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Title: Elements of Linear Representations of Finite Groups

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This paper is a survey of elementary concepts of the theory of representation of finite groups; wherein abstract groups are realized as groups of linear transformations or matrices. Basic definitions and examples of the above are given, as well as a notion of an equivalence relation for representations. The regular representation is presented through the concept of a group algebra. Other properties, such as subrepresentations and irreducible representations, lead to an important result about the reducibility of representations of subgroups known as Clifford's Theorem. The character of a representation is then defined as the trace of the linear transformation or matrix which represents each element of the group. The relationship between characters and representations is developed including: (1) orthogonality relations from Schur's Lemma; and (2) the fact that the number of irreducible representations of a group is equal to its number of conjugacy classes. Finally, the concepts of induced representations and characters are explored, culminating in the Frobenius reciprocity formula.

ELEMENTS OF LINEAR REPRESENTATION OF FINITE GROUPS

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Chapter 1

Basic Definitions and Examples

Insights into the structure of abstract groups can be gained by their realization as familiar groups. In particular, we will consider homomorphisms of finite groups into groups of linear transformations and groups of matrices.

1.1 Representation by Automorphism

In the following, G will be a finite multiplicative group, and V will be a vector space of finite dimension over F, a subfield of $\mathscr{C}$, the complex numbers. Let GL(V) denote the set of all invertible linear transformations of V onto itself. GL(V) acquires a group structure if multiplication is defined as composition: $(f \circ g)(v) = f(g(v))$ for all $f,g \in GL(V)$,
$v \in V$.

<u>1.1 Definition:</u> The map $\rho: G \to GL(V)$ is a <u>linear representation of G over F</u> provided ρ is a group homomorphism from G into GL(V).

So for all $g \in G$, $\rho(g) \in GL(V)$. Of course, the basic properties of group homomorphisms hold for ρ:

1. $\rho(gg') = \rho(g) \circ \rho(g')$ for all $g, g' \in G$;
2. $\rho(1) = Id_V$ (The group identity 1 maps to the identity transformation on V.)

3. $\rho(g^{-1}) = (\rho(g))^{-1}$; and

4. $\rho(g^m) = (\rho(g))^m$ for any integer m.

In the sequel, representation will mean linear representation.

Strictly speaking, a representation of a group G over a field F is given by a pair (V,ρ) and we say that V is a <u>representation space</u> of G. However, when clear from the context, we will simply refer to the representation (V,ρ) as ρ or V. Further, since GL(V) is often called the group of automorphisims of V, we will often refer to ρ as a <u>automorphism representation</u> of G.

<u>1.2 Definition:</u> Let (V,ρ) be a representation of G, with dim V = n. Then (V,ρ) has <u>degree</u> n.

<u>Examples:</u>

① Let G = S_n, the symmetric group of degree n, and let $V=F^n$. As a basis for V, take $B = \{e_1,e_2, \ldots e_n\}$, where $e_i = (0,0,\ldots,1, \ldots,0)$. For any $\sigma \in S_n$, define a
^ i th position
mapping $\rho(\sigma)$ on V by:

1. $\rho(\sigma)(e_i) = e_{\sigma(i)}$, $i = 1,\ldots,n$;

2. $\rho(\sigma)(e_i + e_j) = e_{\sigma(i)} + e_{\sigma(j)}$, $i, j = 1,\ldots,n$;

3. For any $\alpha \in F$, $\rho(\sigma)(\alpha e_i) = \alpha e_{\sigma(i)}$.

In other words, $\rho(\sigma)$ is a linear transformation on V.

Furthermore, for any $\sigma \in S_n$, $(\rho(\sigma^{-1}) \circ (\rho(\sigma))(e_i) = \rho(\sigma^{-1})(\rho(\sigma)(e_i)) = \rho(\sigma^{-1})(e_{\sigma(i)}) = e_{\sigma^{-1}(\sigma(i))} = e_i$.

Similarly, $(\rho(\sigma) \circ \rho(\sigma^{-1}))(e_i) = e_i$, so $\rho(\sigma)$ is invertible and we have $\rho: S_n \to GL(V)$. For any $\sigma,\tau \in G = S_n$, $\rho(\sigma\circ\tau)(e_i) = e_{(\sigma\circ\tau)(i)} = e_{\sigma(\tau(i))} = \rho(\sigma)(e_{\tau(i)}) = \rho(\sigma)(\rho(\tau)(e_i)) = (\rho(\sigma) \circ \rho(\tau))(e_i)$. Hence, ρ is a group homomorphism and a representation of S_n over F. This representation is called the canonical representation of S_n.

It is worthy to note here that while the order of S_n is n!, the degree of the representation is n. The question of whether a representation of degree equal to the order of the group can always be found will be addressed later.

② The trivial representation of degree n: Let G be a finite group and V a vector space of dimension n. Define: $\rho: G \to GL(V)$ by $\rho(g) = Id_V$ for all $g \in G$. Clearly ρ is a representation of G. Both of these examples will surface again in other settings.

1.2 Matrix Representation

It is necessary to recall from linear algebra that there is an nxn matrix associated with a given linear transformation of an n-dimensional vector space and a given basis. If V is such a space over F with basis $B = \{V_1, ..., V_n\}$, and $T \in GL(V)$, then $T(V_j) = \sum_{i=1}^{n} a_{ij} V_i$, $1 \le j \le n$, $a_{ij} \in F$.

The matrix $[T]_B = (a_{ij})$, where the a_{ij} are taken to be the j^{th} column of $[T]_B$, is called the matrix of T relative to the basis B. Clearly, $[T]_B$ is uniquely determined by T and B.

This relationship provides a second major type of representation.

Denote by GL(n,F) the set of all nxn nonsingular matrices with entries from the field F. Under the operation of usual matrix multiplication, GL(n,F) is a group. This group is called the general linear group of degree n over F.

1.3 Definition: The mapping $\theta : G \rightarrow GL(n,F)$ is a matrix representation of G over F of degree n provided θ is a group homomorphism.

So for x, y $\in$ G, $\theta(x)$ and $\theta(y)$ are nxn matrices such that $\theta(xy) = \theta(x)\ \theta(y)$.

Before giving examples, we shall make concrete the connection between automorphism and matrix representations suggested by remarks preceding the definition.

1.4 Theorem: Let $\rho : G \rightarrow GL(V)$ be an automorphism representation of G. Let $B = \{V_1, \ldots, V_n\}$ be a basis for V and let $[\rho(g)]_B$ be the matrix of $\rho(g)$ relative to B for any $g \in G$. Then the mapping $\theta : G \rightarrow GL(n,F)$ given by $\theta(g) = [\rho(g)]_B$ is a matrix representation of G.

Proof: For any $g, g' \in G$, let $[\rho(g)]_B = (a_{ij})$ and $[\rho(g')]_B = (b_{ij})$ where $a_{ij}, b_{ij} \in F$. Then we have $\rho(g)(V_j) = \sum_{i=1}^{n} a_{ij} V_i$ and $\rho(g')(V_j) = \sum_{i=1}^{n} b_{ij} V_i$ for any $V_j \in B$. Now, the k, j entry of $[\rho(g)]_B\ [\rho(g')]_B = (a_{ij})(b_{ij})$ is given by $\sum_{i=1}^{n} a_{ki} b_{ij}$.

On the other hand, to look at the entries in $[\rho(gg')]_B$, we compute: $\rho(gg')(V_j) = (\rho(g) \circ \rho(g'))(v_j) = \rho(g)(\rho(g')(V_j))$

$$= \rho(g)\left(\sum_{i=1}^{n} b_{ij}v_i\right) =$$

$$\sum_{i=1}^{n} b_{ij}\rho(g)(v_i) = \sum_{i=1}^{n} b_{ij}\left(\sum_{k=1}^{n} a_{ki}V_k\right)$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n} b_{ij}\, a_{ki}V_k = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ki}\, b_{ij}\right) V_k$$

So the k,j entry of $[\rho(gg')]_B$ is $\sum_{k=1}^{n} a_{ki}\, b_{ij}$. But this is exactly the k,j entry of $(a_{ij})(b_{ij}) = [\rho(g)]_B\,[\rho(g')]_B$ and θ is a matrix representation.❑

Suppose now that a matrix representation $\theta : G \to GL(n,F)$ is given. Can a corresponding automorphism representation be found? If we let $V = F^n$ and define $\rho : G \to GL(F^n)$ by $\rho(g) : F^n \to F^n$ such that $\rho(g)(v) = \theta(g)v$ (*) for all $g \in G$, $v \in F^n$, ρ is a representation of G. Note that the left-hand side of *, $\rho(g)(v)$, is a linear transformation acting on $v \in F^n$; while the right-hand side of * is the product of an nxn matrix with an nx1 matrix.

To see that ρ is, in fact, a group homomorphism, let g, $g' \in G$ and $v \in F^n$ and compute as follows:

$\rho(gg')(v) = \theta(gg')v = \theta(g)\theta(g')v = \theta(g)(\theta(g')v) = \theta(g)(\rho(g')(v)) = \rho(g)(\rho(g')(v)) = (\rho(g) \circ \rho(g'))(v)$. Note that, for any $g \in G$, the matrix of $\rho(g)$ relative to the standard basis $B = \{e_1, e_2, \ldots, e_n\}$ for F^n is $\theta(g)$. That is, $[\rho(g)]_B = \theta(g)$.

Examples:

③ Let $G = S_n$ as in example ①. Define $\theta: S_n \to GL(n,F)$ by $\theta(\sigma) = (a_{ij})$ where $a_{\sigma(i)j} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$, for all $\sigma \in S_n$.

Notice that $\theta(\sigma)$ has a 1 in the $\sigma(j)$, j positions, for $j = 1,\ldots,n$. That is, the j^{th} column contains a 1 in the $\sigma(j)^{th}$ row. To see that θ is indeed a matrix representation, take the automorphism representation ρ from Example ① and express it in matrix form relative to B, the standard basis of F^n. Then $[\rho(g)]_B = \theta(g)$ for all $g \in G$ and, by Theorem 1.4, θ is a matrix representation. For instance, consider $\sigma = (1\ 3)$ and $\tau = (1234)$ as elements of $G = S_4$. (Note: sometimes σ and τ are written as $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, respectively.)

In $\theta(\sigma)$ we have $j = 1 \Rightarrow \sigma(j) = 3$; $j = 2 \Rightarrow \sigma(j) = 2$; $j = 3 \Rightarrow \sigma(j) = 1$; and $j = 4 \Rightarrow \sigma(j) = 4$. so $\theta(\sigma) = \begin{pmatrix} 0010 \\ 0100 \\ 1000 \\ 0001 \end{pmatrix}$.

In $\theta(\tau)$, $j = 1 \Rightarrow \tau(j) = 2$; $j = 2 \Rightarrow \tau(j) = 3$; $j = 3 \Rightarrow \tau(j) = 4$; $j = 4 \Rightarrow \tau(j) = 1$. Hence, $\theta(\tau) = \begin{pmatrix} 0001 \\ 1000 \\ 0100 \\ 0010 \end{pmatrix}$.

Because the matrices in Example ③ are nxn, the representation is said to be of <u>degree</u> <u>n</u>. Another representation of S_n, this one of degree 1, is given below.

④ Let $G = S_n$ and $F = \mathscr{C}$. Define $\theta : S_n \to GL(1,\mathscr{C})$ by :

$$\theta(\sigma) = \begin{cases} 1 \text{ if } \sigma \text{ is an even permutation} \\ -1 \text{ if } \sigma \text{ is an odd permutation.} \end{cases}$$

(Note that $GL(1,\mathscr{C})$ is just $\mathscr{C}^*$, the multiplicative group of non-zero complex numbers.) For any $\sigma, \tau \in S_n$:

$$\theta(\sigma \circ \tau) = \begin{cases} 1 \text{ if } \sigma \circ \tau \text{ even} \\ -1 \text{ if } \sigma \circ \tau \text{ odd} \end{cases} =$$

$$\begin{cases} 1 \text{ if both } \sigma \text{ and } \tau \text{ even or both } \sigma \text{ and } \tau \text{ odd} \\ -1 \text{ if } \sigma \text{ even and } \tau \text{ odd or } \sigma \text{ odd and } \tau \text{ even} \end{cases} = \theta(\sigma)\theta(\tau)$$ in any case. So θ is a matrix representation of degree 1; in fact, it is known as the <u>alternating representation.</u>

⑤ The set G of rotations about the origin in the real plane is a group under composition of rotations. Note that if g_α, $g_\beta \in G$ are rotations through angles α and β, respectively, then $g_\alpha g_\beta = g_\beta g_\alpha \in G$ is just the rotation through $\alpha + \beta$. Let $\mathscr{R}$ be the real numbers

For $(x,y) \in \mathscr{R}^2$, $g_\alpha(x,y) = (x', y')$ where

$$x' = x \cos \alpha - y \sin \alpha$$

$$y' = x \sin \alpha + y \cos \alpha$$

Define $\theta : G \to GL(2,\mathscr{R})$ by

$$\theta(g_\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}.$$

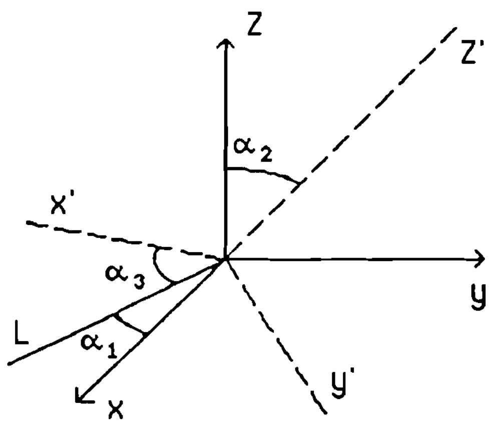
Then θ is a matrix representation of G of degree 2 since:

$$\theta(g_\alpha)\ \theta(g_\beta) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} =$$

(Applying trig identities)

$$\begin{bmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) \end{bmatrix} = \theta(g_\alpha\ g_\beta)\ .$$

⑥ Now let G be the group of rotations about the origin in three dimensional space. Suppose that, as the result of a rotation, the x-, y-, and z-, axes are transformed into the x′-, y′-, and z′- axes.



The rotation can be considered as the product (composition) of three rotations: (1) g_{α_1} → through angle α_1 about the z-axis carrying the x-axis to line L; L is the line of intersection of the x-y plane and the x′-y′ plane.

(2) g_{α_2} → through angle α_2 about L carrying the z-axis to the z′-axis; and

(3) g_{α_3} → through angle α_3 about the z′-axis.

$\theta : G \to (GL\ 3, \mathscr{R})$ is as follows:

$$\theta(g_{\alpha_1}) = \begin{bmatrix} \cos\alpha_1 & -\sin\alpha_1 & 0 \\ \sin\alpha_1 & \cos\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\theta(g_{\alpha_2}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_2 & -\sin\alpha_2 \\ 0 & \sin\alpha_2 & \cos\alpha_2 \end{bmatrix}; \text{ and}$$

$$\theta(g_{\alpha_3}) = \begin{bmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, the entire rotation is represented by:

$$\theta(g_{\alpha_1}\, g_{\alpha_2}\, g_{\alpha_3}) = \theta(g_{\alpha_1})\, \theta(g_{\alpha_2})\, \theta(g_{\alpha_3}) =$$

$$\begin{bmatrix} \cos\alpha_1 \cos\alpha_3 - \sin\alpha_1 \cos\alpha_2 \sin\alpha_3 & -\cos\alpha_1 \sin\alpha_3 - \sin\alpha_1 \cos\alpha_2 \cos\alpha_3 & \sin\alpha_1 \sin\alpha_2 \\ \sin\alpha_1 \cos\alpha_3 + \cos\alpha_1 \cos\alpha_2 \sin\alpha_3 & -\sin\alpha_1 \sin\alpha_3 + \cos\alpha_1 \cos\alpha_2 \cos\alpha_3 & -\cos\alpha_1 \sin\alpha_2 \\ \sin\alpha_2 \sin\alpha_3 & \sin\alpha_2 \cos\alpha_3 & \cos\alpha_2 \end{bmatrix}.$$

Example ⑥ is presented as an important concrete realization of group theoretic results for the physical sciences.

1.3 Representation Modules

Representations can be viewed in a somewhat more general context via the representation module defined below.

<u>1.5 Definition:</u> Let G be a group and let V be a vector space over F. V is called a <u>left G-module</u> if a multiplication is defined on V such that:

for all g, g′ ∈ G; v, v′ ∈ V, a ∈ F,

(1) gv ∈ V

(2) g(v + v′) = gv + gv′

(3) $g(av) = a(gv)$

(4) $(gg')v = g(g'v)$

(5) $1v = v$; where 1 is the identity of G.

A right G-module can be defined in a similar way. For our purposes, G-module will mean left G-module.

The connection between representation modules and the representation of G by automorphism is given by:

1.6 Theorem: The mapping $\rho : G \to GL(V)$ is an automorphism representation of G <=> V is a G-module.

Proof: (=>) If ρ is an automorphism representation, then $\rho(g)$ is a linear transformation on V, for all $g \in G$.

Define the multiplication by $gv \equiv \rho(g)(v)\ \forall\ g \in G,\ v \in V$. It is completely routine to establish from the conditions of the definition that V is a G-module. Parts (2) and (4) are shown here:

(2) $g(v + v') = \rho(g)(v + v') = \rho(g)(v) + \rho(g)(v') = gv + gv'$;

and

(4) $(gg')v = (\rho(g) \circ \rho(g'))(v) = \rho(g)(\rho(g')(v)) = \rho(g)(g'v) = g(g'v)$.

(<=) Let G be a group and V a G-module. Define $\rho(g): V \to V$ by $\rho(g)(v) \equiv gv$. That $\rho(g)$ is a linear transformation is clear from (2) and (3) in the definition of a G-module. Also, $\rho(g)$ is invertible $\forall g \in G$ since $(\rho(g))^{-1} = \rho(g^{-1})$. Hence, $\rho(g) \in GL(V)$ for all $g \in G$.

Finally, for all $v \in V$, $g, g' \in G$: $\rho(gg')(v) = (gg')v = g(g'v) = \rho(g)(\rho(g')(v)) = (\rho(g) \circ \rho(g'))(v)$. Hence, ρ is a group homomorphism from G into GL(V) and, therefore, a representation.❑

1.4 Equivalence

The relationships between representation by automorphisms, matrices, and modules have now been outlined. But what can be said about, say, two automorphism representations of the same group G? Consider this motivation for a definition: Let (V,ρ) be a representation of G over F, and let V' be a vector space isomorphic to V. Now define a map $\rho' : G \to GL(V')$ by $\rho'(g) : V' \to V'$ where $\rho'(g) = \alpha \circ \rho(g) \circ \alpha^{-1}$ where α is a vector space isomorphism from V to V'.

Then, for all $g, g' \in G$: $\rho'(gg') = \alpha \circ \rho(gg') \circ \alpha^{-1} = \alpha \circ \rho(g) \circ \rho(g') \circ \alpha^{-1} = \alpha \circ \rho(g) \circ (\alpha^{-1} \circ \alpha) \circ \rho(g') \circ \alpha^{-1} = (\alpha \circ \rho(g) \circ \alpha^{-1}) \circ (\alpha \circ \rho(g') \circ \alpha^{-1}) = \rho'(g) \circ \rho'(g')$.
Hence (V',ρ') is also a representation of G.

1.7 Definition: Two representations of G over F, (V,ρ) and (V',ρ'), are equivalent provided there exists a vector space isomorphism $\alpha: V \to V'$ such that $\alpha \circ \rho(g) = \rho(g') \circ \alpha$ $\forall g \in G$. In this case, we write $\rho \approx \rho'$. A similar definition applies to matrix representations.

<u>1.8</u> <u>Definition:</u> Two matrix representations, θ and Φ, of G of degree n over F are equivalent ($\theta \approx \Phi$) provided there exists a fixed $P \in GL(n,F)$ such $\theta(g) = P\,\Phi(g)\,P^{-1}$ for all $g \in G$.

Suppose that θ, Φ, and Ψ are matrix representations of G of degree n over F; and that $\theta \approx \Phi$ and $\Phi \approx \Psi$. Then we have $P \in GL(n,F)$ such that $\theta(g) = P\Phi(g)\,P^{-1}$, and $Q \in GL(n,F)$ such that $\Phi(g) = Q\,\Psi(g)\,Q^{-1}$, for all $g \in G$. Combining:

$$\theta(g) = P(Q\,\Psi(g)\,Q^{-1})\,P^{-1} =$$

$$(PQ)\,\Psi(y)\,(Q^{-1}\,P^{-1}) = (PQ)\,\Psi(g)\,(PQ)^{-1}.$$

$\therefore\ \theta \approx \Psi$, and we have transitivity in equivalence of matrix representations.

It is similarly easy to establish, for both matrix and automorphism representations, the other criteria necessary to show:

<u>1.9</u> <u>Theorem:</u> Equivalence of representations is an equivalence relation on the set of all representations of G of degree n over F.

Two observations are in order concerning equivalence:

1. Recall that for a representation (V,ρ), we were able to compute a corresponding matrix representation θ. But θ was dependent upon the choice of basis for V. Another choice of basis would result in another matrix representation θ'. However, from linear algebra, the relationship between the

matrices of a linear transformation relative to two distinct bases is precisely that of the definition of $\theta \approx \theta'$.

2. Similarly, given a matrix representation θ, every choice of a vector space of dimension n yields an equivalent automorphism representation of degree n.

Chapter 2

The Group Algebra of G

2.1 Definition of the Group Algebra

Representation of a group can be given a more algebraic (and abstract) formulation by introducing the group algebra of G.

First, it is necessary to recall the definition of an algebra over a field.

2.1 Definition: A vector space V over a field F is called an algebra over F if a multiplication is defined on V such that, for all u, v, w $\in V$, $\alpha \in F$:

(1) $u(v+w) = uv + uw$;

(2) $(u+v)w = uw + vw$; and

(3) $\alpha(uv) = (\alpha u)v = u(\alpha v)$.

Now let G be a finite group and let F be a field.

2.2 Definition: The group algebra FG is the set of all formal sums:

$$FG = \left\{ \sum_{g \in G} a_g g \mid a_g \in F \right\}$$ with the operations:

(1) $$\sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g+b_g)g;$$

(2) $$\alpha \left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} \alpha a_g g$$ for all $\alpha \in F$; and

(3) $\left(\sum_{g \in G} a_g\, g\right)\left(\sum_{h \in G} b_h\, h\right) = \sum_{g,h \in G} (a_g\, b_h) gh.$

<u>2.3</u> <u>Theorem:</u> FG as defined above is an algebra over F.

<u>Proof:</u> It is routine to verify that FG is a vector space over F under the addition and scalar multiplication in (1) and (2) of the definition (2.2).

To see that the multiplication in (3) of definition 2.2 makes FG and algebra over F, we use definition 2.1: For a_g, b_h, $c_h \in F$,

(1) $$\sum_{g \in G} a_g\, g \left(\sum_{h \in G} b_h\, h + \sum_{h \in G} c_h\, h \right) =$$

$$\sum_{g \in G} a_g\, g \left(\sum_{h \in G} [b_h + c_h] h \right) =$$

$$\sum_{g,h \in G} a_g (b_h + C_h)\; gh = \sum_{g,h \in G} [a_g b_h + a_g c_h]\; gh =$$

$$\sum_{g,h \in G} [a_g\, b_h] gh + \sum_{g,h \in G} [a_g\, c_h] gh =$$

$$\left(\sum_{g \in G} a_g\, g \right) \left(\sum_{h \in G} b_h\, h \right) + \left(\sum_{g \in G} a_g\, g \right) \left(\sum_{h \in G} c_h\, h \right).$$

(2) The proof is similar to the proof of (1).

(3) For any $\alpha \in F$, $\alpha \left(\sum_{g \in G} a_g\, g \right) \left(\sum_{h \in G} b_h\, h \right) =$

$$\alpha \sum_{g,h \in G} [a_g\, b_h] gh = \sum_{g,h \in G} [\alpha a_g]\; [b_h] gh =$$

$$\left(\sum_{g,h \in G} [\alpha a_g] g \right) \left(\sum_{h \in G} b_h\, h \right) = \left(\alpha \sum_{g \in G} a_g\, g \right) \left(\sum_{h \in G} b_h\, h \right).$$

Also, $\sum_{g,h \in G} [a_g]\; [\alpha b_h] gh = \left(\sum_{g \in G} a_g\, g \right) \left(\sum_{h \in G} [\alpha b_h]\; h \right) =$

$\left(\sum_{g \in G} a_g g\right)\left(\alpha \sum_{h \in G} b_h h\right)$, and (3) is established.

Hence, FG is an algebra over F.□

Note that, with the multiplication so defined, FG is also a ring and has as unity the one-term sum $1_{FG} = 1_F 1_G$.

In addition, as a vector space, FG has a most interesting basis. Every element of FG is, by definition, simply a linear combination of the elements of G over F. So the set of elements $\{1_F g\}_{g \in G}$ forms a basis for FG over F. As a consequence, $\dim(FG) = |G|$, where $|G|$ is the order of G.

2.2 Representation of FG

To continue to move toward more algebraic concepts, such as representation of the group algebra FG, we must extend some fundamental notions to answer the question: What, precisely, is meant by a representation of an algebra (vis-à-vis, a group)?

If V is a vector space over F, denote by Hom(V,V) the set of all linear transformation of V into itself. For comparison, notice that $GL(V) \subset \text{Hom}(V,V)$. For all $T_1, T_2, \in \text{Hom}(V,V)$, $v \in V$, $\alpha \in F$, we define addition, scalar multiplication, and multiplication as:

(1) $(T_1 + T_2)(v) = T_1(v) + T_2(v)$;

(2) $(\alpha T_1)(v) = \alpha(T_1(v))$; and

(3) $(T_1 T_2)(v) = (T_1 \circ T_2)(v) = T_1(T_2(v))$, i.e, multiplication is the composition of linear transformations.

With the operations so defined, it is easy to check that Hom(V,V) is an algebra over F.

2.4 Definition: Let A be an algebra with unity over F and V a vector space over F. Then the mapping $\Phi : A \rightarrow$ Hom(V,V) is a representation of A over F provided:

For all a, b $\in$ A, $\alpha \in$ F,

(1) $\Phi(a + b) = \Phi(a) + \Phi(b)$;

(2) $\Phi(\alpha a) = \alpha\Phi(a)$;

(3) $\Phi(ab) = \Phi(a) \circ \Phi(b)$; and

(4) $\Phi(1_A) = Id_V$, where Id_V is the identity transformation on V and 1_A is the mulitplicitive identity of A. So Φ is just an algebra homomorphism.

We wish to define such a representation for the algebra FG. This can be accomplished by a natural and unique extension of the representation of the group G.

Let (V,ρ) be a representation of G. Define $\rho^* :$ FG $\rightarrow$ Hom(V,V) by:

$$\rho^*\left(\sum_{g \in G} a_g\, g\right) = \sum_{g \in G} a_g\, \rho(g) \text{ for all } \sum_{g \in G} a_g\, g \in FG.$$

Since $\rho(g)$ is a linear transformation for each $g \in G$, the image of $\sum_{g \in G} a_g\, g$ under ρ^* is itself a linear transformation which is the sum of scalar multiples of linear transformations.

<u>2.5 Theorem:</u> The mapping ρ^* is an algebra homomorphism and, thus, a representation of FG.

<u>Proof:</u> We refer to conditions (1) - (4) in definition 2.4. For all $\sum_{g \in G} a_g g$, $\sum_{g \in G} b_g g \in FG$, $\alpha \in F$, we have:

(1) $$\rho^*\left(\sum_{g \in G} a_g g + \sum_{g \in G} b_g g\right) = \rho^*\left(\sum_{g \in G} (a_g+b_g) g\right) =$$

$$\sum_{g \in G} [a_g + b_g] \rho(g) = \sum_{g \in G} (a_g\rho(g) + b_g \rho(g)) =$$

$$\sum_{g \in G} a_g \rho(g) + \sum_{g \in G} b_g \rho(g) = \rho^*\left(\sum_{g \in G} a_g g\right) + \rho^*\left(\sum_{g \in G} b_g g\right).$$

(2) $$\rho^* (\alpha \sum_{g \in G} a_g g), = \rho^*\left(\sum_{g \in G} [\alpha a_g] g\right) = \sum_{g \in G} [\alpha a_g] \rho(g) =$$

$$\alpha \sum_{g \in G} a_g \rho(g) = \alpha\rho^*\left(\sum_{g \in G} a_g g\right).$$ (Since α does not depend on $g \in G$.)

(3) $$\rho^*\left(\left(\sum_{g \in G} a_g g\right)\left(\sum_{h \in G} b_h h\right)\right) =$$

$$\rho^*\left(\sum_{g,h \in G} (a_g b_h) gh\right) = \sum_{g,h \in G} (a_g b_h) \rho(gh)) = \sum_{g,h \in G} (a_g b_h) \quad (*)$$

$\rho(g)\rho(h)$.

Let us pause for a moment to see where we are. The sum in (*) calls for the composition of all linear transformations $\rho(g) \circ \rho(h)$ such that the product of the corresponding scalars, $a_g b_h$, is not zero. That is, $a_g \neq 0$ and $b_h \neq 0$.

Now since $\rho(g)$, $\rho(h) \in \mathrm{Hom}(V,V)$ (*) is equal to

$$\sum_{g,h \in G} a_g\rho(g) \circ b_h\rho(h) \quad (**).$$ But we can achieve the same composition of all combinations of $\rho(g)$ and $\rho(h)$ with $a_g \neq 0$ and $b_h \neq 0$ by writing (**) as

$$\sum_{g \in G} a_g\,\rho(g) \circ \sum_{h \in G} b_h\,\rho(h).$$

And $$\sum_{g \in G} a_g\,\rho(g) \circ \sum_{h \in G} b_h\,\rho(h) = \rho^*\Big(\sum_{g \in G} a_g\,g\Big) \circ \rho^*\Big(\sum_{h \in G} b_h\,h\Big).$$

(4) Recall that the unit element of FG is $1_{FG} = 1_F\,1_G$. So $\rho^*(1_{FG}) = \rho^*(1_F\,1_G) = 1_F\rho(1_G) = 1_F\,\mathrm{Id}_V = \mathrm{Id}_V$, since ρ is a group homomorphism from G to GL(V).

Therefore, ρ^* is a representation of FG. ❑

A couple of remarks about this result are in order.

1. The representation ρ^* of FG is said to be the representation corresponding to the representation ρ of G.

2. Recall from definition 1.5 and theorem 1.6 that if $\rho : G \to GL(V)$ is a representation, then V is called a G-module. Now, if for any $\gamma = \sum_{g \in G} a_g\,g \in FG$, $v \in V$, we define $\rho^*(\gamma) = \gamma v$, we have that V is an FG - module.

Suppose now that $\rho^* : FG \to \mathrm{Hom}(V,V)$ is any representation of FG. Recall that the elements of G form a basis for FG as a vector space; hence, $G \subset FG$ (Under the identification $g \leftrightarrow 1_F g$). Our aim now is to define

$\rho: G \to GL(V)$ so that ρ will be a representation of G. Consider the restriction of ρ^* to G and define $\rho = \rho^*|_G$.

Since $GL(V) \subset Hom(V,V)$, we must first be convinced that $\rho(g) \in GL(V)$ for all $g \in G$. Let 1 be the identity of G. Then $\rho(1) = \rho^*(1) = Id_V$. On the other hand $\rho(1) = \rho(gg^{-1}) = \rho^*(gg^{-1}) = \rho^*(g) \circ \rho^*(g^{-1}) = \rho(g) \circ \rho(g^{-1}) = Id_V$, for any $g \in G$. Similarly, $\rho(g^{-1}) \circ \rho(g) = Id_V$, so $\rho(g)^{-1} = \rho(g^{-1})$ and $\rho(g) \in GL(V)$.

To see that ρ is a group homomorphism, let $g, g' \in G$ and compute: $\rho(gg') = \rho^*(gg') = \rho^*(g) \circ \rho^*(g') = \rho(g) \circ \rho(g')$. Hence, we have shown:

<u>2.6</u> <u>Theorem:</u> The mapping $\rho = \rho^*|_G : G \to GL(V)$ is a representation of G with representation space V over F.

So to every representation ρ^* of FG, there corresponds a representation ρ of G. We summarize the discussion concerning theorems 2.5 and 2.6 by observing:

<u>2.7</u> <u>Theorem:</u> There is a one-to-one correspondence between the representations of a finite group G over F with representation space V and the group algebra FG over F with representation space V.

2.3 Regular Representation of a Group

We shall now pursue a question posed in section 1.1: Does every finite group of order n have a representation of degree n?

Let G be a group with $|G| = n$ ($|G|$ is the order of G). Assign to the elements of G some fixed ordering: $1 = g_1, g_2, \ldots, g_n$. Consider the group algebra FG as a vector space of dimension n. In fact, we will use $\mathbf{V} = FG$ as our representation space. The elements of G form a basis for $FG = \mathbf{V}$ and any $v \in \mathbf{V}$ can be uniquely expressed as: $\sum_{i=1}^{n} a_i g_i$ where $a_i \in F$, and $\{g_1, \ldots, g_n\}$ is the basis of $\mathbf{V}$.

Now, for each $x \in G$, define $\rho(x): FG \to FG$ by $\rho(x)(g_i) = xg_i$ on the basis elements and extend $\rho(x)$ linearly:

$$\rho(x)\left(\sum_{i=1}^{n} a_i g_i\right) = \sum_{i=1}^{n} a_i \rho(x)(g_i) = \sum_{i=1}^{n} a_i (xg_i).$$

<u>2.8</u> <u>Theorem:</u> The mapping ρ as defined above is a representation of G over F with representation space FG.

<u>Proof:</u> To see that $\rho(x)$ is invertible for each $x \in G$, note that for any $\sum_{i=1}^{n} a_i g_i \in FG$, $(\rho(x) \circ \rho(x^{-1}))\left(\sum_{i=1}^{n} a_i g_i\right) =$

$$\rho(x)\left(\rho(x^{-1})\left(\sum_{i=1}^{n} a_i g_i\right)\right) = \rho(x)\left(\sum_{i=1}^{n} a_i (x^{-1} g_i)\right) =$$

$$\sum_{i=1}^{n} a_i (xx^{-1} g_i) = \sum_{i=1}^{n} a_i g_i.$$

Similarly, $(\rho(x^{-1}) \circ \rho(x))(\sum_{i=1}^{n} a_i g_i) = \sum_{i=1}^{n} a_i g_i$. So each $\rho(x)$ has an inverse; namely, $\rho(x^{-1})$. Then we have $\rho(x) \in GL(FG)$ for all $x \in G$.

Finally, we establish that $\rho : G \to GL(FG)$ is a group homomorphism. Let $x, y \in G$. Then for each $i = 1, 2, \ldots, n$, $\rho(xy)(g_i) = xyg_i = x(yg_i) = \rho(x)(yg_i) = \rho(x)(\rho(y)(g_i)) = (\rho(x) \circ \rho(y))(g_i)$.

Hence $\rho(xy) = \rho(x) \circ \rho(y)$. Therefore, ρ is a representation of G over F with representation space FG. ❑

2.9 Definition: The representation $\rho : G \to GL(FG)$ above is called the (left) regular representation of G over F.

Remarks concerning the regular representation:

1. We can similarly define the right regular representation by defining $\rho(x)(g_i) \equiv g_i x$, for all $x \in G$, $i=1,\ldots,n$.

2. Since the degree of the regular representation equals $\dim(FG) = n$, we have an affirmative answer to the question at the outset of this section.

The basis $\{1=g_1, g_2,\ldots,g_n\}$ of FG is nothing but the set of images of $1 \in FG$ under the $\rho(g_i)$, $i=1,\ldots,n$. That is, $\rho(g_1)(1) = 1 = g_1$, $\rho(g_2)(1) = g_2, \ldots, \rho(g_n)(1) = g_n$. We will show that any representation, with the property that the images of one vector from the representation space form a basis of that space, is equivalent to the regular representation (FG,ρ).

Let (W,ρ') be a representation of G over F such that there exists a $w \in W$ and $\{\rho'(g)(w) \mid g \in G\}$ form a basis of W.

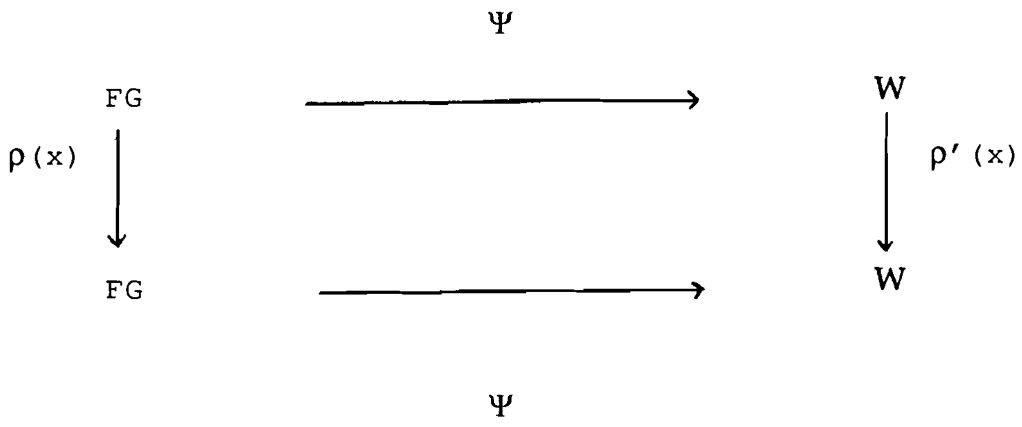
To show that $(FG,\rho) \approx (W,\rho')$, we have need of a vector space isomorphism from FG to W. Define $\Psi : FG \to W$ by $\Psi(g_i) = \rho'(g_i)(w)$, and extend Ψ linearly so that $\Psi\left(\sum_{i=1}^{n} a_i g_i\right) = \sum_{i=1}^{n} a_i \Psi(g_i) = \sum_{i=1}^{n} a_i \rho'(g_i)(w)$.

Now let $\sum_{i=1}^{n} a_i g_i$, $\sum_{i=1}^{n} b_i g_i \in FG$ and suppose $\Psi\left(\sum_{i=1}^{n} a_i g_i\right) = \Psi\left(\sum_{i=1}^{n} b_i g_i\right)$. Then $\sum_{i=1}^{n} a_i \rho(g_i)(w) = \sum_{i=1}^{n} b_i \rho'(g_i)(w)$.

We know that the $\rho'(gi)(w)$, for $i = 1,\ldots,n$ form a basis for the vector space W. But since each element of W, such as $\sum_{i=1}^{n} a_i \rho'(g_i)(w) = \sum_{i=1}^{n} b_i \rho'(g_i)(w)$, can be uniquely written as a linear combination of basis elements, we have that $a_i = b_i$ for $i = 1,\ldots,n$. Hence, $\sum_{i=1}^{n} a_i g_i = \sum_{i=1}^{n} b_i g_i$ and Ψ is one-to-one.

It is a basic property of linear transformations that, since Ψ is one-to-one and $\dim(FG) = \dim W$, then Ψ is an isomorphism from FG to W.

To establish $(FG,\rho) \approx (W,\rho')$, we must now show that, for any $x \in G$, $\Psi \circ \rho(x) = \rho'(x) \circ \Psi$. That is, we show that this diagram commutes:



If g_i is any basis element of FG, $(\Psi \circ \rho(x))(g_i) = \Psi(\rho(x)(g_i)) = \Psi(xg_i) = \rho'(xg_i)(w) = (\rho'(x) \circ \rho'(gi))(w) = \rho'(x)(\rho'(gi)(w)) = \rho'(x)(\Psi(g_i)) = (\rho'(x) \circ \Psi)(g_i)$. Hence; $(FG,\rho) \approx (W,\rho')$.

So far, we have examined the regular representation only in terms of representation by automorphism. As in Chapter 1, for any choice of basis of FG, we have a corresponding matrix representation of G over F. If we stick with our basis G = $\{g_1,\ldots,g_n\}$ of FG, then $\Phi : G \rightarrow GL(n,F)$ will be defined by $\Phi(x) = [\rho(x)]_G$ for any $x \in G$, where $[\rho(x)]_G$ is the matrix of the linear transformation $\rho(x)$ relative to G. Recall that the jth column of $[\rho(x)]_G$ is just the coefficients of $\rho(x)(g_j)$. But since $\rho(x)(g_j) = xg_j$, another basis vector of FG, the j^{th} column will consist of zeros except for a 1 in the i^{th} row where i is such that $g_i = xg_j$. The matrix $[\rho(x)]_G$ can be expressed as $[\rho(x)]_G = (\delta_{g_i,xg_i})$, where δ_{gi,xg_i} is Kronecker delta. This matrix is sometimes called a <u>permutation matrix.</u>

Examples:

① Let $G = \{1, x, x^2\}$ and let $F = \mathscr{R}$, the real numbers. If we denote the elements of G by $g_1 = 1$, $g_2 = x$, $g_3 = x^2$, then a typical element of $FG = \mathscr{R}G$ has the form: $a_1 1 + a_2 x + a_3 x^2 = a_1 g_1 + a_2 g_2 + a_3 g_3$, where $a_1, a_2, a_3 \in \mathscr{R}$. (Note that $\mathscr{R}G$ is isomorphic to $\mathscr{R}^3$ as a vector space.)

If $(\mathscr{R}G, \rho)$ is the regular representation of G via automorphism, then the matrix representing $g_k \in G$, $k = 1, 2, 3$, is given by $[\rho(g_k)]_G = (\delta_{g_i, g_k g_j})$.

For instance, $[\rho(g_3)]_G = (\delta_{g_i, g_3 g_j})$.

We compute the entries of $[\rho(g_3)]_G$

by: $\rho(g_3)(1) = 0_{gi} + 0_{g2} + 1_{g3}$

$\rho(g_3)(x) = 1g_1 + 0g_2 + 0g_3$

$\rho(g_3)(x^2) = 0g_1 + 1g_2 = 0g_3$.

so $[\rho(g_3)]_G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

by similar computations;

$[\rho(g_2)]_G = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$; and

$[\rho(g_1)]_G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

② Let $G = C_4$, the cyclic group of order 4. So $G = \{x, x^2, x^3, x^4 = 1\}$. As in example ①, let us denote the elements by $g_1 = x^4 = 1$; $g_2 = x$, $g_3 = x^2$, $g_4 = x^3$.

Also, let $(\mathscr{R}G, \rho)$ be the regular representation by automorphism, and let $\Phi(g_k) = [\rho(g_k)]_G$ be the matrix representing g_k, $k = 1,2,3,4$. So we have, again, $\Phi(g_k) = (\delta_{g_i}, g_k g_j)$.

But since we know that $x = g_2$ is the generator of $G = C_4$, we need only compute $\Phi(g_2)$. The computations are essentially the same as in example ①, save that $\Phi(g_k)$ will be a 4x4 matrix. The computations yield:

$$\Phi(x) = \Phi(g_2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \quad \text{Also,}$$

$$\Phi(x^2) = \Phi(g_3) = \Phi(x)\Phi(x) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix};$$

$$\Phi(x^3) = \Phi(g_4) = \Phi(x)\Phi(x^2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}; \text{ and}$$

$$\Phi(1) = \Phi(x^4) = \Phi(g_1) = \Phi(x)\Phi(x^3) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

After examining examples ① and ②, it is not difficult to obtain the regular matrix representation of C_n for any integer $n \geq 3$.

③ Let $G = S_3 = \{(1), (12), (13), (23), (123), (132)\}$ and assign the ordering $g_1 = (1)$, $g_2 = (12)$, $g_3 = (13)$, $g_4 = (23)$,

$g_5 = (123)$, and $g_6 = (132)$. Let $(\mathscr{R}G, \rho)$ be the regular representation by automorphism. We wish to specify the regular matrix representation $\Phi : G \to GL(6, \mathscr{R})$.

Since Φ is a group homomorphism, it is sufficient to give the matrices corresponding to a set of generators of

$G = S_3$, such as $\{(12), (123)\} = \{g_2, g_5\}$. We display the computations for the first three columns of $\Phi((12))$:

$\rho(g_2)((1)) = (12)(1) = (12) = 0g_1 + 1g_2 + 0g_3 + \ldots + 0g_6$;

$\rho(g_2)((12)) = (12)(12) = (1) = 1g_1 + 0g_2 + \ldots + 0g_6$; and

$\rho(g_2)((13)) = (12)(13) = (132) = 0g_1 + \ldots + 0g_5 + 1g_6$.

The remaining columns of $\Phi((12))$ and those of $\Phi((123))$ are computed in a similar manner, yielding:

$$\Phi((12)) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}; \text{ and}$$

$$\Phi((123)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

We now return to the general development.

Another method that can be used to obtain the regular representation works through permutations. Let S_n be the symmetric group of degree n and let V be any vector space of dimension n with basis $\{v_1, v_2, \ldots, v_n\}$.

For every $\sigma \in S_n$, define a mapping $\eta(\sigma) : V \to V$ by $\eta(\sigma)(v_i) = v_{\sigma(i)}$ for $i = 1,\ldots,n$. By comparing to example ① in section 1.1, two things are clear:

① For all $\sigma \in S_n$, $\eta(\sigma) \in GL(V)$. so $\eta : S_n \to GL(V)$; and

② η is a group homomorphism.

Hence, η is a representation of S_n.

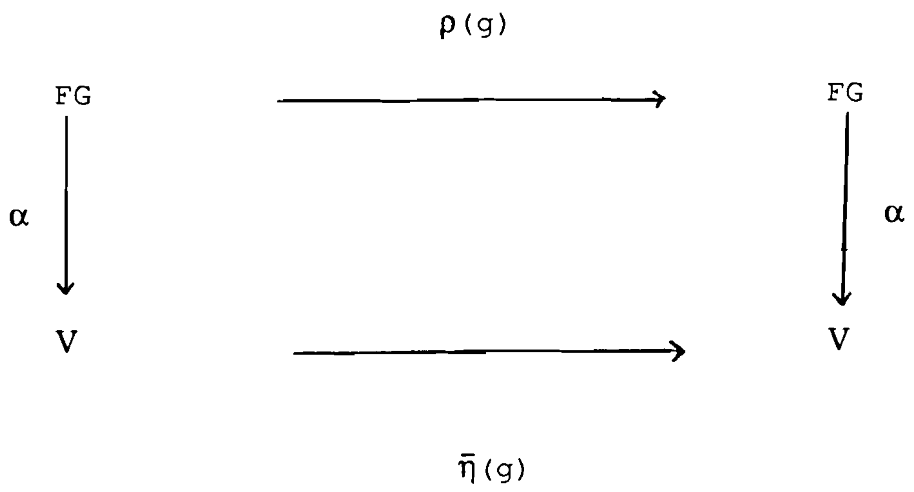
Now let $G = \{g_1,\ldots,g_n\}$ be a group of order n. A classic result of group theory due to Cayley assures us that G is isomorphic to a subgroup of S_n. For every $g \in G$, define $\pi : G \to S_n$ by $\pi(g) = \begin{pmatrix} g_1,\ldots,g_n \\ gg_1,\ldots,gg_n \end{pmatrix}$.

(It is easy to check that, for g, $g' \in G$, $\pi(gg') = \pi(g)\pi(g')$; so π is a group homomorphism. The mapping π is called the left-permutation representation of G.)

Our goal here is to find a linear representation of G, i.e., we need a homomorphism from G to GL(V). Define $\bar{\eta} : G \xrightarrow{\pi} S_n \xrightarrow{\eta} GL(V)$ to be the composition:)$\bar{\eta} = \eta \circ \pi$. It is routine to check that $\bar{\eta}$ is a representation of G over F. In fact, we have:

2.10 Theorem: Let (FG,ρ) be the regular representation of G over F, and let $(V, \bar{\eta})$be as defined above. Then $\rho \approx \bar{\eta}$.

Proof: We require a vector space isomorphism $\alpha : FG \to V$ such that, for all $g \in G$, the following diagram commutes:



Recall that the elements of G form a basis for FG and let $\{v_1, .., v_n\}$ be a basis of V. Define $\alpha : FG \to V$ by $\alpha\left(\sum_{i=1}^{n} a_i\, g_i\right) = \sum_{i=1}^{n} a_i\, v_i$. (In terms of basis elements, $\alpha(g_i) = v_i$.) Then for any $\sum_{i=1}^{n} a_i\, g_i$, $\sum_{i=1}^{n} b_i\, g_i \in FG$, γ, $\beta \in F$; $\alpha\left(\gamma \sum_{i=1}^{n} a_i\, g_i + \beta \sum_{i=1}^{n} b_i\, g_i\right) = \alpha\left(\sum_{i=1}^{n} (\gamma a_i + \beta b_i) g_i\right) =$
$\sum_{i=1}^{n} (\gamma a_i + \beta b_i)\alpha(g_i) = \sum_{i=1}^{n} (\gamma a_i\ \beta b_i) v_i =$
$\gamma\alpha\left(\sum_{i=1}^{n} a_i\, g_i\right) + \beta\alpha\left(\sum_{i=1}^{n} b_i\, g_i\right)$; so α is a linear transformation from FG to V. To see that α is 1-1, and hence an isomorphism, let $\alpha\left(\sum_{i=1}^{n} a_i\, g_i\right) =$ $\alpha\left(\sum_{i=1}^{n} b_i\, g_i\right)$. Then $\sum_{i=1}^{n} a_i\, V_i = \sum_{i=1}^{n} b_i\, V_i$, and since any vector can be uniquely written in V as a linear combination

of the v_i, $i=1,..,n$, $a_i = b_i$ for $i = 1,...,n$. So $\sum_{i=1}^{n} a_i g_i = \sum_{i=1}^{n} b_i g_i$.

To see that $\alpha \circ \rho(g) = \bar{\eta}(g) \circ \alpha$ for any $g \in G$, let $g_i \in FG$ and compute as follows: $(\alpha \circ \rho(g))(g_i) = \alpha(\rho(g)(g_i)) = \alpha(gg_i) = \alpha(g_j) = V_j$, where we have let $gg_i = g_j$ for some $j = 1,...,n$.

On the other hand, $(\bar{\eta}(g) \circ \alpha))(g_i) = \bar{\eta}(g)(\alpha(g_i)) = \bar{\eta}(g)(V_i) = \eta(\pi(g))(V_i) = V_{\pi(g)(i)}$. The only hurdle remaining is the fact that two different common notations have been used here for permutations acting on groups. The "i" in the subscript $\pi(g)(i)$ represents $g_i \in G$. So $\pi(g)(i) \equiv \pi(g)(g_i) = gg_i = g_j \equiv j$, where, as above, we let $gg_i = g_j$. Hence, $V_{\pi(g)(i)} = V_j$, and the diagram commutes as required. ❑

Chapter 3

Properties of Representations

Given a representation (V,ρ) of G over F, some natural questions can arise: Are there nontrivial subspaces of V that can serve, under the action of ρ, as representation spaces of G? How are representations on such subspaces of V related to each other and to (V,ρ)? How is the representation of a subgroup of G related to the representation of G? Much of the material in Chapters 3 and 4 concerns these questions.

3.1 Subrepresentations

3.1 Definition: Let (V,ρ) be a representation of G over F. A vector subspace W of V is invariant under the action of G provided $\rho(g)(w) \in W$ for all $g \in G$ and $w \in W$.

Equivalently, W is said to be stable under G or under ρ.

3.2 Definition: For W invariant under ρ, let ρ^W be the restriction of ρ to W, i.e., $\rho^W = \rho|_W : G \to GL(W)$.

In light of these definitions we have:

3.3 Theorem: If W is invariant under ρ, then $\rho^W: G \to GL(W)$ is a representation of G over F.

Proof: It is clear that $\rho^W(g)$ is a linear transformation on W for all $g \in G$. All that needs to be shown, then, is that ρ^W is a group homomorphism.

Let $g, g' \in G$ and $w \in W$.

Then: $\rho^W(gg')(w) = \rho(gg')(w) = \rho(g)(\rho(g')(w))$, since $w \in V$ and (V,ρ) is a representation of G. But $\rho(g)(\rho(g')(w)) = (\rho^W(g) \circ \rho^W(g'))(w)$, since W is invariant under ρ and $w \in W$. Therefore ρ^W is a representation of G over F. ❑

It will be common in the remainder of this paper to denote a representation by the representation space. For instance, in the discussion above, we have that V and W are representations of G over F. In fact, we say that W is a subrepresentation of V, or W is a G-subspace of V.

From linear algebra, if W is a subspace of V, then the quotient space V/W is also a vector space. Given a representation (V,ρ) of G, we can define $(V/W,\bar{\rho})$ the quotient representation of G induced by ρ.

3.4 Theorem: Let (V,ρ) be a representation of G over F, and let W be a subrepresentation of V. For all $g \in G$, define $\bar{\rho}(g): V/W \to V/W$ by $\bar{\rho}(g)(v + W) = \rho(g)(v) + W$ for each $v + W \in V/W$. Then:

(1) $\bar{\rho}(g)$ is a linear transformation of V/W;

(2) $\bar{\rho}(g)$ is one-to-one (hence, $\bar{\rho}(g)$ is invertible and onto, i.e., $\bar{\rho}(g) \in GL(V/W)$); and

(3) $(V/W, \bar{\rho}))$ is a representation of G over F.

<u>Proof</u>:

(1) That $\bar{\rho}(g) : V/W \to V/W$ is a linear transformation is a direct consequence of the vector space properties of V/W, and the fact that $\rho(g) : V \to V$ is a linear transformation.

Let $\alpha, \beta \in F$ and $v_1, v_2 \in V$, $g \in G$. Then:
$\bar{\rho}(g)(\alpha(v_1 + W) + \beta(v_2 + W)) = \bar{\rho}(g)((\alpha v_1 + \beta v_2) + W) =$
$(\alpha\rho(g)(v_1) + \beta\rho(g)(v_2)) + W = (\alpha\rho(g)(v_1) + W) + (\beta\rho(g)(v_2) + W) =$
$\alpha\bar{\rho}(g)(v_1 + W) + \beta\bar{\rho}(g)(v_2 + W)$.
So $\bar{\rho}(g)$ is a linear transformation on V/W.

(2) Let $v_1, v_2 \in V$, $g \in G$, and suppose that
$\bar{\rho}(g)(v_1 + W) = \bar{\rho}(g)(v_2 + W)$. Then
$\rho(g)(v_1) + W = \rho(g)(v_2) + W$; which means
$\rho(g)(v_1) - \rho(g)(v_2) \in W$. So $\rho(g)(v_1-v_2) \in W$.
But this means that $\rho(g^{-1})(\rho(g)(v_1-v_2)) \in W$ since W is a subrepresentation of G. Now, $\rho(g^{-1})(\rho(g)(v_1-v_2)) = v_1 - v_2$, so we have $v_1 - v_2 \in W$, which implies that
$v_1 + W = v_2 + W$. Hence $\bar{\rho}(g)$ is one-to-one and $\bar{\rho}(g) \in GL(V/W)$.

(3) Let $g, g' \in G$ and $v + W \in V/W$.
Then: $\bar{\rho}(gg')(v + W) = \rho(gg')(v) + W =$
$(\rho(g) \circ \rho(g'))(v) + W = \rho(g)(\rho(g')(v)) + W =$
$\bar{\rho}(g)(\rho(g')(v) + W) = \bar{\rho}(g)(\bar{\rho}(g')(v + W)) =$

$(\bar{\rho}(g) \circ \bar{\rho}(g'))(v + W)$. Therefore $(V/W, \bar{\rho})$ is a representation of G over F. ❑

If G is group with $|G| \geq 2$ ($|G| \equiv$ order of G), the regular representation (FG, ρ) always has a nontrivial subrepresentation (W, ρ^W). Let W be the one-dimensional subspace of FG with basis element $x = \sum_{g \in G} g$.

Then $W = \{ax \mid a \in F.\}$. To see that W is, indeed, invariant under ρ, let $g \in G$, $w \in W$ and compute: $\rho^W(g)(w) = \rho(g)(w) = \rho(g)(ax)$, for some $a \in F$. But, under the regular representation of G, $\rho(g)(ax) = a(gx)$. However, $gx = x$, so $a(gx) = ax = w \in W$. The subrepresentation W is called the unit representation. It will be of importance later when we, in some sense, characterize all of the subrepresentations of the regular representation.

3.2 Direct Sums

Suppose V is a vector space with subspaces W and W'. Recall from linear algebra that V is the direct sum of W and W' provided each $v \in V$ can be uniquely written as $v = w + w'$, where $w \in W$ and $w' \in W'$. Equivalently, $W \cap W' = \{0\}$ and $\dim V = \dim W + \dim W'$. We write $V = W \oplus W'$ and call W' a complement of W in V. Also, the map $p : V \to W$ given by $p(v) = p(w + w') = w$ is called the projection of V onto W.

In terms of representations, we may ask the following: If V is a representation of G and W is a subrepresentation of V, does there exist a subrepresentation W′ of V such that $V = W \oplus W'$? Since W′ is stable under G, it is called an <u>invariant complement</u> of W in V.

We shall answer this question in a theorem whose proof will require:

<u>3.5 Lemma:</u> There is a one-to-one correspondence between the projections of V onto W and the complements of W in V.

<u>Proof:</u> Given $V = W \oplus W'$, W′ is a complement of W in V. Define $p : V \to W$ as above, i.e., for $v \in V$, $p(v) = p(w + w') = w$ where $w \in W$, $w' \in W'$. The image of p is W and $p(w) = w$ for all $w \in W$. So for each complement W′ of W, we have a projection p of V onto W.

Conversely, suppose $p : V \to W$ is a projection. Then $\ker p = \{v \in V | p(v) = 0\}$. We claim that $V = W \oplus \ker p$, i.e., that ker p is the desired complement of W in V.

It is obvious that $W \oplus \ker p \subset V$. So let $v \in V$ and choose a basis $\{v_1, \ldots, v_n\}$ for V such that $\{v_1, \ldots, v_r\}$ $(r \leq n)$ is a basis for W. Then: $v = a_1v_1 + \ldots + a_rv_r + a_{r+1}v_{r+1} + \ldots + a_nv_n$; $a_i \in F$, $i = 1, \ldots, n$.

Let $a_1v_1 + \ldots + a_rv_r = w \in W$ and $a_{r+1}v_{r+1} + \ldots + a_nv_n = x \notin W$. Now we have $v = w + x$ with $w \in W$ and $x \in \ker p$ since $p(x) = 0$. So $v \in W \oplus \ker p$ and $V \subset W \oplus \ker p$. Hence,
$V = W \oplus \ker p$ and ker p is the complement of W corresponding to the projection p. ❑

3.6 Theorem: Let (V,ρ) be a representation of G and let W be a G - subspace of V. Then there exists an invariant complement W′ of W in V.

Proof: From linear algebra, a subspace W of V has a complement in V. Let W^{o} be any complement of W in V, and by the lemma let p be the corresponding projection of V onto W. Define a map p′ to be the average of the conjugates of p by the elements of G:

$$p' = \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho^{-1}(g).$$

We will show:

① p′ is a projection of V onto W; and

② The complement of W corresponding to p′ is the desired invariant complement W′.

Proof of ①:

Since p maps V into W and $\rho(g)(w) \in W$ for every $g \in G$, $w \in W$, then p′ maps V into W.

Now let $w \in W$. Then $\rho^{-1}(g)(w) \in W$ since W is a G - subspace. So $p \circ \rho^{-1}(g)(w) = \rho^{-1}(g)(w)$. Thus we have, for each $g \in G$,

$$[\rho(g) \circ p \circ \rho^{-1}(g)](w) = [\rho(g) \circ \rho^{-1}(g)](w) = w.$$

Hence, $p'(w) = w$ and p′ is onto. Therefore p′ is a projection of V onto W.

Proof of ②:

Again using lemma 3.5, let W' be the complement of W corresponding to p'. From the proof of the lemma, $W' = \ker p'$. To show that W' is stable under G, we need one more intermediate result, namely, that p' commutes with $\rho(g)$ for every $g \in G$. Let $g' \in G$. Then:

$$\rho(g') \circ p' \circ \rho^{-1}(g') =$$

$$\frac{1}{|G|} \sum_{g \in G} \rho(g') \circ (\rho(g) \circ p \circ \rho^{-1}(g)) \circ \rho^{-1}(g') =$$

$\frac{1}{|G|} \sum_{g \in G} \rho(g'g) \circ p \circ \rho^{-1}(g'g)$. But the last sum can only permute the terms of $\frac{1}{|G|} \sum_{g \in G} \rho(g) \circ p \circ \rho^{-1}(g) = p'$. Hence, $\rho(g') \circ p' = p' \circ \rho(g')$. Now for $w' \in W'$, $g' \in G$, $p'(w') = 0$ since p' is a projection onto W. So $p' \circ \rho(g')(w') = \rho(g') \circ p'(w') = 0$. But this means that $\rho(g')(w') \in W'$; hence W' is a G - subspace of V, and the theorem is proved.❑

A couple of comments concerning theorem 3.6 and direct sums in general:

1. If char $F \mid |G|$, then, for all $a \in F$, $v \in V$, $p'(av) = 0$. We shall henceforth assume char $F \nmid |G|$.

2. Suppose that (V, ρ) is a representation of G and V can be decomposed into a direct sum of any finite number of G - subspaces. That is, $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, where the W_i, $i = 1, \ldots, k$, are invariant under ρ. For all $g \in G$,

$\rho_i(g) \in GL(W_i)$, where $\rho_i(g)$ is the restriction of $\rho(g)$ to W_i. It is easy to check that $\rho_i : G \rightarrow GL(W_i)$ is a representation of G. Then we say that ρ is the direct sum of the ρ_i, $i=1,\ldots,k$, and write $\rho = \rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_k$.

3.3 Matrices of Subrepresentations

Sections 3.1 and 3.2 examined subrepresentations, and the quotient representation, and direct sums in terms of representation by automorphism. It is both interesting and useful for later developments to see how matrix representations are affected.

Let (V,ρ) be a representation of G and let W be a nontrivial G-subspace of V such that $\dim V = n$ and $\dim W = r$ with $0 < r < n$. Then a basis $B = \{v_1, v_2, \ldots, v_n\}$ for V can be found such that $B' = \{v_1, \ldots, v_r\}$ is a basis for W. So for any $j = 1,2,\ldots,r$, and any $g \in G$, $\rho(g)(v_j) = \rho^W(g)(v_j) = \sum_{i=1}^{r} a_{ij}v_i$. The a_{ij} are elements of F that will constitute the j^{th} column of $[\rho^W(g)]_{B'}$. At this point we have, for the matrix of $\rho(g)$ relative to the basis B:

$$[\rho(g)]_B = \begin{array}{c} \\ r \\ \\ n-r \end{array} \overset{\quad r \qquad\qquad n-r}{\begin{bmatrix} [\rho^W(g)]_{B'} & ? \\ & \\ 0 & ? \end{bmatrix}}$$

What happens to $\rho(g)(v_j)$ for $r + 1 \le j \le n$? We are no longer restricted to W, so we can write $\rho(g)(v_j) = \sum_{i=1}^{n} a_{ij}v_i$; $r + 1 \le j \le n$. Hence $\rho(g)(v_j) = a_{1j}v_1 + \ldots + a_{rj}v_r + a_{r+1j}v_{r+1} + \ldots + a_{nj}v_n$.

Now we must note that the set $B'' = \{v_{r+1} + W, v_{r+2} + W, \ldots, v_n + W\}$ is a basis for the quotient space V/W. Hence, for $r + 1 \le j \le n$, $\bar{\rho}(g)(v_j + W) = \rho(g)(v_j) + W$. So the j^{th} column of $[\bar{\rho}(g)]_{B''}$ will consist of $a_{r+1j}, a_{r+2j}, \ldots, a_{nj}$. We can now write:

$$[\rho(g)]_B = \begin{bmatrix} [\rho^W(g)]_{B'} & ? \\ 0 & [\bar{\rho}(g)]_{B''} \end{bmatrix}.$$

Suppose now that $V = W \oplus W'$ where W and W' are subrepresentations of G. Then a basis $B = \{v_1, \ldots, v_n\}$ for V can be chosen so that $B' = \{v_1, \ldots, v_r\}$ and $B'' = \{v_{r+1}, \ldots, v_n\}$ are bases for W and W' respectively.

This leaves $[\rho^W(g)]_{B'}$ as above. But now, since W' is invariant under ρ, for $r + 1 \le j \le n$, $\rho(g)(v_j) = \rho^{W'}(g)(v_j) = \sum_{i=r+1}^{n} a_{ij}v_i$. So the matrix of $\rho(g)$ relative to the basis B

becomes:

$$[\rho(g)]_B = \begin{array}{c} r \\ \\ n-r \end{array} \left[\begin{array}{cc} \overset{r}{[\rho^W(g)]_{B'}} & \overset{n-r}{0} \\ & \\ 0 & [\rho^{W'}(g)]_{B''} \end{array} \right]$$

We can extend to $V = W_1 \oplus \ldots \oplus W_k$, where each W_i, $i=1,\ldots,k$ is a G-subspace. A basis B can be found for V so that $[\rho(g)]_B$ consists of a submatrix corresponding to the restriction of $\rho(g)$ to each W_i along the diagonal, and zeros elsewhere.

Chapter 4

Further Properties

4.1 Irreducible Representations

We have seen that if a representation V has a nontrivial G-subspace W, then there exists a complement W' of W in V which is also invariant under G. Hence, V can be written as the direct sum of two nontrivial representations: $V = W \oplus W'$. V is then said to be a reducible representation. On the other hand:

4.1 Definition: Let (V, ρ) be a representation of G. V (or ρ) is said to be irreducible provided $V \neq \{0\}$ and the only G-subspaces of V are $\{0\}$ and V. Otherwise, V is reducible.

A few observations are immediate:

① If V is irreducible, then it cannot be written as the direct sum of two nontrivial G-subspaces;

② Any representation of degree 1 is irreducible; and

③ If $V=\{0\}$, then it is reducible. We say that V is the direct sum of the empty family of irreducible representations.

If V is a representation, we should like to know if it can be "broken down" into irreducible representations.

4.2 Theorem: (Maschke) Every representation is a direct sum of irreducible representations.

Proof: Suppose V is a representation of G. The proof is by induction on $\dim V$. But let us look at the first few cases before trying the induction:

(i) If $\dim V = 0$, then $V = \{0\}$ and the theorem holds by observation ③ above;

(ii) If $\dim V = 1$ then V is irreducible. So the only sum we may write is $V = V \oplus \{0\}$. But by part (i), the theorem holds for $\{0\}$; hence, it is true for $V \oplus \{0\}$. In fact, the theorem is valid for irreducible representations of any dimension.

(iii) Let V be reducible and $\dim V = 2$. Then V can be written as the direct sum $V = W_1 + W_2$, where W_1 and W_2 are nontrivial G-subspaces. Hence, $\dim W_1 = \dim W_2 = 1$ and the theorem holds.

Now suppose V is reducible with $\dim V = n$ and $n > 2$. Our induction hypothesis is that the theorem holds for all reducible representations of dimension less than n. V, being reducible, can be written as $V = V' \oplus V''$ where $\dim V' < n$ and $\dim V'' < n$. Hence, V' and V'' can be written as direct sums of irreducible representations; and therefore, so can V.❑

Since theorem 4.2 depends on theorem 3.6 we are operating under the assumption that char $F \nmid |G|$.

Recall from theorem 1.4 that if (V,ρ) is a representation of G by automorphism and B is a basis for V, then $\theta(g) = [\rho(g)]_B$, for all $g \in G$, is a matrix representation of G. In particular, if (V,ρ) is irreducible, then we will say that θ is an irreducible matrix representation of G. We then get an interesting corollary to theorem 4.2:

4.3 Corollary: Let $\theta : G \to GL(n,F)$ be a matrix representation of G. Then there exists a fixed matrix $P \in GL(n,F)$ such that for all $g \in G$,

$$P\theta(g)\ P^{-1} = \begin{bmatrix} \theta_1(g) & \cdot & \cdot & \cdot & 0 \\ \vdots & \theta_2(g) & \cdot & \cdot & \cdot \\ \cdot & & & & \\ 0 & \cdot & \cdot & \cdot & \theta_r(g) \end{bmatrix}$$

where the $\theta_i(g)$ are irreducible matrix representations of G.

Proof: As in section 1.2, let $V = F^n$ and define $\rho : G \to GL(F^n)$ by $\rho(g) : F^n \to F^n$ where $\rho(g)(v) = \theta(g)\, v$ for all $v \in V$. Then $(V,\rho) = (F^n,\rho)$ is a representation of G by automorphism.

By theorem 4.2, $V = V_1 \oplus V_2 \oplus \ldots \oplus V_r$, where the V_i, $i = 1,\ldots,r$, are irreducible. Denote the restriction of ρ to V_i by ρ^{V_i}.

Now, from section 3.3, a basis B can be chosen for V such that for $g \in G$,

$$[\rho(g)]_B = \begin{bmatrix} [\rho^{V_1}(g)]_{B_1} & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & [\rho^{V_r}(g)]_{B_r} \end{bmatrix}$$; where $B_i \subset B$ is a basis for V_i with $\bigcup_{i=1}^{r} B_i = B$ and $B_i \cap B_j = \emptyset$ for $i \neq j$.

Note that each $[\rho^{V_i}(g)]_{B_i}$ is an irreducible matrix representation of G. Let $[\rho^{V_i}(g)]_{B_i} = \theta_i(g)$ for $i=1,\ldots,r$. Finally, from section 1.2 again, if E is the standard basis for F^n, then $[\rho(g)]_E = \theta(g)$ for all $g \in G$. And by observation 1 following theorem 1.9, $[\rho(g)]_E = \theta(g) \sim [\rho(g)]_B$. Hence, by the definition of equivalence (1.8), there exists a matrix $P \in GL(n,F)$ such that $P\theta(g)P^{-1} = [\rho(g)]_B$. ❑

4.2 Clifford's Theorem

Clifford's theorem is a result about the reducibility of the representation of H, where H is a subgroup of the group G. However, this development does not rely on theorem 3.6; in fact, we need make no assumptions about the relationship of char F to |H| or |G|.

Some preliminary definitions and lemmas are needed.

<u>4.4 Definition:</u> Let (V,ρ) be a representation of G over F and let H be any subgroup of G. The representation of H by restricting ρ to H is denoted by $\mathrm{Res}_H^G\rho: H \to GL(V_H)$.

The representation space V_H is the same as V as a vector space, but we define only the action of H on V_H. V_H, then, is an H - module.

4.5 Lemma: Let H be a normal subgroup of G ($H \Delta G$), and let $\sigma: H \to GL(W)$ be a representation of H. For any $g \in G$, the map σ^g on H given by $\sigma^g(h) = \sigma(ghg^{-1})$ is a representation of H.

Proof: First note that since $H \Delta G$, $ghg^{-1} \in H$ for any $g \in G$, $h \in H$. Thus, $\sigma^g(h) \in GL(W)$.

Now let $h, h' \in H$. Then we simply check:

$\sigma^g(hh') = \sigma(ghh'g^{-1}) = \sigma(gh[g^{-1}g]h'g^{-1}) = \sigma([ghg^{-1}][gh'g^{-1}]) = \sigma(ghg^{-1}) \circ \sigma(gh'g^{-1}) = \sigma^g(h) \circ \sigma^g(h')$. So σ^g is a group homomorphism from H to GL(W) and, hence, a representation of H. ❑

4.6 Definition: The representation of σ^g (or any equivalent to it) is called a conjugate of σ.

4.7 Lemma: Let $\sigma: H \to GL(W)$ be a representation of $H(\Delta G)$. Then:

1. If U is an invariant subspace of W under σ^g for $g \in G$, then U is invariant under σ; and
2. If σ is irreducible, then σ^g is irreducible for any $g \in G$.

Proof: 1. Let U be an invariant subspace of W under σ^g for $g \in G$. Then for all $u \in U$, $h \in H$, $\sigma^g(h)(u) \in U$. Now, for any $u \in U$, $h \in H$, $g \in G$, $\sigma(h)(u) = \sigma(gh'g^{-1})(u)$, where $h' \in H$ such that $gh'g^{-1} = h$. We know that such an h' exists since $H \Delta G \Rightarrow gHg^{-1} = H$.

But $\sigma(gh'g^{-1})(u) = \sigma^g(h')(u) \in U$ since U is invariant under σ^g. Therefore $\sigma(h)(u) \in U$ and U is invariant under σ.

2. Suppose σ is irreducible. The σ^g must also be irreducible. For, from ①, any invariant subspace of W under σ^g must be invariant under σ. But the only invariant subspaces under σ are $\{0\}$ and W. ❑

<u>4.8 Theorem:</u> (Clifford)

Let $H \Delta G$, let (V,ρ) be an irreducible representation of G, and let W be an irreducible H-submodule of V_H via $\mathrm{Res}_H^G\rho \equiv \sigma : H \to GL(W)$. Then V_H is a direct sum of irreducible H-submodules, each of which is conjugate to (W,σ).

<u>Proof</u>: First we show that for each $g \in G$, $\rho(g)W$ (the image of W under $\rho(g)$) is an H-submodule of V_H. To see that $\rho(g)\ W$ is invariant, let $h \in H$, and $y \in W$, Then:

$(\mathrm{Res}_H^G\rho)(h)(\rho(g)(y)) = \rho(h)(\rho(g)(y)) = \rho(hg)(y) = \rho(gg^{-1}hg)(y)$

$= \rho(g)(\rho(g^{-1}hg)(y)) = \rho(g)(\rho(h')(y))$; where $h' = g^{-1}hg$ and $h' \in H$ since $H \Delta G$. But $\rho(g)(\rho(h')(y)) = \rho(g)\big((\mathrm{Res}_H^G\rho)(h')(y)\big) \in \rho(g)W$; since $y \in W$, an invariant subspace of V_H under $\mathrm{Res}_H^G\rho$. Let $\sigma_g' : H \to GL(\rho(g)W)$ be the representation of H with representation space $\rho(g)W$. The relationship $(\mathrm{Res}_H^G\rho)(h)(\rho(g)(y)) = \rho(g)((\mathrm{Res}_H^G\rho)(h')(y))$, where h' is conjugate to h, will be needed again, call it <u>equation 4.9</u>.

Next we show that $\sigma_g' \approx \sigma^g$. For $g \in G$, define a map $\alpha : W \rightarrow \rho(g)W$ by $\alpha(y) = \rho(g)(y)$ for all $y \in W$. Since $\rho(g)W$ is just the image of W under $\rho(g)$, α is clearly a vector space isomorphism. To establish the desired equivalence, we must show that, for all $h \in H$, $\alpha \circ \sigma^g(h) = \sigma_g'(h) \circ \alpha$.

For any $y \in W$, $(\alpha \circ \sigma^g(h))(y) = \alpha(\sigma(ghg^{-1})(y)) = \rho(g)(\sigma(ghg^{-1})(y)) = \rho(g)\ [(\mathrm{Res}_H^G\rho)(ghg^{-1})(y)] =$

$(\mathrm{Res}_H^G\rho)(h)(\rho(g)(y))$ by equation 4.9. Now, since $\rho(g)(y) \in \rho(g)W$, $(\mathrm{Res}_H^G\rho)(h)(\rho(g)(y)) = (\sigma g'(h) \circ \alpha)(y)$. So

we have $(\rho(g)W, \sigma_g') \approx (W, \sigma^g)$.

Now, by hypothesis, (W,σ) is irreducible, and since $(\rho(g)W, \sigma_g') \approx (W,\sigma^g)$, $(\rho(g)W, \sigma_g')$ is conjugate to (W, σ). Hence, applying lemma 4.7, $(\rho(g)W, \sigma_g')$ is irreducible.

All that remains is to show that $V_H = V = \bigoplus_{g \in G} \rho(g)W$. Since (V,ρ) is an irreducible representation of G, it suffices to show that $\bigoplus_{g \in G} \rho(g)W$ is invariant under the action of G.

Let $|G| = n$ and fix some ordering of the elements of $G : g_1, g_2, \ldots, g_n$. Let $w_i \in \rho(g_i)W$. Then for any $g_j \in G$, $w \in \bigoplus_{g_i \in G} \rho(g_i)\ W$:

$\rho(g_j)(w) = \rho(g_j)(w_1 + w_2 + \ldots + w_n) = \rho(g_j)(w_1) + \rho(g_j)(w_2) + \ldots + \rho(g_j)(w_n) = \rho(g_j)(\rho(g_1)(w^{(1)})) + \rho(g_j)(\rho(g_2)(w^{(2)})) + \ldots + \rho(g_j)(\rho(g_n)(w^{(n)})) =$

$\rho(g_jg_1)(w^{(1)}) + \rho(g_jg_2)(w^{(2)}) + \ldots + \rho(g_jg_n)(w^{(n)}) \in \bigoplus_{g_i \in G} \rho(g_i)W$; where $w^{(i)} \in W$ such that $\rho(g_i)(w^{(i)}) = w_i$.

So $\bigoplus_{g \in G} \rho(g)W = V_H$, where each $(\rho(g)W, \sigma_g')$ is irreducible and conjugate to (W,σ). ❑

Note from the hypotheses of Clifford's theorem that we were given one irreducible H-submodule. In that case, we can now know all of the irreducible H-submodules of V_H.

4.3 Schur's Lemma

This section is more a foreshadowing of Chapter 5 than a continuation of material in Chapter 4. Still, getting some of the very technical computations done here will facilitate a smoother flow of results about characters in Chapter 5.

We first record some definitions and properties (without proof) of the trace of a matrix or linear transformation.

4.10 Definition: If $A = (a_{ij})$ is any nxn matrix over a field F, the we denote the trace of A by tr(A) and $tr(A) = \sum_{i=1}^{n} a_{ii}$.

4.11 Theorem: For nxn matrices A, B over F and $\lambda \in F$:

1. $tr(\lambda A) = \lambda\, tr(A)$;
2. $tr(A + B) = tr(A) + tr(B)$; and

3. tr(AB) = tr(BA).

A direct consequence of 3 above is

4.12 Corollary: If $A \in GL(n, F)$ (i.e. A is invertible), then $tr(ACA^{-1}) = tr(C)$ for an nxn matrix C over F.

4.13 Definition: If V is a finite dimensional vector space with basis B and T is any linear transformation on V, then we define the trace of T by $tr(T) = tr[T]_B$.

From definition 4.13 and corollary 4.12, we have that tr(T) does not depend on the choice of basis for V.

For the proof of Schur's lemma, it is also necessary to recall that $\lambda \in F$ is an eigenvalue of T if there exists a $v \in V$ with $v \neq 0$ and $T(v) = \lambda v$.

We shall need the following properties of eigenvalues:

4.14 Theorem: If T is a linear transformation on V over F, then tr(T) is equal to the sum of the eigenvalues of T counted with their multiplicities.

4.15 Theorem: If λ is an eigenvalue of an invertible linear transformation T, then λ^{-1} is an eigenvalue of T^{-1}.

4.16 Lemma: (Schur) Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be irreducible representations of G over $F = \mathscr{C}$. Let $f : V_1 \to V_2$ be a linear transformation such that $\rho^2(s) \circ f = f \circ \rho^1(s)$ for all $s \in G$. Then:

① If ρ^1 and ρ^2 are not equivalent, then $f = 0$; and

② If $V_1 = V_2$ and $\rho^1 = \rho^2$, then f is a scalar multiple of the identity map on V_1.

Proof: We prove the contrapositive of ①. Suppose $f \neq 0$. Let $W_1 = \ker f = \{x \in V_1 \mid f(x) = 0\}$. If $x \in W_1$, then $f(\rho^1(s)(x)) = \rho^2(s)(f(x)) = \rho^2(s)(0) = 0$. So for $x \in W_1$, $\rho^1(s)(x) \in W_1$ and W_1 is invariant under ρ^1.

Since V_1 is irreducible, either $W_1 = \{0\}$ or $W_1 = V_1$. But if $W_1 = V_1$, then $f = 0$, contrary to our assumption. So $W_1 = \{0\}$, which means that f is one-to-one and an isomorphism of V_1 into V_2.

Now let $W_2 = \operatorname{Im} f = \{f(x) \mid x \in V_1\}$. For $y \in W_2$, there is an $x \in V_1$ such that $f(x) = y$. So, for $s \in G$, $(\rho^2(s) \circ f)(x) = \rho^2(s)(y) = (f \circ \rho^1(s))(x) = f(\rho^1(s)(x)) \in W_2$. Hence, W_2 is invariant under ρ^2, so $W_2 = V_2$ or $W_2 = \{0\}$. But if $W_2 = \{0\}$, then $f = 0$. It must be then, that $W_2 = V_2$. Now f is an isomorphism of V_1 onto V_2, whence $(V_1,\rho^1) \approx (V_2,\rho^2)$.

For ②, suppose that $V_1 = V_2$ and $\rho^1 = \rho^2$. Let λ be an eigenvalue of f. Define $f' : V_1 \to V_2 (= V_1)$ by $f' = f - \lambda$ $(= f - \lambda \mathrm{Id}_V)$. Now since λ is an eigenvalue of f, there is a $v \in V_1$ with $v \neq 0$ and $f(v) = \lambda v$. Then $f'(v) = f(v) - \lambda v = \lambda v - \lambda v = 0$. So $v \in \ker f'$ and $\ker f' \neq \{0\}$.

But we also have, for any $s \in G$ and any $v' \in V_1 = V_2$, $(\rho^2(s) \circ f')(v') = \rho^2(s)(f'(v')) = \rho^2(s)((f - \lambda)(v')) = \rho^2(s)(f(v') - \lambda v') = (*) \rho^2(s)(f(v')) - \rho^2(s)(\lambda v')$. On the other hand, $(f' \circ \rho^1(s))(v') = (f - \lambda)(\rho^1(s)(v')) = f(\rho^1(s)(v')) - \lambda\rho^1(s)(v') = (**) f(\rho^1(s)(v')) - \rho^1(s)(\lambda v')$.

Comparing (*) and (**), $\rho^2(s)(f(v')) = f(\rho^1(s)(v'))$ by hypothesis. And, since $\rho^1 = \rho^2$, we have $\rho^2(s)(\lambda v') = \rho^1(s)(\lambda v')$. Therefore $\rho^2(s) \circ f' = f' \circ \rho^1(s)$.

Now we may argue as in the proof of ① that the ker f' is invariant under ρ^1, and since ker f ≠ {0}, then ker f' = V_1. Hence, f' = 0 and so, by definition of f', f = λ, a scalar multiple of the identity. ❑

We shall prove three corollaries to Schur's lemma. Two of them involve quite technical matrix computations, but will be most useful in the discussion of characters.

4.17 Corollary: Let (V_1, ρ^1) and (V_2, ρ^2) be irreducible representations of G over $F = \mathscr{C}$. Let $h \neq 0$ be a linear transformation of V_1 into V_2 and define:

$$h' = \frac{1}{|G|} \sum_{t \in G} \rho^2(t)^{-1} \circ h \circ \rho^1(t).$$

Then:

① If ρ^1 and ρ^2 are not equivalent, then $h' = 0$; and

② If $V_1 = V_2$ and $\rho^1 = \rho^2$, then $h' = \frac{1}{n}\operatorname{tr}(h)$, where $n = \dim V_1$.

Proof:

① Clearly, h' is a linear transformation from V_1 into V_2. We need that $\rho^2(s) \circ h' = h' \circ \rho^1(s)$ for all $s \in G$. Note that $\rho^2(s)^{-1} \circ h' \circ \rho^1(s) =$

$$\rho^2(s)^{-1} \circ \left(\frac{1}{|G|} \sum_{t \in G} \rho^2(t)^{-1} \circ h \circ \rho^1(t)\right) \circ \rho^1(s) =$$

$$\frac{1}{|G|} \sum_{t \in G} \rho^2(s)^{-1} \circ \rho^2(t)^{-1} \circ h \circ \rho^1(t) \circ \rho^1(s) =$$

$$\frac{1}{|G|} \sum_{t \in G} \rho^2(ts)^{-1} \circ h \circ \rho^1(ts).$$

Now, as t runs over the elements of G, so does the product ts. So $\frac{1}{|G|} \sum_{t \in G} \rho^2(ts)^{-1} \circ h \circ \rho^1(ts) = h'$. Hence, $h' \circ \rho^1(s) = \rho^2(s) \circ h'$. So we can apply part ① of Schur's lemma with $f = h'$. Therefore $h' = 0$.

For ②, let $h' = \lambda$, a scalar multiple of the identity. Now we apply some properties of the trace of a linear transformation: Since $\rho^1 = \rho^2$,

$$\operatorname{tr} h' = \operatorname{tr}\left(\frac{1}{|G|} \sum_{t \in G} \rho^1(t)^{-1} \circ h \circ \rho^1(t)\right) =$$

$$\frac{1}{|G|} \sum_{t \in G} \operatorname{tr}(\rho^1(t)^{-1} \circ h \circ \rho^1(t)) = \frac{1}{|G|} \sum_{t \in G} \operatorname{tr}(h) = \operatorname{tr}(h).$$

On the other hand, since λ is a scalar multiple of the identity, $\operatorname{tr}(h') = \operatorname{tr}(\lambda) = n\lambda$, where $n = \dim V_1$. Comparing the two expressions for $\operatorname{tr}(h')$, we have $\lambda = \frac{1}{n} \operatorname{tr}(h)$. ❑

Now let B_1 and B_2 be bases of V_1 and V_2, respectively. The particular bases chosen have no bearing on the results that follow. We choose them simply because the matrix representations corresponding to ρ^1 and ρ^2 must be computed relative to some basis. For $t \in G$, let $[\rho^1(t)]_{B_1} = (a_{ij}(t))$ and $[\rho^2(t)]_{B_2} = (a_{kl}(t))$.

The matrices of the linear transformations h and h′ from V_1 into V_2 depend on both B_1 and B_2. So let $[h]_{B_1,B_2} = (x_{ki})$ and $[h']_{B_1,B_2} = (x'_{ki})$.

Then the k, i entry of (x'_{ki}) is, by definition of h′:

(4.18) $$x'_{ki} = \frac{1}{|G|} \sum_{t,j,l} a_{kl}(t^{-1}) x_{lj} a_{ji}(t).$$ The composition of maps is now expressed as matrix multiplication. The products $a_{kl}(t^{-1})a_{ji}(t)$ can be considered as coefficients in a linear form with respect to x_{lj}.

We can now state the second corollary to Schur's lemma.

4.19 Corollary: Let ρ^1, ρ^2, h, and h′ be as in corollary 4.17. Suppose ρ^1 and ρ^2 are not equivalent. Then $\frac{1}{|G|} \sum_{t \in G} a_{kl}(t^{-1})a_{ji}(t) = 0$ for all i, j, k, l.

Proof: Under these hypotheses, corollary 4.15 gives $h' = 0$. So $x'_{ki} = 0$ for all k and i. This means that each term on the right side of equation 4.18 is zero. Since $h \neq 0$, we must have the products $a_{kl}(t^{-1})a_{ji}(t) = 0$ for all j and l; whence $\frac{1}{|G|} \sum_{t \in G} a_{kl}(t^{-1})a_{ji}(t) = 0$ for all i, j, k, and l. ❑

And finally:

4.20 Corollary: Let ρ^1, ρ^2, h, and h′ be as in corollary 4.17. Suppose $V_1 = V_2$ and $\rho^1 = \rho^2$. Let $n = \dim V_1$. Then:

$$\frac{1}{|G|} \sum_{t,j,l} a_{kl}(t^{-1})a_{ji}(t) = \frac{1}{n}\delta_{ki}\delta_{lj} = \begin{cases} \frac{1}{n} & \text{if } k = i \text{ and } l = j \\ 0 & \text{otherwise} \end{cases}.$$

Proof: By corollary 4.17, $h' = \lambda$. So $x'_{ki} = \lambda\delta_{ki}$. But $\lambda = \frac{1}{n}\,tr(h) = \frac{1}{n}\sum_{j=1}^{n} x_{jj} = \frac{1}{n}\sum_{l,j}\delta_{ki}\delta_{lj}$. Then:. $x'_{ki} = \delta_{ki}\lambda = \delta_{ki}\left(\frac{1}{n}\sum_{l,j}^{n}\delta_{lj}\,x_{lj}\right) = (*)\ \frac{1}{n}\sum_{l,j}\delta_{ki}\delta_{lj}x_{lj}$; another linear form with respect to x_{lj}.

For any l,j, let us now equate the coefficients from (*) and equation 4.18

$$\frac{1}{|G|}\sum_{t\in G} a_{kl}(t^{-1})a_{ji}(t) = \frac{1}{n}\delta_{ki}\delta_{lj} = \begin{cases}\frac{1}{n} & \text{if } k = i \text{ and } l = j\\ 0 & \text{otherwise}\end{cases}. \square$$

It will soon be evident in Chapter 5 that these results will yield much information about representations.

Chapter 5

Character Theory

5.1 Definition and Basic Properties

In the hypotheses of Schur's lemma (4.16) we let $F = \mathscr{C}$, the field of complex numbers. We shall continue with that assumption. The following definition will result in much information about group representations and their irreducible constituents.

5.1 Definition: Let $\rho: G \to GL(V)$ be a representation of G over F. Then the character of (or afforded by) ρ is the mapping $\chi: G \to F$ given by $\chi(s) = tr(\rho(s))$, for all $s \in G$.

We can already begin to see the importance of characters in:

5.2 Theorem: Let (V_1, ρ^1) and (V_2, ρ^2) be equivalent representations of G with characters χ_1 and χ_2, respectively. Then $\chi_1(s) = \chi_2(s)$ for all $s \in G$, i.e., equivalent representations have the same characters.

Proof: Since $\rho^1 \approx \rho^2$, there exists a vector space isomorphism $\alpha : V_1 \to V_2$ such that $\alpha \circ \rho^1(s) = \rho^2(s) \circ \alpha$, for all $s \in G$. That is,

$$\rho^1(s) = \alpha^{-1} \circ \rho^2(s) \circ \alpha.$$

So, $tr(\rho^1(s)) = tr(\alpha^{-1} \circ \rho^2(s) \circ \alpha)$. But from the properties of trace (corollary 4.12 and definition 4.13), $tr(\alpha^{-1} \circ \rho^2(s) \circ \alpha) = tr(\rho^2(s))$. Hence $\chi_1(s) = \chi_2(s)$ for all

$s \in G$. ❑ (The converse of this theorem is true and its proof will be given later.)

Let us also get our first insight into the behavior of characters of subrepresentations:

5.3 Theorem: Let (V,ρ) be a representation of G with G-subspaces W_1 and W_2 such that $V = W_1 \oplus W_2$. Let χ, χ_1 and χ_2 be the characters of V, W_1, and W_2, respectively. Then $\chi = \chi_1 + \chi_2$.

Proof: As in section 3.3, appropriate bases B, B_1, B_2 for V, W_1, W_2(resp.), can be found so that:

$$[\rho(g)]_B = \begin{bmatrix} [\rho^{W_1}(g)]_{B_1} & 0 \\ 0 & [\rho^{W_2}(g)]_{B_2} \end{bmatrix} \quad \text{for all } g \in G.$$

Now simply note that the trace of the left-hand side is $\chi(g)$, while the trace of the right-hand side is $\chi_1(g) + \chi_2(g)$. ❑

Also as in section 3.3, this result may be extended to any finite number of G-subspaces of V.

We close this section with a lemma that will be called on from time to time:

5.4 Lemma: Let χ be the character of representation (V,ρ) of degree n. Then:

① $\chi(1_G) = n$ where 1_G is the identity of G;

② $\chi(s^{-1}) = \overline{\chi(s)}$ for all $s \in G$, where $\overline{\chi(s)}$ denotes the complex conjugate of $\chi(s)$; and

③ $\chi(tst^{-1}) = \chi(s)$ for all $s, t \in G$.

Proof:

① Since $\rho(1) = Id_V$, we have $\chi(1) = tr(Id_V) = n$.

② For all $s \in G$, $\rho(s)$ is an element of the group $GL(V)$. If the order of $s \in G$ is m, then the same is true of $\rho(s)$:

$[\rho(s)]^m = \rho(s^m) = Id_V$. If $\lambda_1, \ldots, \lambda_n \in \mathscr{C}$ are eigenvalues of $\rho(s)$ with eigenvectors $V_1, \ldots, V_n$, then

$(\rho(s))^m(v_i) = \lambda_i^m V_i = V_i$, for $i = 1, .., n$.

Hence, $\lambda_i^m = 1$, or λ_i is an m^{th} root of unity. But this means that $|\lambda_i| = 1$ and $\lambda^{-1}{}_i = \overline{\lambda_1}$. Then:

$$\overline{\chi(s)} = \overline{tr(\rho(s))} = \overline{\left(\sum_{i=1}^{n}\lambda_i\right)} = \sum_{i=1}^{n} \overline{\lambda_i} = \sum_{i=1}^{n}\lambda^{-1}_i = tr(\rho(s)^{-1})$$

$$= tr(\rho(s^{-1})) = \chi(s^{-1}).$$

We have used theorems 4.14 and 4.15 in the above string of equalities.

③ We have that $\chi(tst^{-1}) = tr(\rho(tst^{-1})) =$ $tr(\rho(t) \circ \rho(s) \circ \rho(t)^{-1}) = tr(\rho(s)) = \chi(s)$ from the properties of the trace function. ❑

5.2 The Space of Class Functions

Any function $f : G \to \mathscr{C}$ such that $f(tst^{-1}) = f(s)$, for $s, t \in G$, is called a class function. The name is due to the fact that f is constant on the elements of a given conjugacy class of G. The character χ of any representation of G is, therefore, a class function. We seek to impose a vector space structure on $CF_G = \{f \mid f \text{ is a class function on } G\}$, which contains the characters of G as a subset.

Let f, h $\in CF_G$ and s $\in$ G. Define f + h by $(f+h)(s) = f(s) + h(s)$. Then if t, s $\in$ G, $(f+h)(tst^{-1}) = f(tst^{-1}) + h(tst^{-1}) = f(s) + h(s) = (f + h)(s)$. Hence $f + h \in CF_G$. Also, for $\alpha \in F = \mathscr{C}$, $f \in CF_G$, $s \in G$, define αf by $(\alpha f)(s) = \alpha(f(s))$. Then, for t, s $\in$ G, $(\alpha f)(tst^{-1}) = \alpha(f(tst^{-1}) = \alpha(f(s)) = (\alpha f)(s)$. So $\alpha f \in CF_G$.

We will omit the routine verification of the vector space axioms for CF_G. However, there is much more to say about this space.

Define $(\ ,\ ) : CF_G \times CF_G \to \mathscr{C}$ by: $(f,h) = \frac{1}{|G|} \sum_{t \in G} f(t)\ \overline{h(t)}$ where f, h $\in CF_G$.

<u>5.5</u> <u>Theorem:</u> For any f,h,k $\in CF_G$, α, $\beta \in$ F, we have:

① $(f,h) = \overline{(h,f)}$;

② $(f,f) \geq 0$ with equality iff $f = 0$; and

③ $(\alpha f + \beta h, k) = \alpha(f,k) + \beta(h,k)$.

That is, $(\ ,\ )$ is an inner product on CF_G.

<u>Proof:</u>

Let f,h,k $\in CF_G$, α, $\beta \in$ F.

① $\overline{(h,f)} = \overline{\frac{1}{|G|} \sum_{t \in G} h(t)\ \overline{f(t)}} =$

$$\frac{1}{|G|}\sum_{t \in G}\overline{h(t)\overline{f(t)}} = \frac{1}{|G|}\sum_{t \in G}\overline{h(t)}\,f(t) = (f,h).$$

② $(f,f) = \frac{1}{|G|}\sum_{t \in G} f(t)\overline{f(t)}$. Since $f : G \to \mathscr{C}$, $f(t) = a + bi$ and $\overline{f(t)} = a - bi$ for $t \in G$ and some $a, b \in \mathscr{R}$. Hence, each term of the sum $\sum_{t \in G} f(t)\overline{f(t)}$ has the form $a^2 + b^2$; so $(f,f) \geq 0$. Further, the sum is zero iff each term is zero iff $f(t)\overline{f(t)} = 0$ for each $t \in G$ iff $f(t) = 0$ for all $t \in G$.

③ $(\alpha f + \beta h, k) = \frac{1}{|G|}\sum_{t \in G} (\alpha f + \beta h)(t)\overline{k(t)} =$

$$\frac{1}{|G|}\sum_{t \in G} (\alpha f(t) + \beta h(t))\overline{k(t)} =$$

$$\frac{1}{|G|}\sum_{t \in G} \left(\alpha f(t)\overline{k(t)} + \beta h(t)\overline{k(t)}\right) =$$

$$\alpha\frac{1}{|G|}\sum_{t \in G} f(t)\overline{k(t)} + \beta\frac{1}{|G|}\sum_{t \in G} h(t)\overline{k(t)} =$$

$\alpha(f,k) + \beta(h,k)$. ❑

Any inner product as defined above also has the following property:

<u>5.6 Corollary:</u> If $f,h,k \in CF_G$ and $\alpha,\beta \in F$, then $(f,\alpha h + \beta k) = \overline{\alpha}(f,h) + \overline{\beta}(f,k)$.

An important observation is necessary before making the connection between the inner product above and characters. The vector space CF_G is actually a subspace of

$A = \{$all functions from G into $\mathscr{C}\}$. For instance, from corollaries 4.19 and 4.20, $a_{kl}(t^{-1})$ and $a_{ji}(t)$ are elements of A.

Furthermore, theorem 5.5 never uses the fact that f, h, and k are class functions. The inner product as defined here holds for all elements of A.

Now let (V, ρ) be an irreducible representation of G of degree n with character χ. In this case, we call χ an <u>irreducible character</u>. Let $\rho(t)$ be given in matrix form by $(a_{ij}(t))$ for $t \in G$. Then $\chi(t) = \sum_{i=1}^{n} a_{ii}(t)$.

Consider the inner product $(\chi,\chi) = \frac{1}{|G|} \sum_{t \in G} \chi(t)\,\overline{\chi(t)}$.

By lemma 5.4 ③, $\overline{\chi(t)} = \chi(t^{-1})$, so $(\chi,\chi) = \frac{1}{|G|} \sum_{t \in G} \chi(t)\chi(t^{-1})$. Now using the definitions of χ and $(\ ,\ )$, along with theorem 5.5 and corollary 5.6:

$$(\chi,\chi) = \frac{1}{|G|} \sum_{t \in G} \left(\sum_{i=1}^{n} a_{ii}(t) \right) \left(\sum_{j=1}^{n} a_{jj}(t^{-1}) \right) =$$

$$\left(\sum_{i=1}^{n} a_{ii}(t),\ \overline{\sum_{j=1}^{n} a_{jj}(t^{-1})} \right) =$$

$$\left(\sum_{i=1}^{n} a_{ii}(t),\ \sum_{j=1}^{n} \overline{a_{jj}(t^{-1})} \right) =$$

$$\sum_{i,j}^{n} \left(a_{ii}(t),\ \overline{a_{jj}(t^{-1})} \right) = \sum_{i,j}^{n} \left(\frac{1}{|G|} \sum_{t \in G} a_{ii}(t) a_{jj}(t^{-1}) \right).$$

We may now apply corollary 4.20 since we have $\rho = \rho^1 = \rho^2$ and $V = V_1 = V_2$. Hence, $\frac{1}{|G|} \sum_{t \in G} a_{ii}(t)a_{jj}(t^{-1}) = \frac{1}{n}\delta_{ij}$.

Finally, then:

$$(\chi,\chi) = \sum_{i,j} \frac{1}{n}\delta_{ij} = \frac{n}{n} = 1$$

So we have proven:

<u>5.7 Theorem:</u> If χ is the character of an irreducible representation of G, then $(\chi,\chi) = 1$.

Since equivalent representations have the same characters (theorem 5.2), we immediately have:

<u>5.8 Corollary:</u> If χ and Ψ are irreducible characters of equivalent representations of G, then $(\chi,\Psi) = (\Psi,\chi) = 1$.

On the other hand suppose χ and Ψ are irreducible characters of two representation ρ^1 and ρ^2 of G that are not equivalent. For $t \in G$, let $\rho^1(t)$ and $\rho^2(t)$ be given in matrix form by $(a_{ij}(t))$ and $(a_{kl}(t))$, respectively. Now we may argue as in the proof of theorem 5.7 and arrive at:

$$(\chi,\Psi) = \sum_{i,l}^{n} \left(\frac{1}{|G|} \sum_{t \in G} a_{ii}(t)a_{ll}(t^{-1}) \right).$$

But corollary 4.19 applies and we have

$$\frac{1}{|G|} \sum_{t \in G} a_{ii}(t)a_{ll}(t^{-1}) = 0 \text{ for all } i,\ l.$$

Hence, $(\chi,\Psi) = 0$ and we have shown:

5.9 Theorem: If χ and Ψ are irreducible characters of two nonequivalent representations of G, then $(\chi,\Psi) = 0$.

Theorems 5.7 and 5.9 give us orthogonality relations for the characters in CF_G. From theorem 5.7, we have that the norm of an irreducible character χ, $||\chi||$, is $||\chi|| = \sqrt{(\chi,\chi)} = 1$. And in theorem 5.8, we say that χ and Ψ are orthogonal since $(\chi,\Psi) = 0$.

5.3 Applications of the Orthogonality Relations

Let V be a representation of G. We know from Maschke's theorem(4.2) that V can be written as $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$, where W_i, i = 1, ... k, is an irreducible representation of G. If we fix one of the irreducibles and call it W, we can then ask: how many of the W_i are equivalent to W? This is answered in:

5.10 Theorem: Let V, W, and the W_i , i = 1,...,k, be as above. Let V have character Φ and W have character χ. Then the number of W_i equivalent to W is (Φ,χ).

Proof: Let χ_i be the character of W_i, i = 1,..., k. By theorem 5.3, $\Phi = \chi_1 + \chi_2 + \ldots + \chi_k$. So by the linearity of the inner product (theorem 5.5 ③), $(\Phi,\chi) = (\chi_1, \chi) + (\chi_2, \chi) + \ldots + (\chi_k,\chi)$. But each term of the sum is 0 (if W_i is not equivalent to W) or 1 (if W_i is equivalent to W). Hence (Φ,χ) equals the number of W_i equivalent to W. ❑

Now since (Φ,χ) does not depend on the decomposition of V, we immediately have:

5.11 Corollary: The number of W_i equivalent to W is independent of the chosen decomposition of V.

We call (Φ,χ_i) the multiplicity of χ_i in Φ or, more intuitively, the number of times that W_i is contained in V. In fact, if we let $\chi_1,\ldots,\chi_h$ be the distinct irreducible characters corresponding to $W_1,\ldots,W_h$ (we are possibly changing some subscript from above), and let $m_i = (\Phi,\chi_i)$, $i = 1,\ldots,h$, then we can write: $V = m_1W_1 \oplus\ldots\oplus m_hW_h$.

Each m_iW_i represents the direct sum of the irreducible representations equivalent to W_i. Furthermore, we can also write:

$\Phi = m_1\chi_1 +\ldots+ m_h\chi_h$; which yields:

5.12 Corollary: Keeping the notation above, $(\Phi,\Phi) = \sum_{i=1}^{h} m_i^2$.

Proof:

$(\Phi,\Phi) = (m_1\chi_1 + m_2\chi_2 + \ldots + m_h\chi_h,\ m_1\chi_1 +\ldots+m_h\chi_h)$; of which the only non-zero terms are:

$m_1\overline{m_1}\ (\chi_1,\chi_1) + m_2\overline{m_2}(\chi_2,\chi_2) + \ldots + m_h\overline{m_h}(\chi_h,\chi_h) =$

$m_1\overline{m_1} + m_2\overline{m_2} + \ldots + m_h\overline{m_h} = \sum_{i=1}^{n} m_i^2$, since m_i is a positive integer for each $i = 1,\ldots,h$. ❑

The harvest of results from theorem 5.10 is rich indeed. Another is the converse of theorem 5.2:

5.13 Theorem: If two representations of G have the same character, then they are equivalent.

Proof: Let V_1 and V_2 be representations of G with character Φ. Let $U_1, \ldots, U_r$ be the set of all nonequivalent irreducible representations contained in V_1 or V_2 (i.e., equivalent representation contained in V_1 and V_2 are denoted by a single U_i, $i = 1, \ldots, r$). Then $V_1 \approx \bigoplus_{i=1}^{r} a_i U_i$ and $V_2 \approx \bigoplus_{i=1}^{r} b_i U_i$; a_i, b_i integers.

So $\Phi = \sum_{i=1}^{r} a_i \chi_i = \sum_{i=1}^{r} b_i \chi_i$, where χ_i is the character of U_i. But by theorem 5.10, $(\Phi, \chi_i) = a_i = b_i$ for each $i=1,\ldots,r$; whence $\bigoplus_{i=1}^{r} a_i U_i = \bigoplus_{i=1}^{r} b_i U_i$. Thus $V_1 \approx V_2$. ❑

Also, a direct consequence of corollary 5.12 is a criterion for irreducibility:

5.14 Theorem: Let Φ be the character of a representation V of G. Then $(\Phi,\Phi) = 1$ iff V is irreducible.

Proof: Let $V = m_1 W_1 \oplus \ldots \oplus m_h W_h$ where W_i is irreducible and m_i is the number of times that W_i is contained in V. Then $(\Phi,\Phi) = \sum_{i=1}^{h} m_i^2 = 1$ iff $m_j = 1$ for some $1 \le j \le h$ (and all other terms are 0) iff $V = W_j$. ❑

We can glean more useful results from the character of the regular representation of G as described in section 2.3.

Denote the regular representation by (FG, ρ_R) and recall that its degree is $|G|$. Using the elements of G as a basis for FG, we wrote the matrices of ρ_R as:

$[\rho(s)]_G = (\delta g_i, sg_j)$ for $s \in G$.

Now denote the character of ρ_R by χ_R and we have:

$\chi_R(s) = tr([\rho(s)]_G)$ for $s \in G$.

But $(\delta g_i, sg_j)$ is a permutation matrix. The j^{th} column will have a 1 in the j^{th} row (i.e. on the diagonal) iff $g_j = sg_j$ iff $s = 1_G$, the identity of G.

5.15 Theorem: With the above notation,

$\chi_R(1_G) = |G|$ and $\chi_R(s) = 0$ if $s \neq 1_G$.

Proof: By the preceding comments,

$$tr([\rho(s)]_G) = \begin{cases} \text{degree of } \rho_R = |G| \text{ if } s = 1_G \\ 0 \text{ if } s \neq 1_G \end{cases}. \square$$

In chapter 6, we will describe all of the irreducible representations of some groups by displaying their characters in an organized table. It is desirable, then, to have some place to look for irreducible representations, and how many of each to expect.

5.16 Theorem: Every irreducible representation W_i of G is contained in the regular representation of G with multiplicity equal to its degree n_i.

Proof: Let χ_R be the character of the regular representation of G and let χ_i be the character of W_i.

By theorem 5.10, the number of times that W_i is contained in the regular representation is:

$$(\chi_R, \chi_i) = \frac{1}{|G|} \sum_{t \in G} \chi_R(t)\chi_i(t^{-1}) = \frac{1}{|G|} |G| \chi_i(1_G) = \chi_i(1_G) =$$

n_i, since $t = 1_G$ gives the only non-zero term of the sum. ❑

We should note that saying any irreducible representation W_i is contained in the regular representation is intended in the broader sense that W_i is equivalent to an irreducible representation contained in the regular representation. There are yet other results related to theorem 5.16 which will aid us in determining the characters of the irreducible representations. Let $\{W_1, \ldots, W_r\}$ be all of the nonequivalent irreducible representations of G and let n_i and χ_i be the degree and character of W_i, respectively.

<u>5.17</u> <u>Corollary:</u> With the hypotheses above:

① $\sum_{i=1}^{r} n_i^2 = |G|$; and

② $\sum_{i=1}^{r} n_i\chi_i(s) = 0$, where $s \in G$ but $s \neq 1_G$.

<u>Proof:</u> ① If χ_R is the character of the regular representation of G, then $\chi_R(s) = \sum_{i=1}^{r} n_i\chi_i(s)$ for all $s \in G$. In particular, if $s = 1_G$: $\chi_R(1_G) = \sum_{i=1}^{r} n_i\chi_i(1_G) = \sum_{i=1}^{r} n_i n_i = \sum_{i=1}^{r} n_i^2$; where $\chi_i(1_G) = n_i$ from lemma 5.4 ①.

On the other hand, theorem 5.15 gives $\chi_R(1_G) = |G|$.

② Let $s \in G$ but $s \neq 1_G$. Then $\chi_R(s) = 0$ (theorem 5.15) and, hence, $\sum_{i=1}^{r} n_i\chi_i(s) = 0$. ❑

Chapter 6

Applications of Characters

6.1 Conjugacy Classes

and

Irreducible Representations

In section 5.2, we introduced the vector space CF_G, the space of class functions on G. That is, $CF_G = \{f \mid f(tst^{-1}) = f(s) \text{ for all } s, t \in G\}$. Included in CF_G are the characters of any representation of G. In particular, $\{\chi_1, \chi_2, \ldots, \chi_h\}$, the complete set of characters of nonequivalent irreducible representations is a subset of CF_G. We intend to show that this subset is, in fact, an orthonormal basis of CF_G. First, a somewhat technical lemma is needed:

<u>6.1</u> <u>Lemma:</u> Let $f \in CF_G$ and let (V,ρ) be a representation of G. Define a linear transformation $\rho(f) : V \to V$ by $\rho(f) = \sum_{t \in G} f(t)\rho(t)$. If V is irreducible of degree n and character χ, then $\rho(f) = \lambda$, a scalar multiple of the identity and:

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t)\chi(t) = \frac{|G|}{n} (f, \overline{\chi}).$$

<u>Proof:</u> We will apply part ② of Schur's lemma (4.16), but we must have $\rho(f) \circ \rho(s) = \rho(s) \circ \rho(f)$ for $s \in G$. To that end, we compute:

$$\rho(s)^{-1} \circ \rho(f) \circ \rho(s) = \rho(s)^{-1} \circ \sum_{t \in G} f(t)\rho(t) \circ \rho(s) =$$

$\sum_{t \in G} f(t)(\rho(s)^{-1} \circ \rho(t) \circ \rho(s))$, since $f(t)$ is just a scalar and $\rho(s)$ and $\rho(s)^{-1}$ do not depend on $t \in G$. Now, since ρ is a group homomorphism, $\sum_{t \in G} f(t)(\rho(s)^{-1} \circ \rho(t) \circ \rho(s)) =$

$\sum_{t \in G} f(t)\rho(s^{-1}ts)$. Now substitute $u = s^{-1}ts$ noting: (i) then $t = sus^{-1}$; and (ii) as t runs over the elements of G, so does u. So we have:

$$\rho^{-1}(s) \circ \rho(t) \circ \rho(s) = \sum_{u \in G} f(sus^{-1})\rho(u) =$$

$\sum_{u \in G} f(u)\rho(u) = \rho(f)$, since $f \in CF_G$. Hence, $\rho(f) \circ \rho(s) = \rho(s) \circ \rho(f)$, and we may apply part ② of 4.16.

Then $\rho(f)$ is a scalar λ and $tr(\rho(f)) = tr(\lambda) = n\lambda$. By theorem 4.11 ②, $tr(\rho(f)) = tr(\sum_{t \in G} f(t)\rho(t)) =$

$\sum_{t \in G} f(t)tr(\rho(t))$. But $tr(\rho(t)) = \chi(t)$. So, setting the two expressions for $tr(\rho(f))$ equal to each other:

$$n\lambda = \sum_{t \in G} f(t)tr(\rho(t)) = \sum_{t \in G} f(t)\chi(t) \Rightarrow \lambda =$$

$\frac{1}{n} \sum_{t \in G} f(t)\chi(t)$. Finally, by definition of (,),

$$\frac{1}{n} \sum_{t \in G} f(t)\chi(t) = \frac{|G|}{n} (f,\overline{\chi}). \square$$

Now we can prove:

<u>6.2 Theorem:</u> The set $\{\chi_1, \chi_2, .., \chi_h\}$ forms an orthonormal basis of CF_G.

Proof: From the orthogonality relations, we already know that $\{\chi_1,\ldots,\chi_h\}$ is an orthonormal set, i.e.,

$(\chi_i,\chi_j) = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ otherwise} \end{cases}$. Then $(\overline{\chi}_i,\overline{\chi}_j) = \frac{1}{|G|}\sum_{t\in G} \overline{\chi_i(t)\ \overline{\chi_j}}(t)$

$= (\chi_j,\chi_i) = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ otherwise} \end{cases}$.

So $B = \{\overline{\chi}_1, \overline{\chi}_2,\ldots,\overline{\chi}_h\}$ is also an othonormal set in CF_G. We will actually show B is a basis of CF_G by establishing that the only element of CF_G orthogonal to the $\overline{\chi}_i$, $i = 1,\ldots,h$, is the zero class function.

So let $f \in CF_G$ be such that $(f,\overline{\chi}_i) = 0$, $i=1,\ldots,h$. Let (V,ρ) be any representation of G and define $\rho(f)\colon V \to V$ by $\rho(f) = \sum_{t\in G} f(t)\rho(t)$. There are two cases to consider:

(i) If (V,ρ) is irreducible, then by lemma 6.1, $\rho(f) = \frac{|G|}{n}(f,\overline{\chi})$, where χ is the irreducible character of (V,ρ). Then $\rho(f) = 0$ since $(f,\overline{\chi}) = 0$;

(ii) If (V,ρ) is reducible, let $W_1,\ldots,W_k$ be the nonequivalent irreducible constituents of V, with characters $\chi_1,\ldots,\chi_k$, $k \leq h$. Then, by lemma 6.1:

$$\rho(f)|_{W_i} = \sum_{t\in G} f(t)\rho|_{W_i}(t) = \frac{|G|}{n}(f,\overline{\chi}_i), = 0 \text{ for all } i = 1,\ldots,k.$$

Since $\rho(f) = 0$ on each W_i, then $\rho(f) = 0$ on all of V. So we have, in either case, $\rho(f) = 0$.

Finally, we show $\rho(f) = 0$ implies that $f = 0$. We do so in (FG, ρ_R), the regular representation of G, since it contains every irreducible representation of G. Let g_i be a basis element of FG. Then:

$$\rho_R(f)(g_i) = \sum_{t \in G} f(t)\rho_R(t)(g_i) = \sum_{t \in G} f(t)(tg_i).$$

On the other hand, $\rho(f) = 0$ in any representation, so $\rho_R(f)(g_i) = 0$. This can only be true if $f(t) = 0$ for all $t \in G$. Hence, $f = 0$.

So we have that $B = \{\overline{\chi}_1, \ldots, \overline{\chi}_h\}$, is an orthonormal basis for CF_G. But $\{\chi_1, \ldots, \chi_h\}$ is a set of h orthonormal functions in CF_G, so they must also form an orthonormal basis. ❑

Having determined one basis of CF_G, we shall immediately construct another! The comparison of the two bases will give us a final important result that we need to write down all of the irreducible characters for some specific examples.

From group theory, we know that $t, t' \in G$ are conjugate provided there is an $s \in G$ such that $t' = sts^{-1}$. Conjugacy is an equivalence relation and, thus, partitions G into distinct equivalence classes. Denote the classes by $G_1, G_2, \ldots, G_k$.

If $f \in CF_G$, then $f(t') = f(sts^{-1}) = f(t)$, i.e., f is constant on a given G_i, $i=1,\ldots,k$. Denote the value of f on G_i by λ_i.

Now define functions $f_1, f_2, \ldots, f_k$: $G \to \mathscr{C}$ by:

$$f_i(t) = \begin{cases} 1 \text{ if } t \in G_i \\ 0 \text{ if } t \in G_j \ j \neq i \end{cases}.$$

Is such a function a class function?

For s, $t \in G$, $sts^{-1} \in G_i$ iff $t \in G_i$. So, for $i = 1,\ldots,k$:

$$f_i(sts^{-1}) = \begin{cases} 1 \text{ if } sts^{-1} \in G_i \\ 0 \text{ if } sts^{-1} \in G_j \ j\neq i \end{cases} = \begin{cases} 1 \text{ if } t \in G_i \\ 0 \text{ if } t \in G_j \ j\neq i \end{cases} = f_i(t).$$

Hence, $f_i \in CF_G$ for all $i = 1,\ldots k$.

Furthermore, it is clear that the f_i are linearly independent. And, finally, any $f \in CF_G$ can be uniquely written as a linear combination of the f_i over $\mathscr{C}$. For if $t \in G$, then $t \in G_i$ for some $i=1,\ldots,k$ and $f(t) = 0\, f_1(t) +\ldots+ \lambda_i f_i(t) +\ldots+ 0\, f_k(t) = \lambda_i$.

So we have that $\{f_1,\ldots,f_k\}$ form a basis of CF_G. (This really amounts to nothing more that the standard basis for this space.) Hence, the dimension of CF_G is equal to the number of equivalence classes of G. But, by theorem 6.2, dim CF_G is precisely the number of nonequivalent irreducible representations of G. We have shown:

6.3 Theorem: The number of nonequivalent irreducible representations of G is equal to the number of conjugacy classes of G.

6.4 Corollary: G is abelian iff all irreducible representations of G have degree 1.

Proof: G is abelian iff G has $|G|$ classes iff G has $|G|$ nonequivalent irreducible representations (denote their degrees by $n_1, \ldots, n_{|G|}$) iff $n_i = 1$, $i = 1, \ldots, |G|$ since $|G| = \sum_{i=1}^{|G|} n_i^2$. □

6.2 Character Tables

A character table displays all of the irreducible characters of representations of G. If $G_1, \ldots, G_h$ are the distinct classes of G, let us pick a representative element $g_i \in G_i$ for each $i = 1, \ldots, h$. Let $\{\chi_1, \ldots, \chi_h\}$ be the corresponding set of characters of nonequivalent irreducible representations of G. Then the character table of G has the following form:

	g_1	g_2		g_h
χ_1	$\chi_1(g_1)$	$\chi_1(g_2)$	. . .	$\chi_1(g_h)$
χ_2	$\chi_2(g_1)$	$\chi_2(g_2)$	. . .	$\chi_2(g_h)$
	.	.		.
	.	.		.
	.	.		.
χ_h	$\chi_h(g_1)$	$\chi_h(g_2)$	. . .	$\chi_h(g_h)$

The regular representation FG of G will be a useful guide in constructing the table, as it contains equivalents of every irreducible representation of G. In particular, recall from section 3.1 that if $|G| \geq 2$, it will always have an irreducible representation of degree 1. In the regular representation, the invariant 1-dimensional subspace is generated by the element $x = \sum_{g \in G} g$, and is called the unit representation. It occurs, of course, with multiplicity 1.

Some of the techniques for determining other irreducible characters are perhaps best seen by example.

① As in example ③ of section 2.3, let $G = S_3$. There are three nonequivalent irreducible representations since $\{(1)\}$, $\{(12),(13),(23)\}$, and $\{(123),(132)\}$ are the three conjugacy classes of G. The character of the class $\{(1)\}$ is easy: $\chi_i(1) = n_i$, where n_i is the degree of the representation corresponding to χ_i. As representatives from the other two classes, let $g_2 = (12)$ and $g_3 = (123)$; remembering that the resulting characters are independent of the choice of representatives.

We know that G has at least one representation of degree 1, i.e., the unit representation. Are there any others? If ρ is any representation of G of degree 1, then $\rho(g) = \lambda \in \mathscr{C}$ for $g \in G$. But then we also have $[\rho(g)] = [\lambda]$, so the character of $\rho(g)$ is $\chi_\rho(g) = \lambda = \rho(g)$. That is, for degree 1, the character can be thought of as the representation.

In our case, $g_2{}^2 = (1)$ and $g_3{}^3 = (1)$, so $\chi_\rho(g_2{}^2) = (\chi_\rho(g_2))^2 = 1$ and $\chi_\rho(g_3{}^3) = (\chi_\rho(g_3))^3 = 1$. Hence, $\chi_\rho(g_2) = \pm 1$, and $\chi_\rho(g_3) = 1$, $e^{(2\pi i)/3}$, $e^{(4\pi i)/3}$.

However, there is another relationship between g_2 and g_3: $g_2 g_3 = g_3{}^2 g_2$. So $\chi_\rho(g_2 g_3) = \chi_\rho(g_3{}^2 g_2)$; which gives $\chi_\rho(g_2)\ \chi_\rho(g_3) =$

$(\chi_\rho(g_3))^2 \ \chi_\rho(g_2)$.

Hence $\chi_\rho(g_3)$ can only be 1, eliminating the two complex roots above. But since $\chi_\rho(g_2)$ has two values, we will have two nonequivalent irreducible representations of degree 1. So far, then, the character table for G = S_3 looks like this:

	$g_1 = 1$	g_2	g_3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	$\chi_3(g_1)$	$\chi_3(g_2)$	$\chi_3(g_3)$

The character χ_1 corresponds to the unit representation, while χ_2 is from the alternating representation of example ④ in section 1.2. In the regular representation, the one-dimensional subspace of the alternating representation is generated by $\sum_{g \in G} (\pm)_g \, g$; where $(\pm)_g$ is + if g is an even permutation and - if it is odd.

Now, the degree (and multiplicity) of the representation corresponding to χ_3 is n_3 where $1 + 1 + n_3^2 = 6$. Thus, $n_3 = 2$. Then $\chi_3((1)) = 2$. And from corollary 5.17, $\sum_{i=1}^{3} n_i\chi_i(g_2) = 0$ and $\sum_{i=1}^{3} n_i\chi_i(g_3) = 0$. These yield $\chi_3(g_2) = 0$ and $\chi_3(g_3) = -1$.

So we now have the complete character table for S_3:

	$1 = g_1$	g_2	g_3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

② As a second example, let us consider D_4, the dihedral group of order 8. This group is usually thought of as the group of all symmetries of a square. It consists of rotations about the center through an angle of $\frac{\pi}{2}$, together with reflections about four lines forming angles which are multiples of $\frac{\pi}{4}$. These are just the rotations and reflections which preserve the square. If r is such a rotation and s is any one of the reflections (e.g., about the horizontal axis), then each element of D_4 can be uniquely written as either r^k or sr^k for k = 0, 1, 2, 3.(Note, then, that D_4 has a cyclic subgroup of order 4.)

Furthermore, as with S_3, the relationships among the generators of the group are vital in determining the irreducible characters of degree 1. For D_4 we have: $r^4 = 1$; $s^2 = 1$; and $srs = r^{-1}$.

So if ρ is any irreducible representation of degree 1, $\chi_\rho(r^4) = (\chi_\rho(r))^4 = 1$. Then $\chi_\rho(r) = 1, -1, e^{(\pi i)/2}, e^{\pi i}$.

And $\chi_\rho(s^2) = (\chi_\rho(s))^2 = 1$, yielding $\chi_\rho(s) = \pm 1$. From $srs = r^{-1}$, we have $(sr)^2 = 1$. So $\chi_\rho(sr) = \pm 1$, from which we glean $\chi_\rho(s) = \pm \chi_\rho(r)^{-1}$. But since $\chi_\rho(s) = \pm 1$, we must have, $\chi_\rho(r) = \pm 1$, and we eliminate the imaginary values for $\chi_\rho(r)$.

By taking all combinations of ± 1 for $\chi_\rho(s)$ and $\chi_\rho(r)$, we obtain four irreducible characters of degree one for D_4. They will be denoted by χ_1, χ_2, χ_3, and χ_4 in the table.

Now we know that $\sum_{i=1}^{h} n_i^2 = 8$, where the n_i are the degrees of nonequivalent representations. Hence, there is one representation of degree 2 not yet accounted for. Let its character be χ_5. Then $\chi_5(r^4) = \chi_5(r^o) = \chi_5(1) = 2$. For any other element g of D_4, $\sum_{i=1}^{5} n_i\chi_i(g) = 0$, and we can write down the complete character table. For k = 0, 1, 2, 3:

	r^k	sr^k
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$
χ_5	$-1 + (-1)^{k+1}$	0

The group D_4, then, must have 5 equivalence classes.

Chapter 7

Induced Representations

7.1 Basic Notions and Examples

Given any representation of a group G, we can obtain a representation of a subgroup H of G by a simple restriction (cf. Def. 4.4). It is a bit more complicated to reverse the process. That is, starting with a representation of H, we shall "extend" to a representation of G.

Recall that if $g \in G$ and H is a subgroup of G, then $Hg = \{hg \mid h \in H\}$ is a right coset of H in G. Left cosets are similarly defined. Let $n = \frac{|G|}{|H|}$ be the index of H in G.

A couple of other elementary group theoretic results are used in this section, namely:

① n = the number of right (or left) cosets of H in G; and

② G may be decomposed into mutually exclusive and exhaustive right (or left) cosets. For instance, if $G = Ht_1 \cup Ht_2 \cup \ldots \cup Ht_n$ is a decomposition of G into right cosets, the elements $t_1, t_2, \ldots, t_n$ are called a <u>system of representatives</u> of the right cosets of H in G.

Suppose that $\theta : H \to GL(m,F)$ is a matrix representation of H. We define an extension:

$\bar{\theta} : G \to GL(m, F)$ by:

$$\bar{\theta}(x) = \begin{cases} \theta(x) & \text{if } x \in H \\ 0_{m\times m} & \text{if } x \in G - H \end{cases} .$$

Since the zero matrix is singular, $\bar{\theta}$ will not, in general, be a matrix representation of G. So we define a different map for $x \in G$: $\Phi(x) =$

$$(\bar{\theta}(t_i\ x\ t_j^{-1})) = \begin{bmatrix} \bar{\theta}(t_1xt_1^{-1}) & \bar{\theta}(t_1xt_2^{-1}) & . & . & . & \bar{\theta}(t_1xt_n^{-1}) \\ \bar{\theta}(t_2xt_1^{-1}) & . & . & . & & \vdots \\ \vdots & & & & & \\ \bar{\theta}(t_nxt_1^{-1}) & . & . & . & & \bar{\theta}(t_nxt_n^{-1}) \end{bmatrix},$$

where the $t_1, t_2, \ldots, t_n$ are as in ② above. So $\Phi(x)$ is an $mn \times mn$ matrix consisting of n^2 submatrices each of size $m \times m$.

The proof that $\Phi : G \to GL(mn,F)$ is a matrix representation of G is computational in nature and is facilitated by:

<u>7.1</u> <u>Lemma:</u> Let H by a subgroup of G and let $t_1, t_2, \ldots, t_n$ be a system of representatives of the right cosets of H in G. Then for fixed $i, j = 1, \ldots, n$ and $x, y \in G$, there exists at most one value of k $(k=1,\ldots,n)$ such that $t_i\ x\ t_k^{-1} \in H$ and $t_kyt_j^{-1} \in H$.

<u>Proof:</u> Actually, it suffices to show there is at most one value of k $(k = 1,\ldots,n)$ such that $t_i\ x\ t_k^{-1} \in H$. For if there is such a value of k, it may or may not be true that $t_kyt_j^{-1} \in H$. In either case, the lemma will be true.

So let k_1 and k_2 $(k_1, k_2 = 1,\ldots,n)$ be such that $t_i x t_{k_1}^{-1} \in H$ and $t_k x t_{k_2}^{-1} \in H$. Then $t_i x$ and t_{k_1} are in the same coset, as are $t_i x$ and t_{k_2}. Hence, t_{k_1} and t_{k_2} are in the same coset; which yields $k_1 = k_2$ since t_{k_1} and t_{k_2} were from the system of representatives. ❑

7.2 Theorem: Let $x \in G$ and $i, j = 1,\ldots,n$. Then the map $\Phi: G \to GL(mn, F)$ given by $\Phi(x) = (\bar{\theta}(t_i x t_j^{-1}))$ $(x \in G, i, j = 1,\ldots, n)$ is a matrix representation of G.

Proof: Let x and y be elements of G. Then $\Phi(x)\,\Phi(y) =$

$$\begin{bmatrix} \sum_{k=1}^{n}\bar{\theta}(t_1 x t_k^{-1})\bar{\theta}(t_k y t_1^{-1}) & \cdots & \sum_{k=1}^{n}\bar{\theta}(t_1 x t_k^{-1})\bar{\theta}(t_k y t_n^{-1}) \\ \vdots & & \vdots \\ \sum_{k=1}^{n}\bar{\theta}(t_n x t_k^{-1})\bar{\theta}(t_k y t_1^{-1}) & \cdots & \sum_{k=1}^{n}\bar{\theta}(t_n x t_k^{-1})\bar{\theta}(t_k y t_n^{-1}) \end{bmatrix}.$$

In general, the mxm submatrix in the i, j position of $\Phi(x)\,\Phi(y)$ is $\sum_{k=1}^{n}\bar{\theta}(t_i x t_k^{-1})\bar{\theta}(t_k y t_j^{-1})$, $i, j = 1,\ldots, n$.

On the other hand, then mxm submatrix in the i,j position of $\Phi(xy)$ is $\bar{\theta}(t_i x y t_j^{-1}) = \bar{\theta}((t_i x t_k^{-1})(t_k y t_j^{-1}))$. The theorem will be proved if we can establish that the corresponding submatrices of $\Phi(x)\,\Phi(y)$ and $\Phi(xy)$ are equal. In the sum $\sum_{k=1}^{n}\bar{\theta}(t_i x t_k^{-1})\bar{\theta}(t_k y t_j^{-1})$, the only non-zero matrices occur for values of k such that $t_i x t_k^{-1} \in H$ and $t_k y t_j^{-1} \in H$. This follows from the definition of $\bar{\theta}$ and the fact that

$\theta : H \to GL(m, F)$ is a matrix representation of H. From lemma 7.1, there is at most one k ($k = 1,\ldots,n$) such that $t_i x t_k^{-1} \in H$ and $t_k y t_j^{-1} \in H$. This leaves us two cases:

① Suppose there is no value of k ($k = 1, \ldots, n$) such that $t_i x t_k^{-1} \in H$ and $t_k y t_j^{-1} \in H$. Then $\sum_{k=1}^{n} \bar{\theta}(t_i x t_k^{-1})\bar{\theta}(t_k y t_j^{-1}) = 0_{mxm}$. But we also have $(t_i x t_k^{-1})(t_k y t_j^{-1}) = t_i x y t_j^{-1} \notin H \Rightarrow \bar{\theta}(t_i x y t_j^{-1}) = 0_{mxm}$. Hence, in this case, the corresponding submatrices are equal.

② Suppose there is exactly one value of k ($k = 1,\ldots,n$) such that $t_i x t_k^{-1} \in H$ and $t_k y t_j^{-1} \in H$. Denote the value by k_1.

Then $\sum_{k=1}^{n} \bar{\theta}(t_i x t_k^{-1})\bar{\theta}(t_k y t_j^{-1}) = \bar{\theta}(t_i x t_{k_1}^{-1})\bar{\theta}(t_{k_1} y t_j^{-1}) = \theta(t_i x t_{k_1}^{-1})\theta(t_{k_1} y t_j^{-1}) = \theta((t_i x t_{k_1}^{-1})(t_{k_1} y t_j^{-1})) = \theta(t_i x y t_j^{-1}) = \bar{\theta}(t_i x y t_j^{-1})$, since $t_i x y t_j^{-1} \in H$. Hence, the corresponding submatrices are equal.

Therefore, in either case, we have $\Phi(x)\,\Phi(y) = \Phi(xy)$ and the theorem is proved. ❑

This matrix approach to induced representations has the advantage of being intuitively straight forward. However, it is inconvenient to establish properties (e.g., that the map Φ is independent of the choice of the system of representatives) using matrices, so we will not attempt to do so here.

As an example of an induced representation, consider D_4 as outlined in section 6.2. We have $H = \{1, r, r^2, r^3\}$ as a cyclic subgroup of D_4, so the index of H in D_4 is 2. A representation of degree 2 of H is given by $\theta : H \to GL(2, \mathscr{R})$ where:

$$\theta(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \theta(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix};$$

$$\theta(r^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}; \text{ and } \qquad \theta(r^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let $t_1 = r$ and $t_2 = rs$ be a system of representatives of the right cosets of H in D_4. Then we have $D_4 = Hr \cup Hrs$ where $Hr = H$ and $Hrs = \{rs, r^2s, r^3s, s\}$. Hence $\Phi: D_4 \to GL(4, \mathscr{R})$ is a matrix representation of D_4. The matrices corresponding to r and s are computed below:

$$\Phi(r) = \begin{bmatrix} \overline{\theta}(rrr^{-1}) & \overline{\theta}(rr(rs)^{-1}) \\ \overline{\theta}(rsrr^{-1}) & \overline{\theta}((rs)r(rs)^{-1}) \end{bmatrix} =$$

$$\begin{bmatrix} \overline{\theta}(r) & \overline{\theta}(r^2s\ r^3) \\ \overline{\theta}(rs) & \overline{\theta}(r^3) \end{bmatrix} = \begin{bmatrix} \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} & 0_{2x2} \\ 0_{2x2} & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{bmatrix}.$$

Similarly, $\Phi(s) = \begin{bmatrix} \overline{\theta}(rsr^{-1}) & \overline{\theta}(rs(rs)^{-1}) \\ \overline{\theta}(rssr^{-1}) & \overline{\theta}(rss(rs)^{-1}) \end{bmatrix} =$

$$\begin{bmatrix} \bar{\theta}(rsr^{-1}) & \bar{\theta}(1) \\ \bar{\theta}(1) & \bar{\theta}(rsr^{-1}) \end{bmatrix} = \begin{bmatrix} 0_{2x2} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0_{2x2} \end{bmatrix}.$$

7.2 Character of an Induced Representation

We can derive the character of an induced representation in a manner inspired by the construction in Section 7.1. Let Φ be a matrix representation of a group G induced by the representation θ of a subgroup H of G. Also let $\chi_\theta : H \to \mathscr{C}$ be the character of θ and let $G = Ht_1 \cup Ht_2 \cup \ldots \cup Ht_n$ be a decomposition of G into right cosets.

Extend χ_θ to all of G by :

$$\overline{\chi_\theta}(x) = \begin{cases} \chi_\theta(x) & \text{if } x \in H \\ 0 & \text{if } x \in G - H \end{cases}.$$

We may be tempted to stop here and simply write $\overline{\chi_\theta}(x) = \chi_\Phi(x)$, the character of Φ, for all $x \in G$. But we must be careful about deciding that $\chi_\Phi(x) = 0$ for $x \in G - H$. We must allow for the possibility that $t_i x \in Ht_i$ for some $i = 1,\ldots,n$. In that case, $t_i x t_i^{-1} \in H$ and $tr(\bar{\theta}(t_i x t_i^{-1}))$ may <u>not</u> be 0.

So let $\Psi(x) = \sum_{i=1}^{n} \overline{\chi_\theta}(t_i x t_i^{-1})$ for all $x \in G$. From the definitions of $\bar{\theta}$ and $\overline{\chi_\theta}$ note that: $\overline{\chi_\theta}(t_i x t_i^{-1}) = tr(\bar{\theta}(t_i x t_i^{-1}))$. So $\Psi(x) = \sum_{k=1}^{n} \overline{\chi_\theta}(t_i x t_i^{-1}) =$

$\sum_{i=1}^{n} tr(\overline{\theta}(t_i x t_i^{-1}))) = tr(\Phi(x)) = \chi_\Phi(x)$ for all $x \in G$. We call χ_Φ the character induced by χ_θ and write $\mathrm{Ind}_H^G \chi_\theta$.

We can write the formula for $\Psi(x) = \chi_\Phi(x)$ is a slightly different way. Recall that $\overline{\chi_\theta}\ (t_i x t_i^{-1})$ can only be non-zero if $t_i\, x \in Ht_i$. If $t_i\, x \in Ht_i$, then $ht_i x \in Ht_i$ for all $h \in H$ and $\overline{\chi_\theta}(t_i x t_i^{-1}) = \overline{\chi_\theta}\ ((ht_i)x(ht_i)^{-1})$ since $t_i x t_i^{-1}$ and $(ht_i)x(ht_i)^{-1}$ are conjugate elements. Finally, since $hxh^{-1} \in H$ iff $x \in H$, we may write:

$$\chi_\Phi(x) = \frac{1}{|H|} \sum_{g \in G} \overline{\chi_\theta}(gxg^{-1}) \text{ for all } x \in G.$$

We conclude this section with a look back at the example (D_4) in Section 7.1. It is clear from the matrices that $\chi_\Phi(r) = \chi_\Phi(s) = 0$. Similarly, in $\Phi(r^2s)$, the submatrices on the diagonal would be $\overline{\theta}(rr^2sr^{-1})$ and $\overline{\theta}(rsr^2s(rs)^{-1})$; each of the form $\overline{\theta}(t_i x t_i^{-1})$. But in both cases, $t_i x \notin Ht_i$. Hence, $\overline{\chi_\theta}\ (t_i x t_i^{-1}) = 0$ for $i = 1, 2$ and $\chi_\Phi(r^2s) = 0$.

But in $\Phi(r^2)$, we do have $t_i x \in Ht_i$ for $i = 1$ and 2. In particular, $t_i r^2 t_i^{-1} = r^2$ for $i = 1$ and 2. Then $\overline{\chi_\theta}\ (t_i r^2 t_i^{-1}) = -2$, so $\chi_\Phi(r^2) = -4$. The characters of the other elements of this representation of degree 4 can be similarly computed.

7.3 Induced Representation by Automorphism

Automorphism representations corresponding to the matrix representations in Section 7.1 can be found as in Section 1.2.

However, we give a brief and direct description here of the conditions under which an automorphism representation of a group is said to be induced by a representation of a subgroup. The notation established will help us state and interpret the Frobenius reciprocity formula in the final section.

Let $G = t_1H \cup t_2H \cup \ldots \cup t_nH$ be a decomposition of G into left cosets and let (ρ, V) be a linear representation of G. Then by definition 4.4, $\mathrm{Res}_H^G\, \rho$ is a representation of H. Now let W be a subrepresentation of $\mathrm{Res}_H^G\, \rho$, and denote this representation of H by $\overline{\rho} : H \to GL(W)$.

For any $s \in G$ consider $\rho(s)(W)$, the image of W under $\rho(s)$. Since s is in some left coset of G, we can write $s = t_ih$ for some t_i from the system of representatives and some $h \in H$. Then $\rho(s)(W) = \rho(t_ih)(W) = \rho(t_i)\circ\rho(h)(W)$, since ρ is a representation of G. But $\rho(h)(W) = W$ because W is stable under $\rho(h)$ for $h \in H$. So $\rho(s)(W) = \rho(t_i)(W)$. In other words, the image depends only on the left coset from which s comes. Let $\rho(s)(W) = W_i$ for $i = 1,..,n$ where $s \in t_iH$ and W_i is a subspace of V. Then we say that (ρ, V) is induced by $(\overline{\rho}, W)$ iff $V = \bigoplus_{i=1}^{n} W_i$. We denote the induced representation by $\mathrm{Ind}_H^G\, \overline{\rho}$ or $\mathrm{Ind}_H^G\, W$.

7.4 The Frobenius Reciprocity Formula

The Frobenius formula will be developed in the general setting of class functions. The result of most interest to us will then be an immediate corollary in terms of the special case of characters.

Let H be a subgroup of G and let f be a class function on H. Since the irreducible characters of H form a basis for the vector space CF_H of class functions on H, f can be thought of as a linear combination of characters. Now define f_0 on G by:

$$f_0(x) = \begin{cases} f(x) & \text{if } x \in H \\ 0 & \text{if } x \in G - H \end{cases};$$

and f′ on G by:

$$f'(x) = \frac{1}{|H|} \sum_{t \in G} f_0(txt^{-1}).$$

The only (possibly) non-zero terms in the sum are those where $txt^{-1} \in H$, and such terms are equal to $f(txt^{-1})$. Hence, f′ is a class function on G. We say that f′ is the class function on G induced by f and write $f' \equiv \operatorname{Ind}_H^G f$.

If we now recall the inner product defined on class functions in Section 5.2, we can state:

<u>7.3 Theorem:</u> (Frobenius) Let H be a subgroup of G, let f be a class function on H, and let g be a class function on G. Then $(f, g|_H) = (\operatorname{Ind}_H^G f, g)$, where $g|_H$ is the restriction of g to H.

Proof: As in the comments above, let

$$f_0(x) = \begin{cases} f(x) \text{ if } x \in H \\ 0 \text{ if } x \in G - H \end{cases}.$$

Then: $(\text{Ind}_H^G f, g) = \frac{1}{|G|} \sum_{t \in G} \text{Ind}_H^G f(t)\, \overline{g(t)} =$

$$\frac{1}{|G|} \sum_{t \in G} \left(\frac{1}{|H|} \sum_{x \in G} f_0(xtx^{-1})\right) \overline{g(t)} =$$

$\frac{1}{|G|}\frac{1}{|H|} \sum_{t \in G} \sum_{x \in G} f_0(xtx^{-1})\, \overline{g(xtx^{-1})}$, from the definition of $\text{Ind}_H^G f$ and the fact that g is a class function. The double summation will simply be the sum of all possible products of the form $f_0(xtx^{-1})\, \overline{g(xtx^{-1})}$ for all $x, t \in G$. So we may change the order of summation and write:

$$(\text{Ind}_H^G f, g) = \frac{1}{|G|}\frac{1}{|H|} \sum_{x \in G} \sum_{t \in G} f_0(xtx^{-1})\, \overline{g(xtx^{-1})} .$$

Now, for fixed $x \in G$, as t ranges over all elements of G, so does the product xtx^{-1}. So with the substitution $y = xtx^{-1}$ (and slight abuse of notation):

$$(\text{Ind}_H^G f, g) = \frac{1}{|G|}\frac{1}{|H|} \sum_{x \in G} \sum_{y \in G} f_0(y)\, \overline{g(y)} .$$

But since f is a class function on H, the non-zero terms are repeated |G| times; and the only possible non-zero terms occur when $y \in H$. So:

$$(\text{Ind}_H^G f, g) = \frac{1}{|H|} \sum_{y \in G} f_0(y)\, \overline{g(y)} = \frac{1}{|H|} \sum_{y \in H} f(y)\, \overline{g|_H(y)} =$$

$(f, g|_H)$. ❑

The application to representations may be stated as:

7.4 Corollary: Let χ be the character of an irreducible representation W of H (a subgroup of G), and let Ψ be the character of an irreducible representation of V of G. Then:

$$(\chi, \Psi|_H) = (\text{Ind}_H^G \chi, \Psi), \text{ where } \Psi|_H \text{ is the}$$

character of the representation V restricted to H.

In light of section 5.3, we conclude with a particularly nice intuitive interpretation of this corollary: The number of times that W occurs in the restriction of V to H is equal to the number of times that V is contained in Ind_H^G W.

Chapter 8

Summary

The purpose of this paper was to survey elementary concepts of linear representations of finite groups. Chapter 1 presented fundamental definitions of representation of a group G by automorphisms of a vector space, nxn matrices, and by G-modules. Some concrete examples were displayed, and a notion of equivalence of representations was defined.

For a field F and group G, the group algebra FG was developed in Chapter 2. The important result here was the regular representation of G. Examples were given.

According to Chapter 3, if V is a representation of G, it may have nontrivial subspaces which are also representations of G. Such was the definition of subrepresentations. This led to the quotient representation and to the possibility of writing V as a direct sum of subrepresentations. An important result here was that the regular representation FG has a subrepresentation of degree 1 known as the unit representation.

Chapter 4 stated that every representation is a direct sum of irreducibles (Maschke). Another major theorem here was Clifford's Theorem: Given an irreducible representation V of G and an irreducible H-submodule for H Δ G, we can know all of the irreducible H-submodules of V_H. Schur's Lemma and corollaries paved the way for results about characters.

The character of a representation was defined as the trace of the linear transformation or matrix associated with each element of a group. Chapter 5 developed the properties of group characters through the use of an inner product which determined whether two irreducible representations were equivalent.

Characters were further examined in Chapter 6. A noteworthy result here was that the number of nonequivalent irreducible representations of G is equal to the number of conjugacy classes of G. All of the irreducible characters of S_3 and D_4 were displayed in character tables.

If H is a subgroup of G, Chapter 7 showed how a representation of H induced a representation of G. The concept of inducted characters was presented, culminating in the Frobenius reciprocity formula.

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