis.

Having determined one basis of $CFG$, we shall immediately construct another! The comparison of the two bases will give us a final important result that we need to write down all of the irreducible characters for some specific examples.

From group theory, we know that $t, t' \in G$ are conjugate provided there is an $s \in G$ such that $t' = sts^{-1}$. Conjugacy is an equivalence relation and, thus, partitions $G$ into distinct equivalence classes. Denote the classes by $G_1, G_2, \ldots, G_k$.

If $f \in CFG$, then $f(t') = f(sts^{-1}) = f(t)$, i.e., $f$ is constant on a given $G_i$, $i=1, \ldots, k$. Denote the value of $f$ on $G_i$ by $\lambda_i$.

Now define functions $f_1, f_2, \ldots, f_k$: $G \to \mathbb{C}$ by:
\[ f_i(t) = \begin{cases} 1 & \text{if } t \in G_i \\ 0 & \text{if } t \in G_j \neq i \end{cases} \]

Is such a function a class function?

For \( s, t \in G \), \( st^{-1} \in G_i \) iff \( t \in G_i \). So, for \( i = 1, \ldots, k \):

\[ f_i(st^{-1}) = \begin{cases} 1 & \text{if } st^{-1} \in G_i \\ 0 & \text{if } st^{-1} \in G_j \neq i \end{cases} = \begin{cases} 1 & \text{if } t \in G_i \\ 0 & \text{if } t \in G_j \neq i \end{cases} = f_i(t). \]

Hence, \( f_i \in \text{CFG} \) for all \( i = 1, \ldots, k \).

Furthermore, it is clear that the \( f_i \) are linearly independent. And, finally, any \( f \in \text{CFG} \) can be uniquely written as a linear combination of the \( f_i \) over \( \mathbb{C} \). For if \( t \in G \), then \( t \in G_i \) for some \( i=1, \ldots, k \) and \( f(t) = 0 f_1(t) + \ldots + \lambda_i f_i(t) + \ldots + 0 f_k(t) = \lambda_i \).

So we have that \( \{f_1, \ldots, f_k\} \) form a basis of \( \text{CFG} \). (This really amounts to nothing more than the standard basis for this space.) Hence, the dimension of \( \text{CFG} \) is equal to the number of equivalence classes of \( G \). But, by theorem 6.2, \( \dim \text{CFG} \) is precisely the number of nonequivalent irreducible representations of \( G \). We have shown:

6.3 Theorem: The number of nonequivalent irreducible representations of \( G \) is equal to the number of conjugacy classes of \( G \).

6.4 Corollary: \( G \) is abelian iff all irreducible representations of \( G \) have degree 1.

Proof: \( G \) is abelian iff \( G \) has \( |G| \) classes iff \( G \) has \( |G| \) nonequivalent irreducible representations (denote their
degrees by $n_1, \ldots, n_{|G|}$ iff $n_i = 1$, $i = 1, \ldots, |G|$ since $|G| = 
\sum_{i=1}^{\infty} n_i^2$. 

6.2 Character Tables

A character table displays all of the irreducible characters of representations of $G$. If $G_1, \ldots, G_h$ are the distinct classes of $G$, let us pick a representative element $g_i \in G_i$ for each $i = 1, \ldots, h$. Let $\{\chi_1, \ldots, \chi_h\}$ be the corresponding set of characters of nonequivalent irreducible representations of $G$. Then the character table of $G$ has the following form:

\begin{center}
\begin{array}{c|ccc}
\chi_1 & g_1 & g_2 & \ldots & g_h \\
\chi_2 & \chi_2(g_1) & \chi_2(g_2) & \ldots & \chi_2(g_h) \\
\chi_h & \chi_h(g_1) & \chi_h(g_2) & \ldots & \chi_h(g_h) \\
\end{array}
\end{center}

The regular representation $F G$ of $G$ will be a useful guide in constructing the table, as it contains equivalents of every irreducible representation of $G$. In particular, recall from section 3.1 that if $|G| \geq 2$, it will always have an irreducible representation of degree $1$. In the regular representation, the invariant 1-dimensional subspace is
generated by the element \( x = \sum_{g \in G} g \), and is called the unit representation. It occurs, of course, with multiplicity 1.

Some of the techniques for determining other irreducible characters are perhaps best seen by example.

1. As in example 3 of section 2.3, let \( G = S_3 \). There are three nonequivalent irreducible representations since \{\( (1) \), \( (12), (13), (23) \), and \( (123), (132) \)\} are the three conjugacy classes of \( G \). The character of the class \( \{ (1) \} \) is easy:
\[
\chi_1(1) = n_1, \text{ where } n_1 \text{ is the degree of the representation corresponding to } \chi_1. \]

As representatives from the other two classes, let \( g_2 = (12) \) and \( g_3 = (123) \); remembering that the resulting characters are independent of the choice of representatives.

We know that \( G \) has at least one representation of degree 1, i.e., the unit representation. Are there any others? If \( \rho \) is any representation of \( G \) of degree 1, then \( \rho(g) = \lambda \in \mathbb{C} \) for \( g \in G \). But then we also have \([\rho(g)] = [\lambda]\), so the character of \( \rho(g) \) is \( \chi_\rho(g) = \lambda = \rho(g) \). That is, for degree 1, the character can be thought of as the representation.

In our case, \( g_2^2 = (1) \) and \( g_3^3 = (1) \), so \( \chi_\rho(g_2^2) = (\chi_\rho(g_2))^2 = 1 \) and \( \chi_\rho(g_3^3) = (\chi_\rho(g_3))^3 = 1 \). Hence, \( \chi_\rho(g_2) = \pm 1 \), and \( \chi_\rho(g_3) = 1, e^{(2\pi i)/3}, e^{(4\pi i)/3} \).

However, there is another relationship between \( g_2 \) and \( g_3 \): \( g_2 g_3 = g_3^2 g_2 \). So \( \chi_\rho(g_2 g_3) = \chi_\rho(g_3^2 g_2) \); which gives \( \chi_\rho(g_2) \chi_\rho(g_3) = \ldots \)
$(\chi_p(g_3))^2 \chi_p(g_2)$.

Hence $\chi_p(g_3)$ can only be 1, eliminating the two complex roots above. But since $\chi_p(g_2)$ has two values, we will have two nonequivalent irreducible representations of degree 1. So far, then, the character table for $G = S_3$ looks like this:

<table>
<thead>
<tr>
<th></th>
<th>$g_1 = 1$</th>
<th>$g_2$</th>
<th>$g_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$\chi_3(g_1)$</td>
<td>$\chi_3(g_2)$</td>
<td>$\chi_3(g_3)$</td>
</tr>
</tbody>
</table>

The character $\chi_1$ corresponds to the unit representation, while $\chi_2$ is from the alternating representation of example 4 in section 1.2. In the regular representation, the one-dimensional subspace of the alternating representation is generated by $\sum_{g \in G} (\pm)_g g; \text{ where } (\pm)_g \text{ is } + \text{ if } g \text{ is an even permutation and } - \text{ if it is odd.}$

Now, the degree (and multiplicity) of the representation corresponding to $\chi_3$ is $n_3$ where $1 + 1 + n_3^2 = 6$. Thus, $n_3 = 2$. Then $\chi_3((1)) = 2$. And from corollary 5.17, $\sum_{i=1}^{3} n_i \chi_i(g_2) = 0$ and $\sum_{i=1}^{3} n_i \chi_i(g_3) = 0$. These yield $\chi_3(g_2) = 0$ and $\chi_3(g_3) = -1$.  

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So we now have the complete character table for \( S_3 \):

<table>
<thead>
<tr>
<th>( \chi )</th>
<th>( 1 = g_1 )</th>
<th>( g_2 )</th>
<th>( g_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \chi_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_2 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( \chi_3 )</td>
<td>2</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

(2) As a second example, let us consider \( D_4 \), the dihedral group of order 8. This group is usually thought of as the group of all symmetries of a square. It consists of rotations about the center through an angle of \( \frac{\pi}{2} \), together with reflections about four lines forming angles which are multiples of \( \frac{\pi}{4} \). These are just the rotations and reflections which preserve the square. If \( r \) is such a rotation and \( s \) is any one of the reflections (e.g., about the horizontal axis), then each element of \( D_4 \) can be uniquely written as either \( r^k \) or \( sr^k \) for \( k = 0, 1, 2, 3 \). (Note, then, that \( D_4 \) has a cyclic subgroup of order 4.)

Furthermore, as with \( S_3 \), the relationships among the generators of the group are vital in determining the irreducible characters of degree 1. For \( D_4 \) we have:

\( r^4 = 1; \ s^2 = 1; \text{ and } srs = r^{-1} \).

So if \( \rho \) is any irreducible representation of degree 1, \( \chi_\rho (r^4) = (\chi_\rho (r))^4 = 1 \). Then \( \chi_\rho (r) = 1, -1, \ e^{(\pi i)/2}, \ e^{\pi i} \).
And $\chi_p(s^2) = (\chi_p(s))^2 = 1$, yielding $\chi_p(s) = \pm 1$. From $srs = r^{-1}$, we have $(sr)^2 = 1$. So $\chi_p(sr) = \pm 1$, from which we glean $\chi_p(s) = \pm \chi_p(r)^{-1}$. But since $\chi_p(s) = \pm 1$, we must have, $\chi_p(r) = \pm 1$, and we eliminate the imaginary values for $\chi_p(r)$.

By taking all combinations of $\pm 1$ for $\chi_p(s)$ and $\chi_p(r)$, we obtain four irreducible characters of degree one for $D_4$. They will be denoted by $\chi_1'$, $\chi_2'$, $\chi_3'$, and $\chi_4$ in the table.

Now we know that $\sum_{i=1}^{h} n_i^2 = 8$, where the $n_i$ are the degrees of nonequivalent representations. Hence, there is one representation of degree 2 not yet accounted for. Let its character be $\chi_5$. Then $\chi_5(r^4) = \chi_5(r^0) = \chi_5(1) = 2$. For any other element $g$ of $D_4$, $\sum_{i=1}^{5} n_i \chi_i(g) = 0$, and we can write down the complete character table. For $k = 0, 1, 2, 3$:

<table>
<thead>
<tr>
<th>$\chi_i$</th>
<th>$sr^k$</th>
<th>$r^k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$(-1)^k$</td>
<td>$(-1)^k$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>$(-1)^k$</td>
<td>$(-1)^{k+1}$</td>
</tr>
<tr>
<td>$\chi_5$</td>
<td>$-1 + (-1)^{k+1}$</td>
<td>0</td>
</tr>
</tbody>
</table>

The group $D_4$, then, must have 5 equivalence classes.
Chapter 7
Induced Representations

7.1 Basic Notions and Examples

Given any representation of a group $G$, we can obtain a representation of a subgroup $H$ of $G$ by a simple restriction (cf. Def. 4.4). It is a bit more complicated to reverse the process. That is, starting with a representation of $H$, we shall "extend" to a representation of $G$.

Recall that if $g \in G$ and $H$ is a subgroup of $G$, then $Hg = \{hg | h \in H\}$ is a right coset of $H$ in $G$. Left cosets are similarly defined. Let $n = \frac{|G|}{|H|}$ be the index of $H$ in $G$.

A couple of other elementary group theoretic results are used in this section, namely:

1. $n$ = the number of right (or left) cosets of $H$ in $G$; and
2. $G$ may be decomposed into mutually exclusive and exhaustive right (or left) cosets. For instance, if $G = Ht_1 \cup Ht_2 \cup \ldots \cup Ht_n$ is a decomposition of $G$ into right cosets, the elements $t_1, t_2, \ldots, t_n$ are called a system of representatives of the right cosets of $H$ in $G$.

Suppose that $\theta : H \to GL(m, F)$ is a matrix representation of $H$. We define an extension:

\[ \overline{\theta} : G \to GL(m, F) \text{ by:} \]

\[ \overline{\theta}(x) = \begin{cases} \theta(x) & \text{if } x \in H \\ O_{m \times m} & \text{if } x \in G - H \end{cases} \]

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Since the zero matrix is singular, \( \bar{\theta} \) will not, in general, be a matrix representation of \( G \). So we define a different map for \( x \in G \): \( \Phi(x) = \)

\[
(\bar{\theta}(t_1 x t_j^{-1})) = \begin{bmatrix}
\bar{\theta}(t_1 x t_1^{-1}) & \bar{\theta}(t_1 x t_2^{-1}) & \cdots & \bar{\theta}(t_1 x t_n^{-1}) \\
\bar{\theta}(t_2 x t_1^{-1}) & \bar{\theta}(t_2 x t_2^{-1}) & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
\bar{\theta}(t_n x t_1^{-1}) & \bar{\theta}(t_n x t_2^{-1}) & \cdots & \bar{\theta}(t_n x t_n^{-1})
\end{bmatrix},
\]

where the \( t_1, t_2, \ldots, t_n \) are as in (2) above. So \( \Phi(x) \) is an \( mn \times mn \) matrix consisting of \( n^2 \) submatrices each of size \( m \times m \).

The proof that \( \Phi : G \to GL(mn,F) \) is a matrix representation of \( G \) is computational in nature and is facilitated by:

7.1 Lemma: Let \( H \) be a subgroup of \( G \) and let \( t_1, t_2, \ldots, t_n \) be a system of representatives of the right cosets of \( H \) in \( G \). Then for fixed \( i, j = 1, \ldots, n \) and \( x, y \in G \), there exists at most one value of \( k \) (\( k = 1, \ldots, n \)) such that \( t_i x t_k^{-1} \in H \) and \( t_k y t_j^{-1} \in H \).

Proof: Actually, it suffices to show there is at most one value of \( k \) (\( k = 1, \ldots, n \)) such that \( t_i x t_k^{-1} \in H \). For if there is such a value of \( k \), it may or may not be true that \( t_k y t_j^{-1} \in H \). In either case, the lemma will be true.
So let \( k_1 \) and \( k_2 \) be such that \( t_{i \cdot x \cdot k_1}^{-1} \in H \) and \( t_{k_1 \cdot x \cdot k_2}^{-1} \in H \). Then \( t_{i \cdot x} \) and \( t_{k_1} \) are in the same coset, as are \( t_{i \cdot x} \) and \( t_{k_2} \). Hence, \( t_{k_1} \) and \( t_{k_2} \) are in the same coset; which yields \( k_1 = k_2 \) since \( t_{k_1} \) and \( t_{k_2} \) were from the system of representatives.

7.2 Theorem: Let \( x \in G \) and \( i, j = 1, \ldots, n \). Then the map \( \Phi: G \rightarrow GL(\text{mn}, F) \) given by \( \Phi(x) = (\bar{\theta}(t_{i \cdot x \cdot t_{j}^{-1}}))(x \in G, i, j = 1, \ldots, n) \) is a matrix representation of \( G \).

Proof: Let \( x \) and \( y \) be elements of \( G \). Then \( \Phi(x) \Phi(y) = \)

\[
\begin{bmatrix}
\sum_{k=1}^{n} \bar{\theta}(t_{1 \cdot x \cdot k}^{-1}) \bar{\theta}(t_{k \cdot y \cdot t_{1}^{-1}}) & \cdots & \sum_{k=1}^{n} \bar{\theta}(t_{1 \cdot x \cdot k}^{-1}) \bar{\theta}(t_{k \cdot y \cdot t_{n}^{-1}}) \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{n} \bar{\theta}(t_{n \cdot x \cdot k}^{-1}) \bar{\theta}(t_{k \cdot y \cdot t_{1}^{-1}}) & \cdots & \sum_{k=1}^{n} \bar{\theta}(t_{n \cdot x \cdot k}^{-1}) \bar{\theta}(t_{k \cdot y \cdot t_{n}^{-1}})
\end{bmatrix}
\]

In general, the \( m \times m \) submatrix in the \( i, j \) position of \( \Phi(x) \Phi(y) \) is \( \sum_{k=1}^{n} \bar{\theta}(t_{i \cdot x \cdot k}^{-1}) \bar{\theta}(t_{k \cdot y \cdot t_{j}^{-1}}) \), \( i, j = 1, \ldots, n \).

On the other hand, then \( m \times m \) submatrix in the \( i, j \) position of \( \Phi(xy) \) is \( \bar{\theta}(t_{i \cdot x \cdot y \cdot t_{j}^{-1}}) = \bar{\theta}((t_{i \cdot x \cdot t_{k}^{-1}})(t_{k \cdot y \cdot t_{j}^{-1}})) \).

The theorem will be proved if we can establish that the corresponding submatrices of \( \Phi(x) \Phi(y) \) and \( \Phi(xy) \) are equal.

In the sum \( \sum_{k=1}^{n} \bar{\theta}(t_{i \cdot x \cdot k}^{-1}) \bar{\theta}(t_{k \cdot y \cdot t_{j}^{-1}}) \), the only non-zero matrices occur for values of \( k \) such that \( t_{i \cdot x \cdot t_{k}^{-1}} \in H \) and \( t_{k \cdot y \cdot t_{j}^{-1}} \in H \). This follows from the definition of \( \bar{\theta} \) and the fact that
θ : H → GL(m, F) is a matrix representation of H. From lemma 7.1, there is at most one k (k = 1, ..., n) such that
\( t_i x t_k^{-1} \in H \) and \( t_k y t_j^{-1} \in H \). This leaves us two cases:

1. Suppose there is no value of k (k = 1, ..., n) such that
   \( t_i x t_k^{-1} \in H \) and \( t_k y t_j^{-1} \in H \). Then
   \[ \sum_{k=1}^{n} \bar{\theta}(t_i x t_k^{-1}) \bar{\theta}(t_k y t_j^{-1}) = 0_{mxm}. \]
   But we also have \( (t_i x t_k^{-1})(t_k y t_j^{-1}) = t_i x y t_j^{-1} \in H \Rightarrow \bar{\theta}(t_i x y t_j^{-1}) = 0_{mxm}. \)
   Hence, in this case, the corresponding submatrices are equal.

2. Suppose there is exactly one value of k (k = 1, ..., n) such that
   \( t_i x t_k^{-1} \in H \) and \( t_k y t_j^{-1} \in H \). Denote the value by \( k_1 \).
   Then
   \[ \sum_{k=1}^{n} \bar{\theta}(t_i x t_k^{-1}) \bar{\theta}(t_k y t_j^{-1}) = \bar{\theta}(t_i x t_{k_1}^{-1}) \bar{\theta}(t_{k_1} y t_j^{-1}) = \]
   \[ \theta(t_i x t_{k_1}^{-1}) \theta(t_{k_1} y t_j^{-1}) = \theta((t_i x t_{k_1}^{-1})(t_{k_1} y t_j^{-1})) = \theta(t_i x y t_j^{-1}) \]
   = \( \bar{\theta}(t_i x y t_j^{-1}) \), since \( t_i x y t_j^{-1} \in H \). Hence, the corresponding submatrices are equal.

Therefore, in either case, we have \( \Phi(x) \Phi(y) = \Phi(xy) \)
and the theorem is proved. \( \square \)

This matrix approach to induced representations has the advantage of being intuitively straightforward. However, it is inconvenient to establish properties (e.g., that the map \( \Phi \) is independent of the choice of the system of representatives) using matrices, so we will not attempt to do so here.
As an example of an induced representation, consider $D_4$ as outlined in section 6.2. We have $H = \{1, r, r^2, r^3\}$ as a cyclic subgroup of $D_4$, so the index of $H$ in $D_4$ is 2. A representation of degree 2 of $H$ is given by $\theta : H \to \text{GL}(2, \mathbb{R})$ where:

\[
\theta(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \theta(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};
\]

\[
\theta(r^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \quad \text{and} \quad \theta(r^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Let $t_1 = r$ and $t_2 = rs$ be a system of representatives of the right cosets of $H$ in $D_4$. Then we have $D_4 = Hr \cup Hrs$ where $Hr = H$ and $Hrs = \{rs, r^2s, r^3s, s\}$. Hence $\Phi : D_4 \to \text{GL}(4, \mathbb{R})$ is a matrix representation of $D_4$. The matrices corresponding to $r$ and $s$ are computed below:

\[
\Phi(r) = \begin{bmatrix}
\overline{\theta}(rr^{-1}) & \overline{\theta}(rr(rs)^{-1}) \\
\overline{\theta}(rsr^{-1}) & \overline{\theta}((rs)r(rs)^{-1})
\end{bmatrix} = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}.
\]

\[
\begin{bmatrix}
\overline{\theta}(r) & \overline{\theta}(r^2s, r^3) \\
\overline{\theta}(rs) & \overline{\theta}(r^3)
\end{bmatrix} = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
0 & -1
\end{bmatrix}.
\]

Similarly, $\Phi(s) = \begin{bmatrix}
\overline{\theta}(rs^{-1}) & \overline{\theta}(rs(rs)^{-1}) \\
\overline{\theta}(rss^{-1}) & \overline{\theta}(rss(rs)^{-1})
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
0 & 1 \\
0 & -1
\end{bmatrix}$. 

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7.2 Character of an Induced Representation

We can derive the character of an induced representation in a manner inspired by the construction in Section 7.1. Let $\Phi$ be a matrix representation of a group $G$ induced by the representation $\theta$ of a subgroup $H$ of $G$. Also let $\chi_\theta : H \to \mathbb{C}$ be the character of $\theta$ and let $G = H_1 \cup H_2 \cup \ldots \cup H_n$ be a decomposition of $G$ into right cosets.

Extend $\chi_\theta$ to all of $G$ by:

$$
\chi_\theta (x) = \begin{cases} 
\chi_\theta (x) & \text{if } x \in H \\
0 & \text{if } x \in G - H
\end{cases}
$$

We may be tempted to stop here and simply write $\chi_\theta (x) = \chi_\Phi (x)$, the character of $\Phi$, for all $x \in G$. But we must be careful about deciding that $\chi_\Phi (x) = 0$ for $x \in G - H$. We must allow for the possibility that $t_i x \in H_t$ for some $i = 1, \ldots, n$. In that case, $t_i x t_i^{-1} \in H$ and $\text{tr}(\bar{\theta}(t_i x t_i^{-1}))$ may not be 0.

So let $\Psi(x) = \sum_{i=1}^{n} \chi_\theta(t_i x t_i^{-1})$ for all $x \in G$. From the definitions of $\bar{\theta}$ and $\chi_\theta$ note that:

$$
\chi_\theta (t_i x t_i^{-1}) = \text{tr}(\bar{\theta}(t_i x t_i^{-1})).
$$

So $\Psi(x) = \sum_{k=1}^{n} \chi_\theta(t_i x t_i^{-1}) = \ldots$
\[ \sum_{i=1}^{n} \text{tr}(\bar{\Theta}(t_i x t_i^{-1})) = \text{tr}(\Phi(x)) = \chi_\Phi(x) \text{ for all } x \in G. \]

We call \( \chi_\Phi \) the character induced by \( \chi_\Theta \) and write \( \text{Ind}_H^G \chi_\Theta \).

We can write the formula for \( \Psi(x) = \chi_\Phi(x) \) is a slightly different way. Recall that \( \bar{\chi}_\Theta(t_i x t_i^{-1}) \) can only be non-zero if \( t_i x \in H t_i \). If \( t_i x \in H t_i \), then \( h t_i x \in H t_i \) for all \( h \in H \) and \( \bar{\chi}_\Theta(t_i x t_i^{-1}) = \chi_\Theta((ht_i)x(ht_i)^{-1}) \) since \( t_i x t_i^{-1} \) and \( (ht_i)x(ht_i)^{-1} \) are conjugate elements. Finally, since \( hxh^{-1} \in H \) iff \( x \in H \), we may write:

\[ \chi_\Phi(x) = \frac{1}{|H|} \sum_{g \in G} \bar{\chi}_\Theta(g x g^{-1}) \text{ for all } x \in G. \]

We conclude this section with a look back at the example \( D_4 \) in Section 7.1. It is clear from the matrices that \( \chi_\Phi(r) = \chi_\Phi(s) = 0 \). Similarly, in \( \Phi(r^2 s) \), the submatrices on the diagonal would be \( \bar{\Theta}(rr^2sr^{-1}) \) and \( \bar{\Theta}(rsr^2s(rs)^{-1}) \); each of the form \( \bar{\Theta}(t_i x t_i^{-1}) \). But in both cases, \( t_i x \notin H t_i \). Hence, \( \bar{\chi}_\Theta(t_i x t_i^{-1}) = 0 \) for \( i = 1, 2 \) and \( \chi_\Phi(r^2 s) = 0 \).

But in \( \Phi(r^2) \), we do have \( t_i x \in H t_i \) for \( i = 1 \) and \( 2 \). In particular, \( tr^2 t_i^{-1} = r^2 \) for \( i = 1 \) and \( 2 \). Then \( \bar{\chi}_\Theta(tr^2 t_i^{-1}) = -2 \), so \( \chi_\Phi(r^2) = -4 \). The characters of the other elements of this representation of degree 4 can be similarly computed.

### 7.3 Induced Representation by Automorphism

Automorphism representations corresponding to the matrix representations in Section 7.1 can be found as in Section 1.2.
However, we give a brief and direct description here of the conditions under which an automorphism representation of a group is said to be induced by a representation of a subgroup. The notation established will help us state and interpret the Frobenius reciprocity formula in the final section.

Let $G = t_1H \cup t_2H \cup \ldots \cup t_nH$ be a decomposition of $G$ into left cosets and let $(\rho, V)$ be a linear representation of $G$. Then by definition 4.4, $\text{Res}^G_H \rho$ is a representation of $H$. Now let $W$ be a subrepresentation of $\text{Res}^G_H \rho$, and denote this representation of $H$ by $\overline{\rho} : H \to \text{GL}(W)$.

For any $s \in G$ consider $\rho(s)(W)$, the image of $W$ under $\rho(s)$. Since $s$ is in some left coset of $G$, we can write $s = t_ih$ for some $t_i$ from the system of representatives and some $h \in H$. Then $\rho(s)(W) = \rho(t_ih)(W) = \rho(t_i) \circ \rho(h)(W)$, since $\rho$ is a representation of $G$. But $\rho(h)(W) = W$ because $W$ is stable under $\rho(h)$ for $h \in H$. So $\rho(s)(W) = \rho(t_i)(W)$. In other words, the image depends only on the left coset from which $s$ comes. Let $\rho(s)(W) = W_i$ for $i = 1, \ldots, n$ where $s \in t_iH$ and $W_i$ is a subspace of $V$. Then we say that $(\rho, V)$ is induced by $(\overline{\rho}, W)$ iff $V = \bigoplus_{i=1}^n W_i$. We denote the induced representation by $\text{Ind}^G_H \rho$ or $\text{Ind}^G_H W$. 

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7.4 The Frobenius Reciprocity Formula

The Frobenius formula will be developed in the general setting of class functions. The result of most interest to us will then be an immediate corollary in terms of the special case of characters.

Let $H$ be a subgroup of $G$ and let $f$ be a class function on $H$. Since the irreducible characters of $H$ form a basis for the vector space $CF_H$ of class functions on $H$, $f$ can be thought of as a linear combination of characters. Now define $f_0$ on $G$ by:

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in H \\ 0 & \text{if } x \in G - H \end{cases}$$

and $f'$ on $G$ by:

$$f'(x) = \frac{1}{|H|} \sum_{t \in G} f_0(txt^{-1}).$$

The only (possibly) non-zero terms in the sum are those where $txt^{-1} \in H$, and such terms are equal to $f(txt^{-1})$.

Hence, $f'$ is a class function on $G$. We say that $f'$ is the class function on $G$ induced by $f$ and write $f' = \text{Ind}_H^G f$.

If we now recall the inner product defined on class functions in Section 5.2, we can state:

7.3 Theorem: (Frobenius) Let $H$ be a subgroup of $G$, let $f$ be a class function on $H$, and let $g$ be a class function on $G$. Then $(f, g|_H) = (\text{Ind}_H^G f, g)$, where $g|_H$ is the restriction of $g$ to $H$. 
Proof: As in the comments above, let
\[ f_0(x) = \begin{cases} f(x) & \text{if } x \in H \\ 0 & \text{if } x \in G - H \end{cases} \]

Then:
\[ \left( \text{Ind}_H^G f, g \right) = \frac{1}{|G|} \sum_{t \in G} \text{Ind}_H^G f(t) \overline{g(t)} = \frac{1}{|G|} \sum_{t \in G} \left( \frac{1}{|H|} \sum_{x \in G} f_0(xtx^{-1}) \overline{g(xtx^{-1})} \right) \]

\[ = \frac{1}{|G|} \frac{1}{|H|} \sum_{t \in G} \sum_{x \in G} f_0(xtx^{-1}) \overline{g(xtx^{-1})} \]

from the definition of \( \text{Ind}_H^G f \) and the fact that \( g \) is a class function. The double summation will simply be the sum of all possible products of the form \( f_0(xtx^{-1}) \overline{g(xtx^{-1})} \) for all \( x, t \in G \). So we may change the order of summation and write:
\[ \left( \text{Ind}_H^G f, g \right) = \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{t \in G} f_0(xtx^{-1}) \overline{g(xtx^{-1})} \]

Now, for fixed \( x \in G \), as \( t \) ranges over all elements of \( G \), so does \( xtx^{-1} \). So with the substitution \( y = xtx^{-1} \) (and slight abuse of notation):
\[ \left( \text{Ind}_H^G f, g \right) = \frac{1}{|G|} \frac{1}{|H|} \sum_{x \in G} \sum_{y \in G} f_0(y) \overline{g(y)} \]

But since \( f \) is a class function on \( H \), the non-zero terms are repeated \( |G| \) times; and the only possible non-zero terms occur when \( y \in H \). So:
\[ \left( \text{Ind}_H^G f, g \right) = \frac{1}{|H|} \sum_{y \in G} f_0(y) \overline{g(y)} = \frac{1}{|H|} \sum_{y \in H} f(y) \overline{g|_H(y)} = (f, g|_H). \qed \]
The application to representations may be stated as:

7.4 Corollary: Let $\chi$ be the character of an irreducible representation $W$ of $H$ (a subgroup of $G$), and let $\Psi$ be the character of an irreducible representation of $V$ of $G$. Then:

$$(\chi, \Psi|_H) = (\text{Ind}^G_H \chi, \Psi),$$

where $\Psi|_H$ is the character of the representation $V$ restricted to $H$.

In light of section 5.3, we conclude with a particularly nice intuitive interpretation of this corollary: The number of times that $W$ occurs in the restriction of $V$ to $H$ is equal to the number of times that $V$ is contained in $\text{Ind}^G_H W$. 
Chapter 8

Summary

The purpose of this paper was to survey elementary concepts of linear representations of finite groups. Chapter 1 presented fundamental definitions of representation of a group $G$ by automorphisms of a vector space, $n \times n$ matrices, and by $G$-modules. Some concrete examples were displayed, and a notion of equivalence of representations was defined.

For a field $F$ and group $G$, the group algebra $FG$ was developed in Chapter 2. The important result here was the regular representation of $G$. Examples were given.

According to Chapter 3, if $V$ is a representation of $G$, it may have nontrivial subspaces which are also representations of $G$. Such was the definition of subrepresentations. This led to the quotient representation and to the possibility of writing $V$ as a direct sum of subrepresentations. An important result here was that the regular representation $FG$ has a subrepresentation of degree 1 known as the unit representation.

Chapter 4 stated that every representation is a direct sum of irreducibles (Maschke). Another major theorem here was Clifford's Theorem: Given an irreducible representation $V$ of $G$ and an irreducible $H$-submodule for $H \Delta G$, we can know
all of the irreducible $H$-submodules of $V_H$. Schur's Lemma and corollaries paved the way for results about characters.

The character of a representation was defined as the trace of the linear transformation or matrix associated with each element of a group. Chapter 5 developed the properties of group characters through the use of an inner product which determined whether two irreducible representations were equivalent.

Characters were further examined in Chapter 6. A noteworthy result here was that the number of nonequivalent irreducible representations of $G$ is equal to the number of conjugacy classes of $G$. All of the irreducible characters of $S_3$ and $D_4$ were displayed in character tables.

If $H$ is a subgroup of $G$, Chapter 7 showed how a representation of $H$ induced a representation of $G$. The concept of induced characters was presented, culminating in the Frobenius reciprocity formula.
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